Further centres of mass 3A
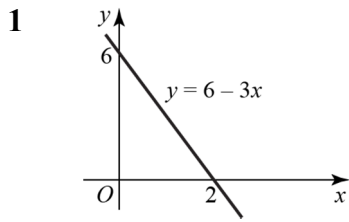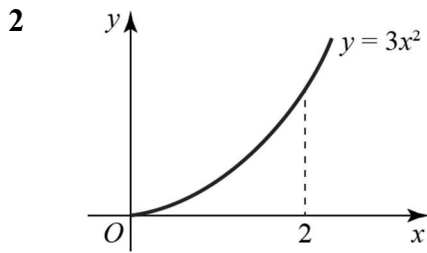
1

Mass $= M = \dfrac{1}{2}\rho \times 2 \times 6 = 6\rho$, where ρ is the mass per unit area.

$$M\overline{x} = \int_0^2 \rho x(6 - 3x)\,\mathrm{d}x \qquad M\overline{y} = \int_0^2 \rho \dfrac{1}{2}(6 - 3x)^2\,\mathrm{d}x$$

$$= \rho\int_0^2 6x - 3x^2\,\mathrm{d}x \qquad\qquad = \dfrac{1}{2}\rho\int_0^2 36 - 36x + 9x^2\,\mathrm{d}x$$

$$= \rho\left[3x^2 - x^3\right]_0^2 \qquad\qquad = \dfrac{1}{2}\rho\left[36x - 18x^2 + 3x^3\right]_0^2$$

$$= \rho[4 - 0] \qquad\qquad\qquad = \dfrac{1}{2}\rho[24 - 0]$$

$$= 4\rho \qquad\qquad\qquad\qquad = 12\rho$$

$$\therefore \overline{x} = \dfrac{4\rho}{M} = \dfrac{4\rho}{6\rho} \qquad\qquad \therefore \overline{y} = \dfrac{12\rho}{M} = \dfrac{12\rho}{6\rho}$$

$$= \dfrac{2}{3} \qquad\qquad\qquad\qquad = 2$$

The centre of mass is at the point with coordinates $\left(\dfrac{2}{3}, 2\right)$

2

$$M = \int_0^2 \rho y \, dx$$

$$= \rho \int_0^2 3x^2 dx$$

$$= \rho \left[x^3 \right]_0^2$$

$$= 8\rho$$

$$M\overline{x} = \int_0^2 \rho x 3x^2 \, dx$$

$$= \rho \int_0^2 3x^3 dx$$

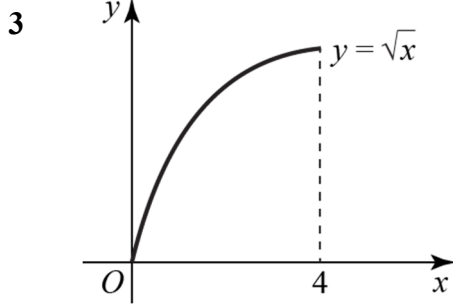$$= \rho \left[\frac{3}{4} x^4 \right]_0^2$$

$$= 12\rho$$

$$\therefore \overline{x} = \frac{12\rho}{M} = \frac{12\rho}{8\rho}$$

$$= \frac{3}{2} = 1.5$$

$$M\overline{y} = \int_0^2 \frac{1}{2} \rho \left(3x^2 \right)^2 \, dx$$

$$= \frac{1}{2} \rho \int_0^2 9x^4 \, dx$$

$$= \frac{9}{2} \rho \left[\frac{x^5}{5} \right]_0^2$$

$$= \frac{9 \times 32}{10} \rho$$

$$= \frac{144}{5} \rho$$

$$\therefore \overline{y} = \frac{144\rho}{5M} = \frac{144\rho}{40\rho}$$

$$= 3.6$$

The centre of mass is at the point with coordinates (1.5, 3.6).

3

$$M = \rho \int_0^4 y \, dx$$

$$= \rho \int_0^4 x^{\frac{1}{2}} \, dx$$

$$= \left[\frac{2}{3} \rho x^{\frac{3}{2}} \right]_0^4$$

$$= \frac{16\rho}{3}$$
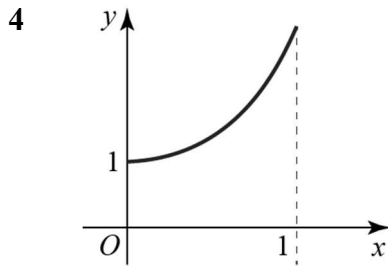
$$M\bar{x} = \rho \int_0^4 x x^{\frac{1}{2}} \, dx$$

$$= \rho \int_0^4 x^{\frac{3}{2}} \, dx$$

$$= \left[\frac{2}{5} \rho x^{\frac{5}{2}} \right]_0^4$$

$$= \frac{64\rho}{5}$$

$$\therefore \bar{x} = \frac{64\rho}{5M}$$

$$= \frac{64\rho}{5} \div \frac{16\rho}{3}$$

$$= \frac{64\rho}{5} \times \frac{3}{16\rho}$$

$$= \frac{12}{5} = 2.4$$

$$M\bar{y} = \frac{1}{2} \rho \int_0^4 \left(x^{\frac{1}{2}} \right)^2 \, dx$$

$$= \frac{1}{2} \rho \int_0^4 x \, dx$$

$$= \frac{1}{2} \rho \left[\frac{x^2}{x} \right]_0^4$$

$$= 4\rho$$

$$\therefore \bar{y} = \frac{4\rho}{M}$$

$$= 4\rho \div \frac{16\rho}{3}$$

$$= 4\rho \times \frac{3}{16\rho}$$

$$= \frac{3}{4} = 0.75.$$

The centre of mass is at the point with coordinates (2.4, 0.75).

4

$$M = \rho\int_0^1 (x^3 + 1)\,dx$$

$$= \rho\left[\frac{1}{4}x^4 + x\right]_0^1$$

$$= \frac{5}{4}\rho$$

$$M\bar{x} = \rho\int_0^1 x(x^3 + 1)\,dx \qquad M\bar{y} = \frac{1}{2}\rho\int_0^1 (x^3 + 1)^2\,dx$$

$$= \rho\int_0^1 x^4 + x^4\,dx \qquad = \frac{1}{2}\rho\int_0^1 x^6 + 2x^3 + 1\,dx$$

$$= \rho\left[\frac{1}{5}x^5 + \frac{1}{2}x^2\right]_0^1 \qquad = \frac{1}{2}\rho\left[\frac{1}{7}x^3 + \frac{1}{2}x^4 + x\right]_0^1$$

$$= \frac{7}{10}\rho \qquad = \frac{\rho}{2}\times\frac{23}{14} = \frac{23\rho}{28}$$

$$\therefore \bar{x} = \frac{7}{10}\rho \div M \qquad \therefore \bar{y} = \frac{23\rho}{28} \div M$$

$$= \frac{7}{10}\rho \div \frac{5}{4}\rho \qquad = \frac{23\rho}{28} \div \frac{5}{4}\rho$$

$$= \frac{7\rho}{10}\times\frac{4}{5\rho} \qquad = \frac{23\rho}{28}\times\frac{4}{5\rho}$$

$$= \frac{28}{50} = \frac{14}{25} \qquad = \frac{23\rho}{35\rho} = \frac{23}{35}$$

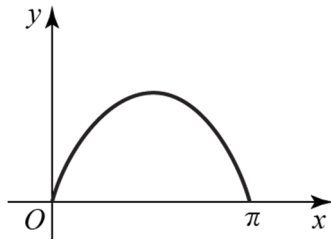The centre of mass is at the point with coordinates $\left(\dfrac{14}{25}, \dfrac{23}{35}\right)$

5
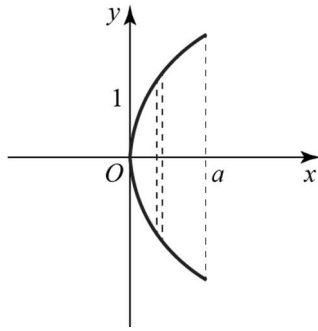
$$M = \rho \int_0^a 2y \, dx$$

$$= \rho \int_0^a 2 \times 2a^{\frac{1}{2}} x^{\frac{1}{2}} \, dx$$

$$= 4\rho a^{\frac{1}{2}} \left[\frac{2}{3} x^{\frac{3}{2}} \right]_0^a$$

$$= \frac{8}{3} \rho a^2$$

$$M\bar{x} = \rho \int_0^a x \times 4a^{\frac{1}{2}} x^{\frac{1}{2}} \, dx$$

$$= \rho \int_0^a 4a^{\frac{1}{2}} x^{\frac{3}{2}} \, dx$$

$$= \rho \times 4a^{\frac{1}{2}} \left[\frac{2}{5} x^{\frac{5}{2}} \right]_0^a$$

$$= \frac{8\rho}{5} a^3$$

$$\therefore \bar{x} = \frac{8\rho a^3}{5} \div M$$

$$= \frac{8\rho a^3}{5} \div \frac{8\rho a^2}{3}$$

$$= \frac{3}{5} a$$

From symmetry $\bar{y} = 0$

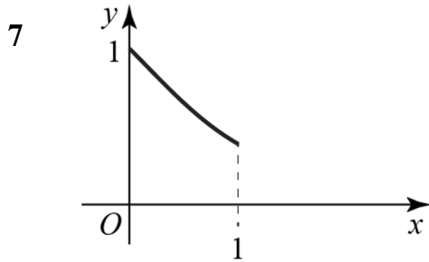$\therefore$ The centre of mass is at the point with coordinates $\left(\frac{3}{5} a, 0 \right)$

6

$$y = \sin x$$

$$M = \rho \int_0^\pi \sin x \, dx$$

$$= \rho \left[-\cos x \right]_0^\pi$$

$$= 2\rho$$

From symmetry $\bar{x} = \dfrac{\pi}{2}$

$$M\bar{y} = \frac{1}{2} \rho \int_0^\pi \sin^2 x \, dx$$

$$= \frac{1}{2} \rho \times \frac{1}{2} \int_0^\pi 1 - \cos 2x \, dx$$

$$= \frac{1}{4} \rho \left[x - \frac{1}{2} \sin 2x \right]_0^\pi$$

$$= \frac{1}{4} \rho \pi$$

$$\therefore \bar{y} = \frac{1}{4} \rho \pi \div M = \frac{1}{4} \rho \pi \div 2\rho$$

$$= \frac{1}{8} \pi$$

The centre of mass is at the point with coordinates $\left(\dfrac{\pi}{2}, \dfrac{\pi}{8} \right)$

7

$$M = \rho \int_0^1 y \, dx$$

$$= \rho \int_0^1 \frac{1}{1+x} \, dx$$

$$= \rho \left[\ln(1+x) \right]_0^1$$

$$= \rho \ln 2$$

$$M\overline{x} = \rho \int_0^1 \frac{x}{1+x} \, dx$$

$$= \rho \int_0^1 1 - \frac{1}{1+x} \, dx$$

$$= \rho \left[x - \ln(1+x) \right]_0^1$$

$$= \rho \left[1 - \ln 2 \right]$$

$$\therefore \overline{x} = \rho \frac{\left[1 - \ln 2 \right]}{M}$$

$$= \rho \frac{\left[1 - \ln 2 \right]}{\rho \ln 2}$$

$$= \frac{1 - \ln 2}{\ln 2}$$

$$M\overline{y} = \frac{1}{2} \rho \int_0^1 \frac{1}{(1+x)} \, dx$$

$$= \frac{1}{2} \rho \left[-(1+x)^{-1} \right]_0^1$$

$$= \frac{1}{2} \rho \left[\frac{-1}{2} + 1 \right]$$

$$= \frac{1}{4} \rho$$

$$\therefore \overline{y} = \frac{1}{4} \rho \div M$$

$$= \frac{\frac{1}{4} \rho}{\rho \ln 2}$$

$$= \frac{1}{4 \ln 2}$$

The centre of mass is at the point with coordinates $\left(\dfrac{1 - \ln 2}{\ln 2}, \dfrac{1}{4 \ln 2} \right)$

8
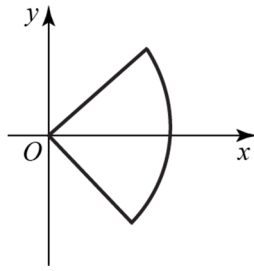
$M = \dfrac{1}{4}\rho\pi r^2$ as this is a quarter of a circle.

$$M\overline{x} = \rho\int_{\frac{-\pi}{4}}^{\frac{\pi}{4}} \frac{1}{2}r^2 \times \frac{2}{3}r\cos\theta \, d\theta$$

$$= \rho \times \frac{1}{3}r^3\int_{\frac{-\pi}{4}}^{\frac{\pi}{4}} \cos\theta \, d\theta$$

$$= \frac{1}{3}\rho r^3\left[\sin\theta\right]_{\frac{-\pi}{4}}^{\frac{\pi}{4}}$$

$$= \frac{1}{3}\rho r^3\left[\frac{1}{\sqrt{2}} - \left(\frac{-1}{\sqrt{2}}\right)\right]$$
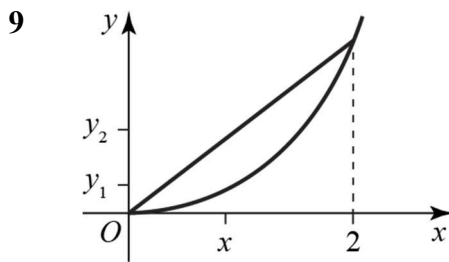
$$= \frac{2\rho r^3}{3\sqrt{2}}$$

$$\therefore \overline{x} = \frac{2\rho r^3}{3\sqrt{2}} \div \frac{1}{4}\rho\pi r^2$$

$$= \frac{8r}{3\pi\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$$

$$= \frac{8\sqrt{2}}{6\pi} = \frac{4\sqrt{2}}{3\pi}$$

Also $\overline{y} = 0$ from symmetry.

The centre of mass is at the point with coordinates $\left(\dfrac{4\sqrt{2}r}{3\pi}, 0\right)$

9

$y = x^3$ meets $y = 4x$ when $x^3 = 4x$ i.e. $x = \pm 2$.

$\therefore$ when $x > 0, x = 2$

The small strip shown has dimensions $(y_2 - y_1)$ by δx and centre of mass at

$$\left(x, \frac{1}{2}(y_1 + y_2)\right)$$

$$M = \rho \int_0^2 (4x - x^3)\, dx$$

$$= \rho \left[2x^2 - \frac{1}{4}x^4 \right]_0^2$$

$$= 4\rho$$

$$M\bar{x} = \rho \int_0^2 x(4x - x^3)\, dx$$

$$= \rho \int_0^2 4x^2 - x^4\, dx$$

$$= \rho \left[\frac{4}{3}x^3 - \frac{1}{5}x^5 \right]_0^2$$

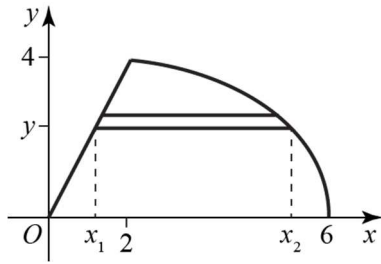$$= \rho \left[\frac{32}{3} - \frac{32}{5} \right]$$

$$= \frac{64\rho}{15}$$

$$\therefore \bar{x} = \frac{64\rho}{15} \div M = \frac{64\rho}{15} \div 4\rho$$

$$= \frac{16}{15}$$

$$M\bar{y} = \frac{1}{2}\rho \int_0^2 (y_1 + y_2)(y_2 - y_1)\, dx$$

$$= \frac{1}{2}\rho \int_0^2 (4x + x^3)(4x - x^3)\, dx$$

$$= \frac{1}{2}\rho \int_0^2 16x^2 - x^6\, dx$$

$$= \frac{\rho}{2} \left[\frac{16}{3}x^3 - \frac{1}{7}x^7 \right]_0^2$$

$$= \frac{\rho}{2} \left[\frac{128}{3} - \frac{128}{7} \right]$$

$$= \frac{256}{21}\rho$$

$$\therefore \bar{y} = \frac{256}{21}\rho \div M = \frac{256}{21}\rho \div 4\rho$$

$$= \frac{64}{21}$$

The centre of mass is at the point with coordinates $\left(\frac{16}{15}, \frac{64}{21}\right)$

10

Using $M = \rho \int_0^4 (x_2 - x_1)\,dy$

$$\therefore M = \rho \int_0^4 \left(6 - \frac{1}{4}y^2 - \frac{y}{2} \right) dy$$

$$= \rho \left[6y - \frac{1}{12}y^3 - \frac{1}{4}y^2 \right]_0^4$$

$$= \rho \left[24 - \frac{16}{3} - 4 \right]$$

$$= \frac{44}{3}\rho$$

Using $M\bar{y} = \rho \int_0^4 y(x_2 - x_1)\,dy$

$$M\bar{y} = \rho \int_0^4 6y - \frac{1}{4}y^3 - \frac{y^2}{2}\,dy$$

$$= \rho \left[3y^2 - \frac{1}{16}y^4 - \frac{1}{6}y^3 \right]_0^4$$

$$= \rho \left[48 - 16 - \frac{64}{6} \right]$$

$$= \frac{64}{3}\rho$$

$$\therefore \bar{y} = \frac{64}{3}\rho \div M = \frac{64}{3} \times \frac{3}{44}$$

$$= \frac{16}{11}$$

$$= 1.5\,(2\text{ s.f.})$$

10 continued

Using $M\bar{x} = \rho \int_0^6 yx\,dx = \rho \int_0^2 yx\,dx + \rho \int_2^6 yx\,dx$

$= \rho \int_0^2 2x^2\,dx + \rho \int_2^6 x\sqrt{24-4x}\,dx$

$\qquad = \rho \left[\dfrac{2}{3}x^3\right]_0^2 - \rho \left[\dfrac{4}{5}(x+4)(6-x)\sqrt{6-x}\right]_2^6$

$\qquad = \rho \left[\dfrac{16}{3} + \dfrac{192}{5}\right] = \rho \left[\dfrac{656}{15}\right]$

$\qquad \therefore \bar{x} = \dfrac{656}{15}\rho \div M = \dfrac{656}{15} \times \dfrac{3}{44}$

$\qquad = 2\dfrac{54}{55}$

$\qquad = 2.98\,(2\text{ s.f.})$

Hence the centre of mass has coordinates $\left(2\dfrac{54}{55}, 1\dfrac{5}{11}\right)$

11   By symmetry, the centre of mass of the uniform lamina will lie on the $y = 0$ axis. The mass of the

lamina is $M = 2\int_0^4 \rho y\,dx$, where ρ is the density. Integrating gives $M = 2\rho \int_0^4 \dfrac{x+5}{(x+2)(2x+1)}\,dx$

$\qquad = 2\rho \int_0^4 \left(\dfrac{3}{2x+1} - \dfrac{1}{x+2}\right)dx$

$\qquad = 2\rho \left[\dfrac{3}{2}\ln(2x+1) - \ln(x+2)\right]_0^4$

$\qquad = 2\rho\big(\ln(27) - \ln(3)\big) = 2\rho \ln(9)$

Now using the formula $M\bar{x} = 2\int_0^4 \rho xy\,dx$,

$M\bar{x} = 2\rho \int_0^4 \dfrac{(x+5)x}{(x+2)(2x+1)}\,dx$

$\qquad = 2\rho \int_0^4 \left(-\dfrac{3}{2(2x+1)} + \dfrac{2}{x+2} + \dfrac{1}{2}\right)dx$

$\qquad = 2\rho \left[-\dfrac{3}{4}\ln(2x+1) + 2\ln(x+2) + \dfrac{1}{2}x\right]_0^4$

$= 2\rho\left(-\tfrac{1}{4}\ln(27) + 2\ln(3) + 2\right) = \rho\big(4 + \ln(3)\big)$

Hence $\bar{x} = \dfrac{\rho\big(4 + \ln(3)\big)}{4\rho\ln(3)} \approx 1.16\,(3\text{ s.f.})$, and $\bar{y} = 0$

12 The mass of the lamina is $M = \int_0^3 \rho y \, dx = \rho \int_0^3 \frac{1}{\sqrt{x^2 + 4}} \, dx$

$= \rho \left[\text{arcsinh} \left(\tfrac{1}{2} x \right) \right]_0^3 = \rho \, \text{arcsinh} \left(\tfrac{3}{2} \right)$

Using the formula $M \bar{x} = \int_0^3 \rho x y \, dx$

$= \rho \int_0^3 \frac{x}{\sqrt{x^2 + 4}} \, dx$. Make a change of variable $u = x^2 + 4$, $du = 2x \, dx \Rightarrow$

$M \bar{x} = \rho \int_4^{13} \frac{1}{2\sqrt{u}} \, du = \rho \left[\sqrt{u} \right]_4^{13} = \rho \left(\sqrt{13} - 2 \right)$.

We also have $M \bar{y} = \int_0^3 \rho \tfrac{1}{2} y^2 \, dx$

$= \rho \int_0^3 \frac{1}{x^2 + 4} \, dx = \tfrac{1}{4} \rho \left[\arctan \left(\tfrac{1}{2} x \right) \right]_0^3$

$= \tfrac{1}{4} \rho \arctan \left(\tfrac{3}{2} \right)$. Hence,

$\bar{x} = \frac{\left(\sqrt{13} - 2 \right)}{\text{arcsinh} \left(\tfrac{3}{2} \right)} \approx 1.34 \, (3 \text{ s.f.})$, and $\bar{y} = \frac{\arctan \left(\tfrac{3}{2} \right)}{4 \, \text{arcsinh} \left(\tfrac{3}{2} \right)} \approx 0.206 \, (3 \text{ s.f.})$

Challenge

First we find the centre of mass of the pendant. The equation of the pendant curve is $x = y^2$.

Because the pendant is symmetric, $\bar{y} = 0$. Using the formula $M \bar{x} = 2 \int_0^4 \rho x y \, dx$

$= 2 \rho \int_0^4 x \sqrt{x} \, dx = 2 \rho \left[\tfrac{2}{5} x^{5/2} \right]_0^4 = \rho \tfrac{128}{5}$. The mass of the pendant is $M = 2 \int_0^4 \rho y \, dx$

$= 2 \rho \int_0^4 \sqrt{x} \, dx = 2 \rho \left[\tfrac{2}{3} x^{3/2} \right]_0^4 = \rho \tfrac{32}{3}$, hence $\bar{x} = 2.4$. If the pendant is suspended at any point P on its perimeter, no part of the pendant should be higher than P. If we connect P to the centre of mass with a line ℓ_1, and draw a line ℓ_2 at P which is perpendicular to ℓ_1, no parts of pendant should be above that line. Some possible points are immediately obvious: these are on the symmetry axis $y = 0$, $(0, 0)$ and $(4, 0)$. Suppose another such point is (x_0, y_0). The equation for ℓ_1 is $\frac{y - \bar{y}}{x - \bar{x}} = -\frac{1}{k}$, and for ℓ_2 is

$\frac{y - y_0}{x - x_0} = k = \left[\frac{dy}{dx} \right]_{x_0} = \frac{1}{2\sqrt{x_0}}$. Hence, at (x_0, y_0) we have $\frac{y_0 - \bar{y}}{x_0 - \bar{x}} = -2\sqrt{x_0}$. Note that $y_0 = \sqrt{x_0}$.

Now we can find $x_0 = 1.9$, and $y_0 = \sqrt{1.9}$. By symmetry, point $\left(1.9, -\sqrt{1.9} \right)$ will also be a solution.

We also need to check the edges, where the tangent to the pendant is not defined. This gives us two more points $(4, \pm 2)$.