Review exercise 1

1 Changing the units:

diameter $= 7\,\text{cm} \Rightarrow$ radius $(r) = 3.5\,\text{cm} = 0.035\,\text{m}$

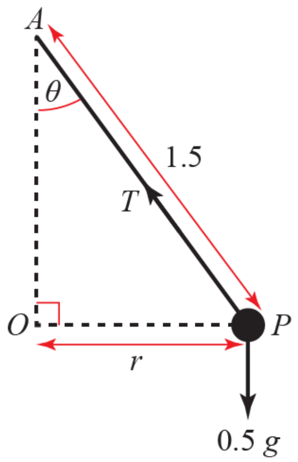
angular speed $(\omega) = 1000$ revolutions per minute $= \dfrac{1000 \times 2\pi}{60}$ radians per second

Using $v = r\omega$ gives:

$$v = 0.035 \times \frac{1000 \times 2\pi}{60} = 3.67\,\text{m s}^{-1} \text{ (3 s.f.)}$$

So the speed is $3.67\,\text{m s}^{-1}$ (3 s.f.)

2 **a** Let the tension in the string be T and let the string make an angle θ with the vertical.



Let $OP = r$, then $r = 1.5\sin\theta$

Acceleration towards the centre of the circle $= r\omega^2 = 1.5\sin\theta \times 2.7^2$

Force towards the centre of the circle $= T\sin\theta$

So using $F = ma$ gives:

$T\sin\theta = 0.5 \times 1.5\sin\theta \times 2.7^2$

$\Rightarrow T = 0.5 \times 1.5 \times 2.7^2 = 5.4675 = 5.5\,\text{N}$ (2 s.f.)

b Resolving the forces vertically $\text{R}(\uparrow)$ and substituting for T gives:

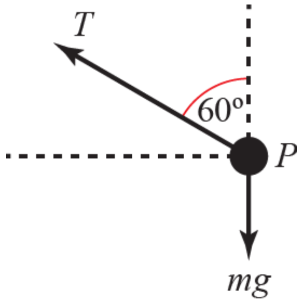
$T\cos\theta - 0.5g = 0$

$\Rightarrow T\cos\theta = 0.5g$

$\Rightarrow \cos\theta = \dfrac{0.5g}{5.4675} = 0.8962$

So $\theta = \cos^{-1} 0.8962 = 26°$ (to the nearest degree)

3 a These are the forces acting on P.



$R(\uparrow): T\cos 60° - mg = 0$

$\Rightarrow T = \dfrac{mg}{\cos 60°} = 2mg$

b $R(\leftarrow): T\sin 60° = ma = mr\omega^2$ using $F = ma$ and r is the radius of the circular motion

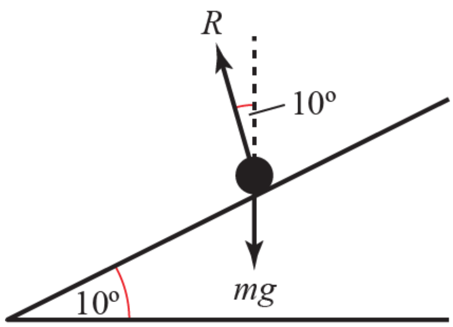
As $r = L\sin 60°$ this gives:

$T\sin 60° = mL\sin 60°\omega^2$

$\Rightarrow \omega^2 = \dfrac{T}{mL} = \dfrac{2mg}{mL} = \dfrac{2g}{L}$ substituting result for T from part **a**

So $\omega = \sqrt{\dfrac{2g}{L}}$

4 The forces acting on the car are its weight mg and the normal reaction R.



$R(\uparrow): R\cos 10° - mg = 0$

$\Rightarrow R = \dfrac{mg}{\cos 10°}$

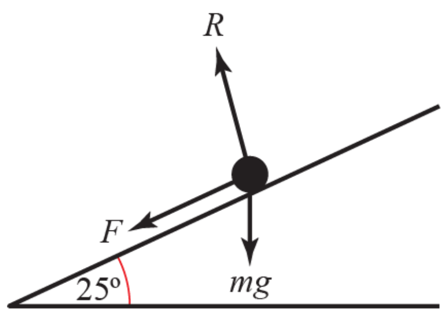
Using $F = ma$ horizontally gives:

$R\sin 10° = \dfrac{mv^2}{r} = \dfrac{m \times 18^2}{r}$

$\Rightarrow \dfrac{mg}{\cos 10°} \times \sin 10° = \dfrac{m \times 18^2}{r}$ substituting for R

$\Rightarrow r = \dfrac{18^2}{g\tan 10°} = 187.4995\ldots = 190\text{ m (2 s.f.)}$

5 The forces acting on the cyclist and bicycle are their weight *mg*, the normal reaction *R* and the friction acting down the slope.



R(↑): $R\cos 25° - F\sin 25° - mg = 0$

If μ is the coefficient of friction between the cycle's tyres and the track, then the maximum friction for which the tyres do not slip is $F = \mu R = 0.6R$. Substituting for *F* gives:

$R\cos 25° - 0.6R\sin 25° - mg = 0$

$\Rightarrow R = \dfrac{mg}{\cos 25° - 0.6\sin 25°}$ **(1)**

R(←): $R\sin 25° + F\cos 25° = \dfrac{mv^2}{r}$ using $F = ma$ and $a = \dfrac{v^2}{r}$

$\Rightarrow R\sin 25° + 0.6R\cos 25° = \dfrac{mv^2}{40}$ as $r = 40$ and $F = 0.6R$ at maximum speed

$\Rightarrow R = \dfrac{mv^2}{40(\sin 25° + 0.6\cos 25°)}$ **(2)**

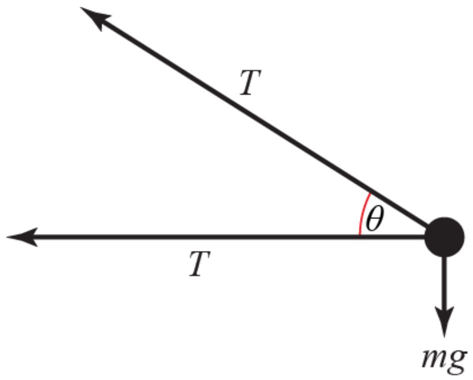
Using equations **(1)** and **(2)** gives:

$\dfrac{mv^2}{40(\sin 25° + 0.6\cos 25°)} = \dfrac{mg}{\cos 25° - 0.6\sin 25°}$

$\Rightarrow v^2 = \dfrac{40g(\sin 25° + 0.6\cos 25°)}{\cos 25° - 0.6\sin 25°} = 580.37$ (2 d.p.)

$\Rightarrow v = 24\,\text{m s}^{-1}$ (2 s.f.)

6 **a** The forces acting on the ring are the the weight of the ring and tensions of equal magnitude T along each section of string.



Note that θ is an an angle in a 3-4-5 triangle, with $\cos\theta = \frac{3}{5}$, $\sin\theta = \frac{4}{5}$

$\text{R}(\uparrow): T\sin\theta - mg = 0$

$\Rightarrow T = \frac{mg}{\sin\theta} = \frac{5mg}{4}$

b $\text{R}(\leftarrow): T + T\cos\theta = \frac{mv^2}{r}$ using $F = ma$ and $a = \frac{v^2}{r}$

$\Rightarrow T\left(1 + \frac{3}{5}\right) = \frac{mv^2}{3l}$ as $r = 3l$ and $\cos\theta = \frac{3}{5}$

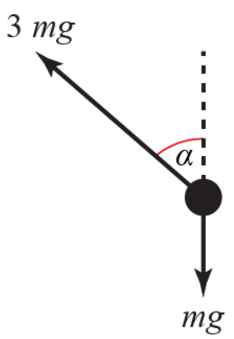
$\Rightarrow \frac{5mg}{4} \times \frac{8}{5} = \frac{mv^2}{3l}$ substituting for T

$\Rightarrow v^2 = 6gl$

$\Rightarrow v = \sqrt{6gl}$

c If the ring is firmly attached to the string, then the tensions in each section of the string cannot be assumed to have the same magnitude. So the approach adopted for parts **a** and **b** would not be valid.

7 **a** The forces acting on the metal ball are the weight of the ball and tension along the string.



$\text{R}(\uparrow): 3mg\cos\alpha - mg = 0$

$\Rightarrow \cos\alpha = \frac{mg}{3mg} = \frac{1}{3}$

$\Rightarrow \alpha = 70.5°$ (3 s.f.)

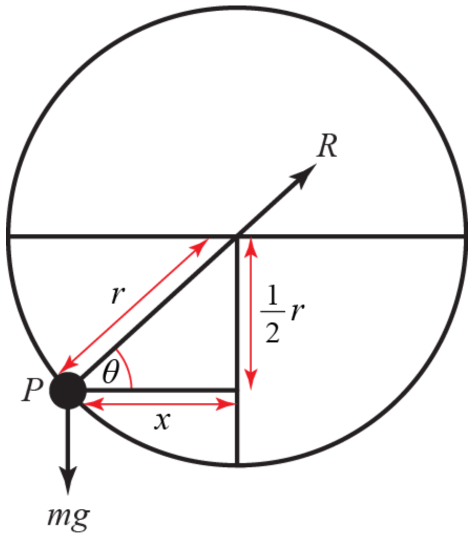
7 b R(←): $3mg\sin\alpha = mr2gk$ using $F = ma$ and $a = r\omega^2$

Let the length of the string be l, then from ΔAOB is is clear that $\sin\alpha = \frac{r}{l}$

So $3mg\frac{r}{l} = mr2gk$

$\Rightarrow l = \frac{3}{2k}$

8 a The forces acting on the particle are its weight and the normal reaction.



Let θ be the angle between the normal reaction and the horizontal.

Then $\sin\theta = \frac{\frac{1}{2}r}{r} = \frac{1}{2} \Rightarrow \theta = 30°$

R(↑): $R\sin 30° - mg = 0$

$\Rightarrow R = \frac{mg}{\sin 30°} = 2mg$

b R(→): $R\cos 30° = mx\omega^2$ using $F = ma$ and $a = r\omega^2$

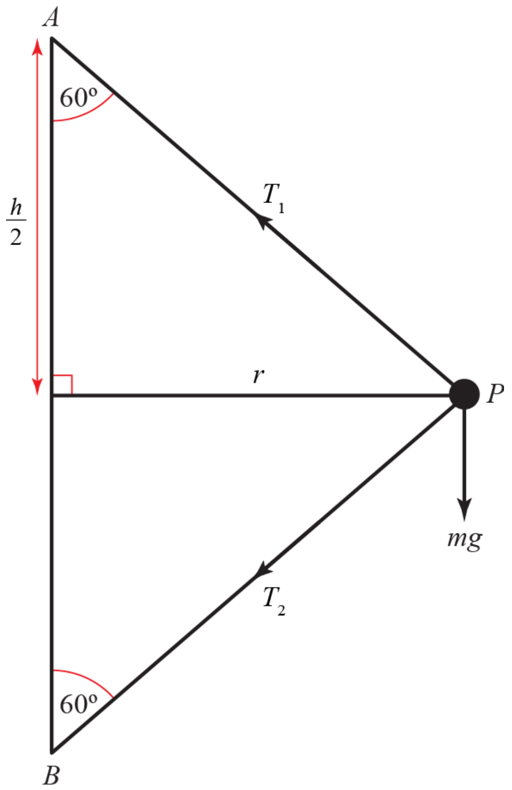
Using the result from part **a** and as $x = r\cos 30°$ this gives:

$2mg = mr\omega^2$

$\Rightarrow \omega = \sqrt{\frac{2g}{r}}$

Time to complete one revolution $= \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{r}{2g}}$

9 a Let T_1 be the tension in AP and T_2 be the tension in BP. The forces acting on P are the tensions in the two strings and its weight.



From the diagram, it can be seen that the equilateral triangle APB can be divided into two right-angled triangles, where:

$$\tan 60° = \frac{r}{\frac{h}{2}}$$

$$\Rightarrow r = \frac{h}{2} \times \tan 60° = \frac{\sqrt{3}h}{2}$$

b Resolving the forces from the diagram in part **a**

$$\text{R}(\uparrow):\ T_1 \cos 60° - T_2 \cos 60° - mg = 0$$

$$\Rightarrow T_1 - T_2 = 2mg \qquad \textbf{(1)}$$

$$\text{R}(\leftarrow):\ T_1 \sin 60° + T_2 \sin 60° = mr\omega^2 \qquad \text{using } F = ma \text{ and } a = r\omega^2$$

$$\Rightarrow \frac{\sqrt{3}}{2}T_1 + \frac{\sqrt{3}}{2}T_2 = m\frac{\sqrt{3}}{2}h\omega^2 \qquad \text{as } r = \frac{\sqrt{3}}{2}h \text{ from part } \textbf{a}$$

$$\Rightarrow T_1 + T_2 = mh\omega^2 \qquad \textbf{(2)}$$

Adding equations **(1)** and **(2)** gives: $2T_1 = 2mg + mh\omega^2 \Rightarrow T_1 = mg + \frac{1}{2}mh\omega^2$

Substituting for T_1 in equation **(2)** gives: $T_2 = \frac{1}{2}mh\omega^2 - mg$

9 c Both strings are taut, therefore $T_1 > 0$ and $T_2 > 0$. From part **b**, $T_1 > 0$ for all values of ω

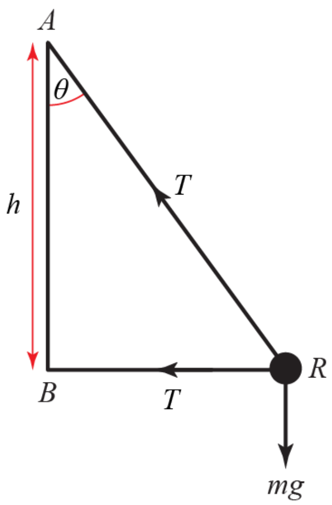
From the formula for T_2 in part **b**, the condition that $T_2 > 0 \Rightarrow \omega > \sqrt{\frac{2g}{h}}$

As $T = \frac{2\pi}{\omega}$, $\omega = \frac{2\pi}{T}$

So $\omega > \sqrt{\frac{2g}{h}} \Rightarrow \frac{2\pi}{T} > \sqrt{\frac{2g}{h}} \Rightarrow T < 2\pi\sqrt{\frac{h}{2g}}$ as $T > 0$ and $\sqrt{\frac{h}{2g}} < 0$

$\Rightarrow T < \pi\sqrt{\frac{2h}{g}}$ as required

10 a The forces acting on the ring are the the weight of the ring and, as the ring is threaded on the strong, tensions of equal magnitude T along each section of string.



$R(\uparrow): T\cos\theta - mg = 0$

$\Rightarrow T = \frac{mg}{\cos\theta}$ **(1)**

$R(\leftarrow): T + T\sin\theta = mr\omega^2$ using $F = ma$ and $a = r\omega^2$

$\Rightarrow \frac{mg}{\cos\theta}(1+\sin\theta) = mr\omega^2$ substituting for T from equation **(1)**

$\Rightarrow \frac{mg}{\cos\theta}(1+\sin\theta) = mh\tan\theta\omega^2 = mh\frac{\sin\theta}{\cos\theta}\omega^2$ as $r = h\tan\theta$ by basic trigonometry

$\Rightarrow \omega^2 = \frac{g}{h}\left(\frac{1+\sin\theta}{\sin\theta}\right)$

b From part **a**, $\omega^2 = \frac{g}{h}\left(\frac{1+\sin\theta}{\sin\theta}\right) = \frac{g}{h}\left(\frac{1}{\sin\theta}+1\right)$

As $\sin\theta < 1$ it follows that $\frac{1}{\sin\theta} > 1$

Therefore $\omega^2 > \frac{g}{h} \times 2$

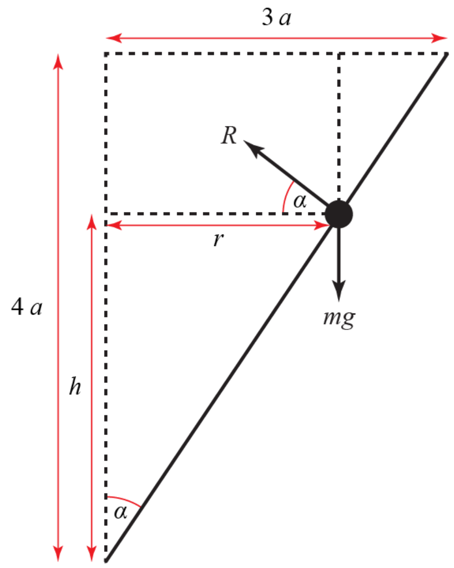
$\Rightarrow \omega > \sqrt{\frac{2g}{h}}$

10 c Given $\omega = \sqrt{\frac{3g}{h}}$, then $\frac{g}{h}\left(\frac{1+\sin\theta}{\sin\theta}\right) = \frac{3g}{h}$

$\Rightarrow 1 + \sin\theta = 3\sin\theta \Rightarrow \sin\theta = \frac{1}{2}$ and so $\cos\theta = \frac{\sqrt{3}}{2}$

From equation **(1)**, $T = \frac{mg}{\cos\theta} = \frac{2}{\sqrt{3}}mg = \frac{2\sqrt{3}}{3}mg$

11 Let α be the angle between the slant side and the axis of the cone, r the radius of the horizontal circle with centre C and h the height of C above V. The forces acting on the particle are its weight and the normal reaction.



From the diagram, $\tan\alpha = \frac{3a}{4a} = \frac{3}{4}$ and $\tan\alpha = \frac{r}{h}$

R($\uparrow$): $R\sin\alpha - mg = 0$

$\Rightarrow R\sin\alpha = mg$ **(1)**

R($\leftarrow$): $R\cos\alpha = mr\omega^2$ using $F = ma$ and $a = r\omega^2$

$\Rightarrow R\cos\alpha = \frac{mr8g}{9a}$ **(2)** using $\omega = \sqrt{\frac{8g}{9a}}$

Dividing equation **(1)** by equation **(2)** gives:

$\tan\alpha = mg \div \frac{8mrg}{9a} = \frac{9a}{8r}$

$\Rightarrow \frac{3}{4} = \frac{9a}{8r}$ using $\tan\alpha = \frac{3}{4}$

$\Rightarrow r = \frac{3a}{2}$

As $\tan\alpha = \frac{3}{4} = \frac{r}{h}$, $h = \frac{4}{3}r$

So $h = \frac{4}{3} \times \frac{3a}{2} = 2a$

The height of C above V is $2a$.

12 a Let the tension in the string be T.

$R(\leftarrow): T\cos 30° = mr\omega^2$ using $F = ma$ and $a = r\omega^2$

$\Rightarrow T\cos 30° = m2a\cos 30° \frac{kg}{3a}$ using $\omega = \sqrt{\frac{kg}{3a}}$ and $r = 2a\cos 30°$ by trigonometry

$\Rightarrow T = \frac{2kmg}{3}$ as required

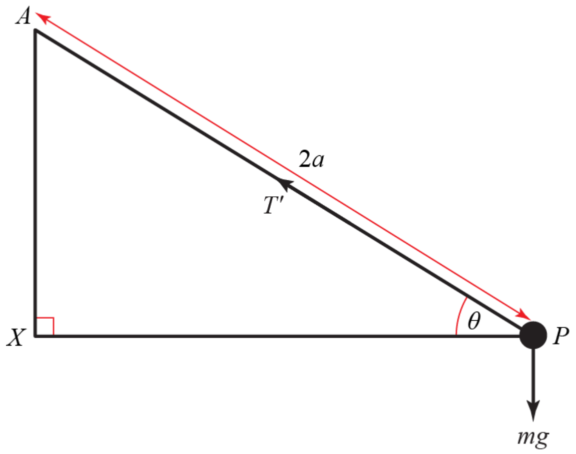
b Let the normal reaction be R and resolve vertically.

$R(\uparrow): R + T\sin 30° - mg = 0$

$\Rightarrow R = mg - \frac{2kmg}{3} \times \frac{1}{2} = mg\left(1 - \frac{k}{3}\right)$

c Using the result from part **b**, as $R > 0, 1 - \frac{k}{3} > 0 \Rightarrow k < 3$

d Let the new tension be T' and let AP make an angle θ with the horizontal. So $PX = 2a\cos\theta$.



$R(\leftarrow): T'\cos\theta = mr\omega^2$ using $F = ma$ and $a = r\omega^2$

$\Rightarrow T'\cos\theta = m2a\cos\theta \frac{2g}{a}$ using $\omega = \sqrt{\frac{2g}{a}}$ and $r = 2a\cos\theta$

$\Rightarrow T' = 4mg$

$R(\uparrow): T'\sin\theta - mg = 0$

$\Rightarrow \sin\theta = \frac{mg}{T'} = \frac{mg}{4mg} = \frac{1}{4}$

Hence $AX = 2a\sin\theta = 2a \times \frac{1}{4} = \frac{a}{2}$

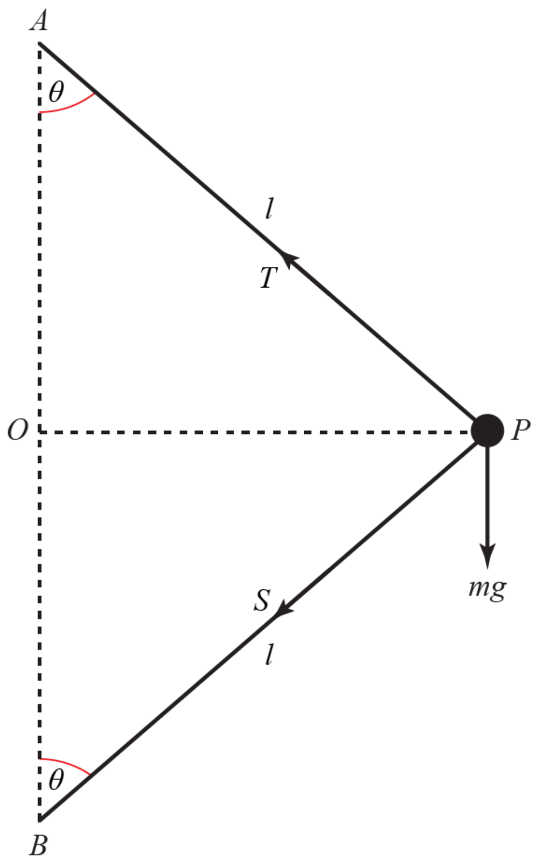
$AO = 2a\sin 30° = 2a \times \frac{1}{2} = a$

Therefore $AX = \frac{AO}{2}$

So the point X is the midpoint of AX.

13 a Let the tension in AP be T and in BP be S. Let $\angle BAP = \theta$. As the triangle APB is an equilateral triangle $\angle BAP = \angle PBA$.

$AO = \frac{3}{2}l \times \frac{1}{2} = \frac{3l}{4}$, so from the right-angled triangle APO, $\cos\theta = \frac{3l}{4l} = \frac{3}{4}$



$R(\uparrow): T\cos\theta - S\cos\theta - mg = 0$

$\Rightarrow T - S = \frac{mg}{\cos\theta} = \frac{4mg}{3}$ **(1)**

$R(\leftarrow): T\sin\theta + S\sin\theta = mr\omega^2$ using $F = ma$ and $a = r\omega^2$

$\Rightarrow T + S = \frac{mr\omega^2}{\sin\theta} = ml\omega^2$ **(2)** as $\sin\theta = \frac{r}{l}$

Adding equations **(1)** and **(2)** gives:

$$2T = \frac{4mg}{3} + ml\omega^2 \Rightarrow T = \frac{1}{6}m(4g + 3l\omega^2) = \frac{1}{6}m(3l\omega^2 + 4g)$$ as required

b Subtracting equation **(1)** from equation **(2)** gives:

$$2S = ml\omega^2 - \frac{4mg}{3} \Rightarrow S = \frac{1}{6}m(3l\omega^2 - 4g)$$

13 c From part b), $T_{BP} = \frac{m}{6}\left(3l\omega^2 - 4g\right)$

Since BP is taut, $T_{BP} \geqslant 0$

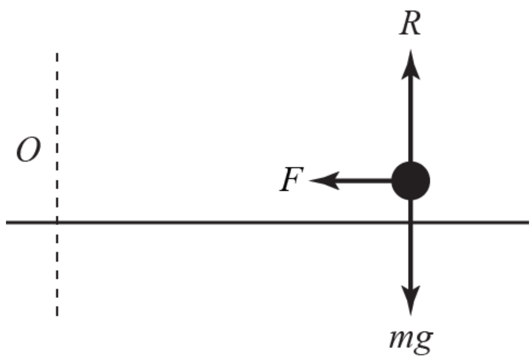
$\frac{m}{6}\left(3l\omega^2 - 4g\right) \geqslant 0$

$3l\omega^2 - 4g \geqslant 0$

$3l\omega^2 \geqslant 4g$

$\omega^2 \geqslant \frac{4g}{3l}$

14 a The forces acting on the particle are its weight, the normal reaction and friction.



R($\uparrow$): $R - mg = 0 \Rightarrow R = mg$

R($\leftarrow$): $F = mr\omega^2$ using $F = ma$ and $a = r\omega^2$

$\Rightarrow F = \frac{4ma\omega^2}{3}$ as $r = \frac{4}{3}a$

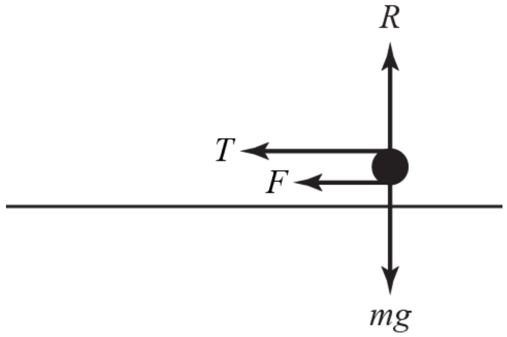
As P remains at rest $F \leqslant \mu R$

So $\frac{4ma\omega^2}{3} \leqslant \frac{3}{5}R$

$\Rightarrow \frac{4ma\omega^2}{3} \leqslant \frac{3mg}{5}$

$\Rightarrow \omega^2 \leqslant \frac{9g}{20a}$ as required

14 b The forces now acting on the particle are its weight, the normal reaction, friction and the tension in the elastic string.



Find T using Hooke's law (Further Mechanics 1, Chapter 3): $T = \frac{\lambda x}{l}$,

where λ is the modulus of elesticity, x is the extension of the string and l is its natural length

So in this case, $T = \frac{2mg}{a} \times \frac{a}{3} = \frac{2mg}{3}$

R($\leftarrow$): $T + F = mr\omega^2$ using $F = ma$ and $a = r\omega^2$

$\Rightarrow \omega^2 = \frac{3}{4ma}\left(\frac{2mg}{3} + F\right)$

Note that the frictional force can act away from O against the pull of the elastic string, or towards O through the force of the acceleration of the particle generated by the circular motion. As P remains at rest $-\mu R \leqslant F \leqslant \mu R$ (where $\mu R = 0.6mg$ from part **a**).

So, from the equation for ω^2, it is maximum when $F = \mu R = \frac{3mg}{5}$

So $\omega_{max}^2 = \frac{3}{4ma}\left(\frac{2mg}{3} + \frac{3mg}{5}\right) = \frac{3}{4ma} \times \frac{19mg}{15} = \frac{19g}{20a}$

Similarly ω^2 is minimum when $F = -\mu R = -\frac{3mg}{5}$

So $\omega_{min}^2 = \frac{3}{4ma}\left(\frac{2mg}{3} - \frac{3mg}{5}\right) = \frac{3}{4ma} \times \frac{mg}{15} = \frac{g}{20a}$

15 a i $AP = \frac{x}{\cos\theta}$, so extension in the string $= \frac{x}{\cos\theta} - x$

Using Hooke's law: $T = \frac{10g}{x}\left(\frac{x}{\cos\theta} - x\right) = \frac{10g}{\cos\theta} - 10g$ **(1)**

R($\uparrow$): $T\cos\theta - 2g = 0 \Rightarrow \cos\theta = \frac{2g}{T}$

Substituting for $\cos\theta$ into equation **(1)** gives:

$T = 5T - 10g$

$\Rightarrow T = \frac{10g}{4} = 2.5g$ N

ii $\cos\theta = \frac{2g}{T} = \frac{2g}{2.5g} = 0.8$

So $\theta = \arccos 0.8 = 36.9°$ (3 s.f.)

15 b R(←): $T\sin\theta = \frac{2v^2}{r}$ using $F = ma$ and $a = \frac{v^2}{r}$

As $\cos\theta = \frac{4}{5}$, APO is a 3-4-5 right-angled triangle, with $\sin\theta = \frac{3}{5}$ and $\tan\theta = \frac{r}{x} = \frac{3}{4} \Rightarrow r = \frac{3}{4}x$

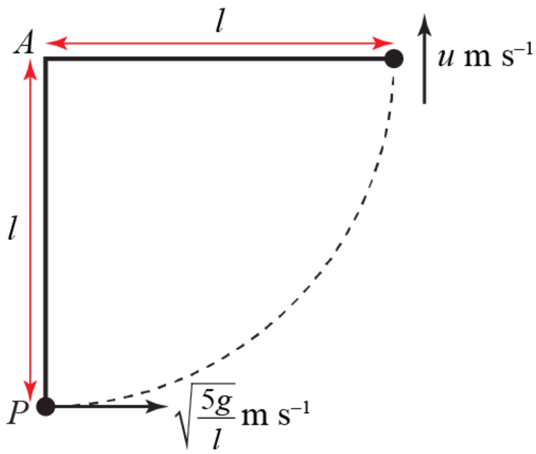
So $T\sin\theta = \frac{2v^2}{r}$

$\Rightarrow 2.5g \times \frac{3}{5} = \frac{4}{3x} \times 2v^2$

$\Rightarrow v^2 = \frac{9gx}{16}$

$\Rightarrow v = \frac{3}{4}\sqrt{gx}$

16 a Let u be the speed of P when the string is horizontal.



Applying the work-energy principle, the sum of the particle's kinetic and gravitational potential energy remains constant, so when it reaches the horizontal the loss of kinetic energy = the gain in potential energy.

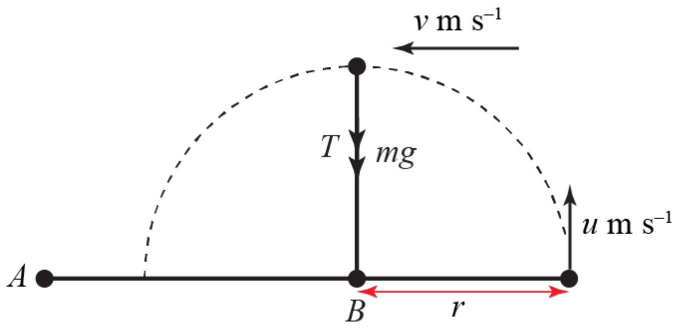
Take the starting point as the zero level for potential energy.

So $\frac{1}{2}m\frac{5gl}{2} - \frac{1}{2}mu^2 = mgl$

$\Rightarrow u^2 = \frac{5gl}{2} - 2gl = \frac{gl}{2}$

$\Rightarrow u = \sqrt{\frac{gl}{2}}$

16 b Let the particle move in a semi-circle about B with radius r.



Taking the line AB as the zero level for potential energy, then applying conservation of energy at the highest point of the semi-circle:

$$\frac{1}{2}m(u^2 - v^2) = mgr$$

$$\Rightarrow v^2 = u^2 - 2gr \qquad (1)$$

Resolving the vertical forces at the highest point:

$$\text{R}(\downarrow): T + mg = \frac{mv^2}{r} \qquad \text{using } F = ma \text{ and } a = \frac{v^2}{r}$$

$$\Rightarrow T = \frac{m(u^2 - 2gr)}{r} - mg \qquad \text{substituting for } v^2 \text{ using equation (1)}$$

$$\Rightarrow T = \frac{mu^2}{r} - 3mg$$

$$\Rightarrow T = \frac{mgl}{2r} - 3mg \qquad \text{substituting for } u^2 \text{ using result from part } \mathbf{a}$$

As the string does not go slack $T > 0$, so

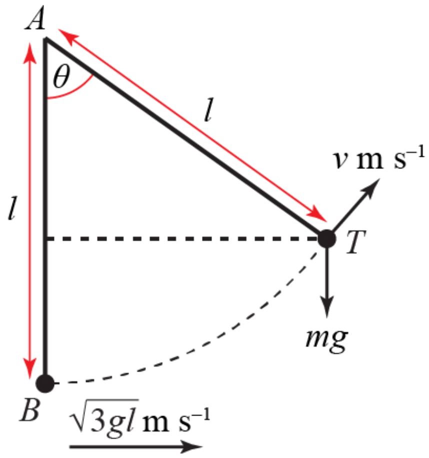
$$\frac{mgl}{2r} - 3mg > 0$$

$$\Rightarrow mgl > 6mgr$$

$$\Rightarrow r < \frac{l}{6}$$

As $AB = l - r$, this shows that $AB > \frac{5l}{6}$

17 a Let the speed of the particle be v when it makes an angle θ with the downward vertical.



Take the horizontal through B as the zero level for potential energy. Using conservation of energy:

$\frac{1}{2}m(3gl - v^2) = mgl(1-\cos\theta)$

$\Rightarrow v^2 = 3gl - 2gl(1-\cos\theta)$

$\Rightarrow v^2 = gl + 2gl\cos\theta$ **(1)**

Resolving along the string:

R(↖): $T - mg\cos\theta = \frac{mv^2}{l}$ using $F = ma$ and $a = \frac{v^2}{r}$ and $r = l$

$\Rightarrow T = mg\cos\theta + \frac{mgl + 2gl\cos\theta}{l}$ substituting for v^2 using equation **(1)**

$\Rightarrow T = mg + 3mg\cos\theta = mg(1+3\cos\theta)$ as required

b The instant the string becomes slack, $T = 0$.

Using the expression for T from part **a**, this occurs when:

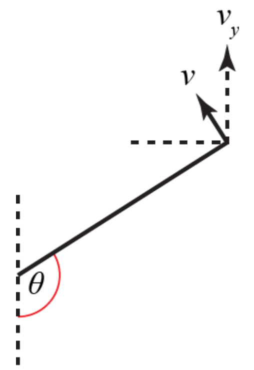
$1 + 3\cos\theta = 0 \Rightarrow \cos\theta = -\frac{1}{3}$

Substituting for $\cos\theta$ in equation **(1)** gives:

$v^2 = gl + 2gl \times -\frac{1}{3} = \frac{gl}{3}$

$\Rightarrow v = \sqrt{\frac{gl}{3}}$

17 c The particle now moves as a projectile under gravity. The maximum height is achieved when the vertical component of the velocity is zero.



$$v_y = v\sin(180° - \theta) = \sqrt{\frac{gl}{3}}\sin\theta$$

If $\frac{\pi}{2} < \theta < \pi$ and $\cos\theta = -\frac{1}{3}$, then $\sin\theta = \frac{2\sqrt{2}}{3}$

So $v_y = \frac{2\sqrt{2}}{3}\sqrt{\frac{gl}{3}}$

Considering the particle's vertical motion, $u = v_y$, $v = 0$, $a = -g$, $s = h$, where h is the height above the point the string becomes slack.

Using $v^2 = u^2 - 2gh$

$$h = \frac{v_y^2}{2g} = \frac{8}{9} \times \frac{gl}{3} \times \frac{1}{2g} = \frac{4l}{27}$$

The height at which string becomes slack $= l + l\cos(180° - \theta) = l(1 - \cos\theta) = \frac{4l}{3}$

So if H is the maximum height above the level of B reached by P

$$H = \frac{4l}{3} + \frac{4l}{27} = \frac{40l}{27}$$

18 a Take the horizontal through l as the zero level for potential energy. When angle $\theta = \alpha$, the particle is momentarily at rest. Using conservation of energy, with the loss of kinetic energy equal to the gain in potential energy:

$$\frac{1}{2}mu^2 = mgl(1 - \cos\alpha) = \frac{mgl}{3} \qquad \text{as } \cos\alpha = \frac{2}{3}$$

$$\Rightarrow u^2 = \frac{2}{3}gl$$

$$\Rightarrow u = \sqrt{\frac{2gl}{3}}$$

18 b Let the speed of the particle at θ be v.
Resolving along the string gives:

R(↖): $T - mg\cos\theta = \frac{mv^2}{l}$ using $F = ma$ and $a = \frac{v^2}{r}$ and $r = l$

$\Rightarrow T = mg\cos\theta + \frac{mv^2}{l}$

Applying conservation of energy:

$\frac{1}{2}mu^2 - \frac{1}{2}mv^2 = mgl(1-\cos\theta)$

$\Rightarrow v^2 = u^2 - 2gl(1-\cos\theta) = \frac{2gl}{3} - 2gl(1-\cos\theta)$ using the result from part **a**

$\Rightarrow v^2 = 2gl\cos\theta - \frac{4gl}{3}$

Substituting for v^2 in the equation for T gives:

$T = mg\cos\theta + 2mg\cos\theta - \frac{4mg}{3}$

$= 3mg\cos\theta - \frac{4mg}{3}$

$= \frac{mg}{3}(9\cos\theta - 4)$

c Maximum value of T is when $\cos\theta = 1$

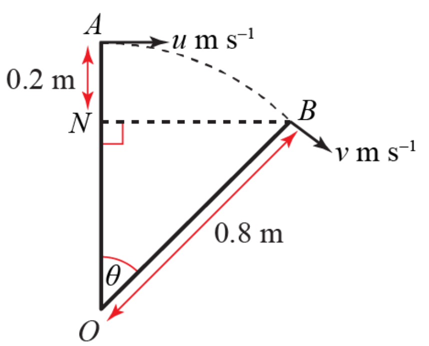
So $T_{max} = \frac{5mg}{3}$

Minimum value of T is when $\cos\theta = \frac{2}{3}$

So $T_{min} = \frac{2mg}{3}$

Hence $\frac{2mg}{3} \leqslant T \leqslant \frac{5mg}{3}$

19 a The lengths on the diagram can be deduced from the question:
$OA = OB = 0.8\text{m}$ $AN = 0.2\text{m}$ $ON = OA - AN = 0.6\text{m}$



$\cos\theta = \frac{ON}{OB} = \frac{0.6}{0.8} = \frac{3}{4}$

19 b As the particle leaves the hemisphere at B the normal reaction $R = 0$.
So resolving along the radius OB gives:

$$\text{R}(\swarrow): mg\cos\theta = \frac{3mg}{4} = \frac{mv^2}{0.8}$$ using $F = ma$ and $a = \frac{v^2}{r}$ and $r = 0.8$

$$\Rightarrow v^2 = \frac{3g}{4} \times 0.8 = 0.6g = 5.88$$

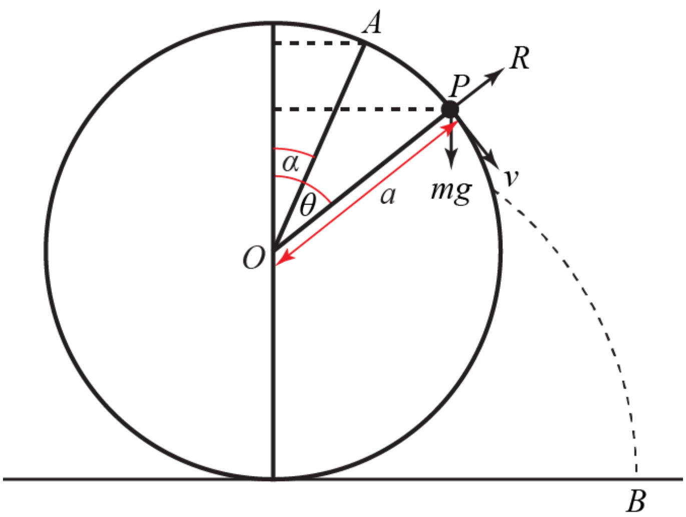
c Take the horizontal through A as the zero level for potential energy. Using conservation of energy, the gain of kinetic energy at B equals the loss in potential energy:

$$\frac{1}{2}m \times 5.88 - \frac{1}{2}mu^2 = mg \times 0.2$$

$$\Rightarrow u^2 = 5.88 - 0.4g = 0.2g$$

So $u = 1.4$

20 a This is a diagram of the problem.



Take the horizontal through A as the zero level for potential energy. Using conservation of energy, the gain of kinetic energy at P equals the loss in potential energy:

$$\frac{1}{2}mv^2 = mg(a\cos\alpha - a\cos\theta)$$

$$v^2 = 2ga(\cos\alpha - \cos\theta)$$

b Resolving along the radius OP:

$$\text{R}(\swarrow): mg\cos\theta - R = \frac{mv^2}{a}$$

At the point when P loses contact with the sphere $R = 0$

$$\Rightarrow mg\cos\theta = \frac{mv^2}{a} = 2gm(\cos\alpha - \cos\theta)$$ substituting for v^2 from part **a**

$$\Rightarrow 3\cos\theta = 2\cos\alpha = \frac{3}{2}$$

$$\Rightarrow \cos\theta = \frac{1}{2}, \text{ so } \theta = 60° \text{ or } \frac{\pi}{3} \text{ radians}$$

20 c Let the speed of P when it hits the table be w. Then taking the horizontal through A as the zero level for potential energy and using conservation of energy, the gain of kinetic energy at B equals the loss in potential energy:

$$\frac{1}{2}mw^2 = mg(a + a\cos\alpha)$$

$$\Rightarrow w^2 = 2ga\left(1+\frac{3}{4}\right) = \frac{7ga}{2}$$

So $w = \sqrt{\frac{7ga}{2}}\ \text{m s}^{-1}$

There are alternative ways to calculate w that use projectiles, but the approach shown above is the shortest method.

21 a Let the speed of P at point B be v. Then taking the horizontal through A as the zero level for potential energy and using conservation of energy, the loss of kinetic energy at B equals the gain in potential energy:

$$\frac{1}{2}mu^2 - \frac{1}{2}mv^2 = mga$$

$$\Rightarrow \frac{1}{2}m\times\frac{7ag}{2} - mga = \frac{1}{2}mv^2$$

$$\Rightarrow v^2 = \frac{3}{2}ga$$

$$\text{R}(\leftarrow):\ R = \frac{mv^2}{a}$$

$$\Rightarrow R = \frac{m}{a}\times\frac{3}{2}ga = \frac{3}{2}mg$$

b Let the speed of P at point C be w. Then taking the horizontal through A as the zero level for potential energy and using conservation of energy, the loss of kinetic energy at C equals the gain in potential energy:

$$\frac{1}{2}m\times\frac{7ga}{2} - \frac{1}{2}mw^2 = mga(1+\cos\theta) \qquad \textbf{(1)}$$

Resolving along the radius OP

$$\text{R}(\swarrow):\ mg\cos\theta + R = \frac{mw^2}{a}$$

At the point when P loses contact with the sphere $R = 0$

So $ga\cos\theta = w^2$

Substituting for w^2 in equation **(1)** gives:

$$\frac{7mga}{4} - \frac{mga\cos\theta}{2} = mga(1+\cos\theta)$$

$$\Rightarrow \frac{3}{2}\cos\theta = \frac{3}{4}$$

$$\Rightarrow \cos\theta = \frac{1}{2},\ \text{hence}\ \theta = 60°$$

21 c Consider the motion of the projectile P as it moves from C to A in the horizontal direction.

The horizontal distance from C to $A = a\sin 60°$

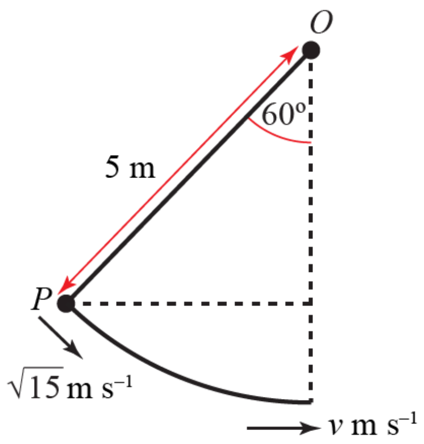
The horizontal component of the speed at C, $w_x = w\cos 60°$

So the time t that P takes to travel from C to A is given by $t = \dfrac{a\sin 60°}{w\cos 60°}$

From part **b**, $w^2 = ag\cos 60° \Rightarrow w = \sqrt{\dfrac{ag}{2}}$

So $t = \dfrac{a\sin 60°}{\cos 60°} \div \sqrt{\dfrac{ag}{2}} = a\sqrt{3} \times \dfrac{\sqrt{2}}{\sqrt{ag}} = \sqrt{\dfrac{6a}{g}}$

22 a Let the speed of the trapeze artist at the lowest point of her path be v. Then taking the horizontal through A as the zero level for potential energy and using conservation of energy, the gain in kinetic energy at this point equals the loss in potential energy:

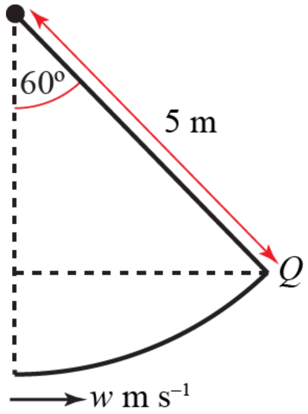


$$\frac{1}{2}mv^2 - \frac{1}{2}m(\sqrt{15})^2 = mg \times 5(1 - \cos 60°)$$

$$\Rightarrow v^2 = 15 + 5g = 64$$

$$\Rightarrow v = 8\,\text{m s}^{-1}$$

22 b Let the velocity after catching the ball be *w*. Then the loss in kinetic energy at point Q where the trapeze artist becomes momentarily stationery equals the gain in potential energy from the lowest point of the trapeze artist's path.



$$\frac{1}{2}(60+m)w^2 - 0 = (60+m)g5(1-\cos 60°)$$
$$\Rightarrow w^2 = 5g = 49, \text{ so } w = 7\,\text{m s}^{-1}$$

So at the instant just prior to catching the ball, the trapeze artist is travelling at $8\,\text{m s}^{-1}$ (part **a**) and the ball is travelling at $3\,\text{m s}^{-1}$ in the opposite direction, and immmediately after catching the ball she is travelling at $7\,\text{m s}^{-1}$. Using conservation of linear momentum at the instant when she catches the ball, this gives:

$$60\times 8 + (-3\times m) = (60+m)\times 7$$
$$\Rightarrow 480 - 3m = 420 + 7m$$
$$\Rightarrow 10m = 60$$
$$\text{So } m = 6\text{ kg}$$

c $\text{R}(\uparrow):\ T - 66g = \frac{66w^2}{r}$ using $F = ma$ and $a = \frac{v^2}{r}$

$$\Rightarrow T = 66g + 66\times\frac{7^2}{5} = 1293.6 = 1300\,\text{N (2 s.f.)}$$

23 a Let the speed at *C* be $v\,\text{m s}^{-1}$. Then taking the horizontal through *B* as the zero level for potential energy and using conservation of energy, the gain in kinetic energy at point *C* equals the loss in potential energy:

$$\frac{1}{2}mv^2 - \frac{1}{2}m\times 20^2 = mg\times 50(1-\cos 60°)$$
$$\Rightarrow v^2 = 20^2 + 50g \Rightarrow v^2 = 890$$
$$\text{So } v = 30\,\text{m s}^{-1}\text{ (2 s.f.)}$$

b At *C*, $\text{R}(\uparrow):\ R - 70g = \frac{70v^2}{50}$ using $F = ma$ and $a = \frac{v^2}{r}$

$$\Rightarrow R = 70g + \frac{70\times 890}{50} = 686 + 1246 = 1900\,\text{N (2 s.f.)}$$

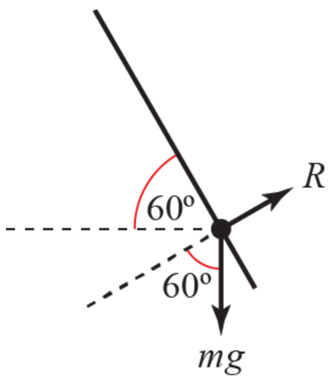
23 c Consider motion C to D. Let the speed at D be $w\,\text{m s}^{-1}$. Then taking the horizontal through C as the zero level for potential energy and using conservation of energy, the loss in kinetic energy at point C equals the gain in potential energy:

$$\frac{1}{2}m\times 890-\frac{1}{2}mw^2=mg\times 50(1-\cos 30°)$$

$$\Rightarrow w^2=890-100g(1-\cos 30°)=759\ (3\text{ s.f.})$$

$$\Rightarrow w=28\,\text{m s}^{-1}\ (2\text{ s.f.})$$

d Resolving perpendicular to the slope at B just before the circular motion and just after circular motion begins:



Before: $R=mg\cos 60°=35g$

After: $R-mg\cos 60°=\dfrac{m\times 20^2}{50}\Rightarrow R=35g+560$

So change in R = 560 N

e Allowing for the influence of friction would mean that the skier would arrive at C with lower speed. From the equation used in part **b**, this would result in a lower normal reaction.

24 a Taking the horizontal through A as the zero level for potential energy and using conservation of energy, the loss in kinetic energy when OP makes an angle θ equals the gain in potential energy:

$$\frac{1}{2}m3ag-\frac{1}{2}mv^2=mga(1+\cos\theta)$$

$$\Rightarrow v^2=ag(1-2\cos\theta)$$

b Resolving along the radius when OP makes an angle θ

$$\text{R}(\swarrow):\ T+mg\cos\theta=\frac{mv^2}{a}$$

$$\Rightarrow T=\frac{mag(1-2\cos\theta)}{a}-mg\cos\theta=mg(1-3\cos\theta)$$

c The string becomes slack when T = 0, i.e. from part **b** when $\cos\theta=\dfrac{1}{3}$

Height above $A=a+a\cos\theta=a+\dfrac{1}{3}a=\dfrac{4}{3}a$

24 d Using result for v^2 from part **a**, and that $\cos\theta = \frac{1}{3}$ at point B, then $v^2 = \frac{ag}{3}$ at point B

Consider the vertical component of the motion of the particle under gravity

Let acceleration be α, to avoid cofusion with a as used in the question.

Using $v^2 = u^2 + 2\alpha s$, where u is the vertical component of motion at $B = \sqrt{\frac{ag}{3}}\sin\theta$, v is the vertical component of motion at C = 0, $\alpha = -g$ and s is the vertical height of C above B

So $0 = \frac{ag}{3}\sin^2\theta - 2gs$

$\Rightarrow s = \frac{a}{6}\sin^2\theta = \frac{a}{6}(1-\cos^2\theta) = \frac{a}{6}\times\frac{8}{9} = \frac{4a}{27}$

The solution can also be found using conservation of energy as follows.

At C the particle has speed $\sqrt{\frac{ag}{3}}\cos\theta$, the horizontal component of the speed at B

Let the vertical height of C above B be h. Taking the horizontal through B as the zero level for potential energy and using conservation of energy, the loss in kinetic energy at C equals the gain in potential energy:

$\frac{1}{2}m\frac{ag}{3} - \frac{1}{2}m\frac{ag}{3}\cos^2\theta = mgh$

$h = \frac{a}{6}\left(1-\frac{1}{9}\right) = \frac{8a}{54} = \frac{4a}{27}$

25 a Taking the horizontal through O as the zero level for potential energy and using conservation of energy, the gain in kinetic energy equals the loss in potential energy:

$\frac{1}{2}mv^2 - \frac{1}{2}mu^2 = mga\sin\theta$

$\Rightarrow v^2 = u^2 + 2ga\sin\theta$

$\Rightarrow v^2 = \frac{3}{2}ga + 2ga\sin\theta$

b Resolving along the radius when OP makes an angle θ with the horizontal:

R($\nwarrow$): $T - mg\sin\theta = \frac{mv^2}{a}$

$\Rightarrow T = mg\sin\theta + \frac{m}{a}\left(\frac{3ga}{2} + 2ga\sin\theta\right) = \frac{3mg}{2} + 3mg\sin\theta = \frac{3mg}{2}(1+2\sin\theta)$

c The string becomes slack when T = 0, i.e. from part **b** when $2\sin\theta = -1$

So $\sin\theta = -\frac{1}{2} \Rightarrow \theta = 210°$

d From part **a**, $v^2 = \frac{3}{2}ga + 2ga\sin\theta$

To complete the circle, $v \neq 0$ before the rod reaches the top point, i.e. $v > 0$ for $0° \leqslant \theta \leqslant 270°$

But v = 0 when $\sin\theta = -\frac{3}{4} \Rightarrow \theta = 229°$ (3 s.f.), so P would not complete a vertical circle.

25 e Using conservation of energy, as there is no change in potential energy when the particle reaches A, as A and O are on the same level, there can be no change in its kinetic energy:

So $v = u = \sqrt{\frac{3ga}{2}}$

f Using the results from parts **a** and **c**, when the string becomes slack $\theta = 210°$ and

$$v^2 = \frac{3}{2}ga + 2ga\left(-\frac{1}{2}\right) = \frac{1}{2}ga$$

Its horizontal component of velocity is $\sqrt{\frac{1}{2}ga}\ \cos 60°$

From part **e**, when it reaches point A, its horizontal component of velocity is $\sqrt{\frac{3}{2}ga}\ \cos\phi$

When P moves freely under gravity, its horizontal component of velocity will remain constant, so

$$\sqrt{\frac{3ga}{2}}\cos\phi = \sqrt{\frac{1}{2}ga}\cos 60°$$

$$\Rightarrow \cos\phi = \frac{\cos 60°}{\sqrt{3}} = \frac{1}{2\sqrt{3}}$$

$$\Rightarrow \phi = 73.2° \text{ (3 s.f.)}$$

26 a Let the velocity of the particle at C be v. Taking the horizontal through A as the zero level for potential energy and using conservation of energy, the gain in kinetic energy at C equals the loss in potential energy:

$$\frac{1}{2}mv^2 - \frac{1}{2}mu^2 = mga(1-\cos\theta)$$

$$\Rightarrow v^2 = u^2 + 2ga(1-\cos\theta) \qquad \textbf{(1)}$$

Resolving along the radius through C:

$$\text{R}(\swarrow):\ -R + mg\cos\theta = \frac{mv^2}{a}$$

As the particle leaves the sphere at C, $R = 0$ so $v^2 = ag\cos\theta$

Substituting for v^2 in equation **(1)** gives:

$$ag\cos\theta = u^2 + 2ga(1-\cos\theta)$$

$$\Rightarrow 3\cos\theta = \frac{u^2}{ag} + 2$$

$$\Rightarrow \cos\theta = \frac{2}{3} + \frac{u^2}{3ag}$$

b Using conservation of energy, the gain in kinetic energy from C to when the particle hits the ground equals the loss in potential energy:

$$\frac{1}{2}m\left(\frac{9ag}{2}\right) - \frac{1}{2}m(ag\cos\theta) = mga(1+\cos\theta)$$

$$\Rightarrow \frac{3}{2}\cos\theta = \frac{9}{4} - 1 = \frac{5}{4} \Rightarrow \cos\theta = \frac{5}{6}$$

So $\theta = 34°$ (2 s.f.)

27 a Taking the horizontal through A as the zero level for potential energy and using conservation of energy, the loss in kinetic energy at B equals the gain in potential energy:

$$\frac{1}{2}mu^2 - \frac{1}{2}mv^2 = mga(1+\cos 60°)$$

$$\Rightarrow v^2 = u^2 - 3ga \qquad \textbf{(1)}$$

b Resolving along the radius through B:

$$\text{R}(\swarrow):\ R + mg\cos 60° = \frac{mv^2}{a} \qquad \textbf{(2)}$$

When $u^2 = 6ga$, $v^2 = 3ga$ from equation **(1)**

$$\text{So } R = \frac{mv^2}{a} - mg\cos 60° = 3mg - \frac{mg}{2} = \frac{5mg}{2}$$

c The least value for u will be that which enables the particle to reach B, but to leave the surface at B so that $R = 0$ at point B.

If $R = 0$ at B, then from equation **(2)** $\frac{mg}{2} = \frac{mv^2}{a} \Rightarrow v^2 = \frac{ag}{2}$

So from equation **(1)** $\frac{ag}{2} = u^2 - 3ga \Rightarrow u^2 = \frac{7ag}{2}$

Hence $u = \sqrt{\frac{7ag}{2}}$

d The distance $BC = 2 \times a\sin 60° = \sqrt{3}a$

The horizontal component of the speed at $B = v\cos 60°$

For motion under gravity the horizontal velocity of the particle in travelling from B to C is constant and using $s = vt$ gives:

$$\sqrt{3}a = v\cos 60° t \Rightarrow t = \frac{2a\sqrt{3}}{v}$$

The vertical component of the speed at $B = v\sin 60°$

The vertical component of the distance the particle travels is 0 (as the particle is now at C, which is level with B)

So using $s = ut + \frac{1}{2}at^2$ gives:

$$0 = v\sin 60° t - \frac{1}{2}gt^2$$

$$\Rightarrow t = \frac{2v\sin 60°}{g} = \frac{v\sqrt{3}}{g}$$

So when the particle leaves the bowl at B and meets it at C

$$\frac{v\sqrt{3}}{g} = \frac{2a\sqrt{3}}{v} \Rightarrow v^2 = 2ag$$

So using equation **(1)** from part **a**, $2ga = u^2 - 3ga \Rightarrow u^2 = 5ga$

Hence $u = \sqrt{5ga}$

28 a The total mass is $3m+5m+\lambda m=(8+\lambda)m$

Taking moments about the *y*-axis:

$$3m\times 4+5m\times 0+\lambda m\times 4=(8+\lambda)m\times 2$$
$$12+4\lambda=16+2\lambda$$
$$2\lambda=4$$
$$\lambda=2$$

b Taking moments about the *x*-axis:

$$3m\times 0+5m\times -3+\lambda m\times 2=(8+\lambda)m\times k$$
$$-15+4=10k \qquad \text{substituting for } \lambda=2$$
$$-11=10k$$

So k = 1.1

29 The total mass is $2M+xM+yM=(2+x+y)M$

Taking moments about the *y*-axis:

$$(2+x+y)M\times 2=2M\times 2+xM\times 1+yM\times 3$$
$$4+2x+2y=4+x+3y$$
$$x-y=0 \qquad \textbf{(1)}$$

Taking moments about the *x*-axis:

$$(2+x+y)M\times 4=2M\times 5+xM\times 3+yM\times 1$$
$$8+4x+4y=10+3x+y$$
$$x+3y=2 \qquad \textbf{(2)}$$

Subtracting equation **(1)** from equation **(2)** gives:

$$4y=2\Rightarrow y=\frac{1}{2}$$

And so from equation **(2)** $x=\frac{1}{2}$

30 Using $\sum m_i\mathbf{r}_i=\mathbf{r}\sum m_i$

$$0.1\begin{pmatrix}2\\-1\end{pmatrix}+0.2\begin{pmatrix}2\\5\end{pmatrix}+0.3\begin{pmatrix}4\\2\end{pmatrix}=(0.1+0.2+0.3)\begin{pmatrix}\bar{x}\\\bar{y}\end{pmatrix}$$
$$\begin{pmatrix}0.2\\-0.1\end{pmatrix}+\begin{pmatrix}0.4\\1\end{pmatrix}+\begin{pmatrix}1.2\\0.6\end{pmatrix}=(0.6)\begin{pmatrix}\bar{x}\\\bar{y}\end{pmatrix}$$
$$\begin{pmatrix}1.8\\1.5\end{pmatrix}=(0.6)\begin{pmatrix}\bar{x}\\\bar{y}\end{pmatrix}$$
$$\begin{pmatrix}3\\2.5\end{pmatrix}=\begin{pmatrix}\bar{x}\\\bar{y}\end{pmatrix}$$

So the centre of mass is (3**i** + 2.5**j**) m

31 a Using $\sum m_i\mathbf{r}_i = \mathbf{r}\sum m_i$

$$2M\begin{pmatrix}6\\0\end{pmatrix}+M\begin{pmatrix}0\\4\end{pmatrix}+kM\begin{pmatrix}2\\-2\end{pmatrix}=(3+k)M\begin{pmatrix}3\\c\end{pmatrix}$$

$$\begin{pmatrix}12\\0\end{pmatrix}+\begin{pmatrix}0\\4\end{pmatrix}+\begin{pmatrix}2k\\-2k\end{pmatrix}=\begin{pmatrix}9+3k\\3c+ck\end{pmatrix}$$

Equating **i** component gives $12+2k=9+3k \Rightarrow k=3$ as required.

b Equating **j** component and substituting for k gives:

$$4-2k=3c+ck \Rightarrow -2=6c \Rightarrow c=-\frac{1}{3}$$

32 a The area of the lamina is $20\times10=200\text{ cm}^2$

The area of the circle is $\pi\times3^2=9\pi\text{ cm}^2$

The area of the plate is $(200-9\pi)\text{ cm}^2$

Let the distance of the centre of mass of the plate from *AD* be $\bar{x}$ cm

As the lamina is uniform, masses are proportional to areas. By symmetry, the centre of mass of the rectangular lamina is 10 cm from *AD* and the centre of mass of the circle is 6 cm from *AD*.

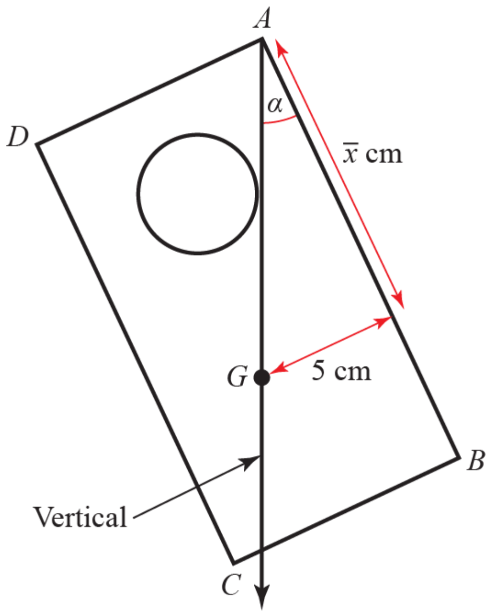
	Lamina	**Circle**	**Plate**
Area	200	9π	$200-9\pi$
Distance of centre of mass from *AD*	10	6	$\bar{x}$

The moment of the plate about AD equals the moment of the complete lamina less the moment of the circle that has been removed:

$$(200-9\pi)\times\bar{x}=200\times10-9\pi\times6$$

$$\Rightarrow \bar{x}=\frac{2000-54\pi}{200-9\pi}=10.658\ldots=10.7\text{ cm (3 s.f.)}$$

32 b Let the angle between AB and the vertical be α. When the plate is suspended freely from A, its centre of mass G is vertically below the point of suspension A. The distance of G from AD was found in part **a** and the distance of G from AB is 5 cm by symmetry.



$$\tan\alpha = \frac{5}{\bar{x}} = \frac{5}{10.658} = 0.469 \text{ (3 s.f.)}$$
$$\Rightarrow \alpha = 25° \text{ (to the nearest degree)}$$

33 a The area of rectangle $ABDE$ is $6a \times 8a = 48a^2$
The centre of mass of rectangle is 4a from X

The area of ΔBCD is $\frac{1}{2} \times 6a \times 4a = 12a^2$

The centre of mass of the triangle is $\frac{1}{3}h = \frac{1}{3} \times 4a$ from the base BD of the triangle.

The area of lamina $ABCDE$ is $48a^2 + 12a^2 = 60a^2$

	Lamina	Rectangle	Triangle
Area	$60a^2$	$48a^2$	$12a^2$
Displacements from X	GX	$4a$	$-\frac{4}{3}a$

Taking moments about X, the rectangle is on one side of X, taken as positive, and the triangle is on the other side of X, taken as negative:

$$60a^2 \times GX = 48a^2 \times 4a + 12a^2 \times \left(-\frac{4}{3}a\right) = 192a^3 - 16a^3 = 176a^3$$

$$GX = \frac{176a^3}{60a^2} = \frac{44}{15}a \qquad \text{as required}$$

33 b With the particle at C, taking moments about X:

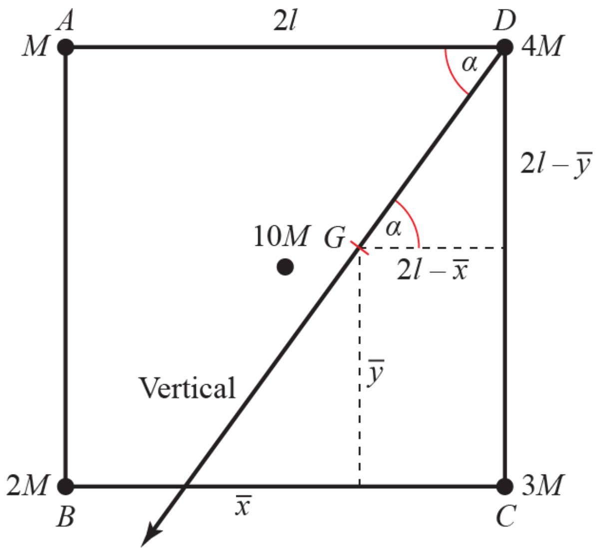
$$Mg \times GX = \lambda Mg \times 4a$$

$$\cancel{M}\cancel{g} \times \frac{44}{15}\cancel{a} = \lambda \cancel{M}\cancel{g} \times 4\cancel{a} \quad \text{substituting for } GX \text{ from part } \mathbf{a}$$

$$\lambda = \frac{44}{15 \times 4} = \frac{11}{15}$$

There is a vertical force at *B* but, as the line of action of this force passes through *X*, its moment about *X* is zero.

34 a Let the distance of the centre of mass, *G*, of the loaded plate from *AB* and *BC* be $\bar{x}$ cm and $\bar{y}$ cm respectively.



Taking *B* as the origin and using $\sum m_i \mathbf{r}_i = \mathbf{r} \sum m_i$

$$M\begin{pmatrix}0\\2l\end{pmatrix} + 2M\begin{pmatrix}0\\0\end{pmatrix} + 3M\begin{pmatrix}2l\\0\end{pmatrix} + 4M\begin{pmatrix}2l\\2l\end{pmatrix} + 10M\begin{pmatrix}l\\l\end{pmatrix} = (1+2+3+4+10)M\begin{pmatrix}\bar{x}\\\bar{y}\end{pmatrix}$$

$$\begin{pmatrix}24l\\20l\end{pmatrix} = 20\begin{pmatrix}\bar{x}\\\bar{y}\end{pmatrix}$$

So distance from $AB = \bar{x} = \frac{24l}{20} = \frac{6l}{5}$

b From working in part **a**, distance from $BC = \bar{y} = \frac{20l}{20} = l$

c When the framework hangs freely from *D*, its centre of mass *G* is vertically below *D*. Let the angle made by *DA* with the downward vertical be α. Then from the diagram in part **a**:

$$\tan\alpha = \frac{2l - \bar{y}}{2l - \bar{x}} = \frac{2l - l}{2l - \frac{6l}{5}} = \frac{l}{\frac{4l}{5}} = \frac{5}{4}$$

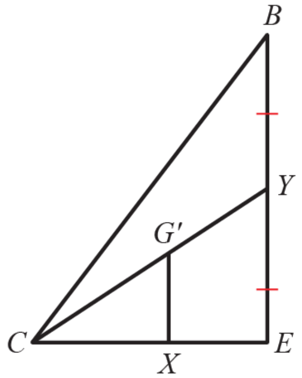
$\Rightarrow \alpha = 51°$ (to the nearest degree)

35 a The area of rectangle *ABCD* is $3a \times 2a = 6a^2$

The area of triangle *BCE* is $\frac{1}{2}a \times 2a = a^2$

The area of lamina *ABCD* is $6a^2 - a^2 = 5a^2$

Let the distance of the centre of mass of the lamina, *G*, from *AD* be $\bar{x}$ cm. The distance of the centre of mass of the rectangle from AD is 1.5*a*. Consider the centre of mass of the triangle, G'. If *Y* is the midpoint of BE then $CG' : G'Y = 2 : 1$.



Using similar triangles, $\frac{CX}{XE} = \frac{CG'}{G'Y} = \frac{2}{1}$

As $CE = a, XE = \frac{1}{3}a$, so G' is $\frac{1}{3}a$ from *BE* and $3a - \frac{1}{3}a = \frac{8}{3}a$ from *AD*.

	Rectangle	Lamina	Triangle
Area	$6a^2$	$5a^2$	a^2
Distance from *AD*	$\frac{3}{2}a$	$\bar{x}$	$\frac{8}{3}a$

The moment of the lamina about *AD* is the moment of the complete rectangle *less* the moment of the triangle which has been removed from the rectangle:

$$5a^2 \times \bar{x} = 6a^2 \times \frac{3}{2}a - a^2 \times \frac{8}{3}a$$

$$5a^2\bar{x} = 9a^3 - \frac{8}{3}a^3 = \frac{19}{3}a^3$$

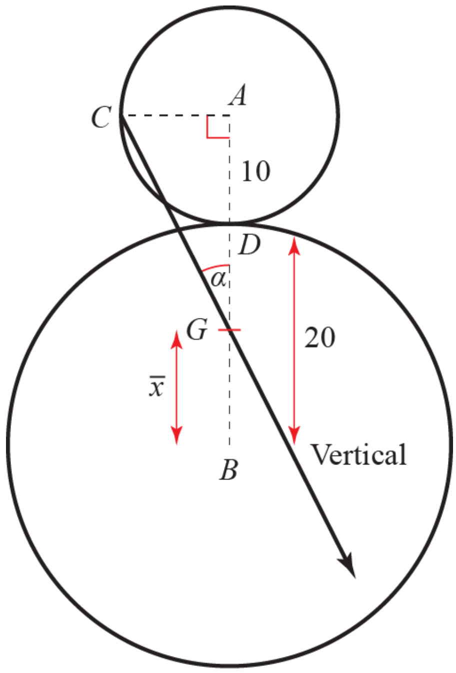
$$\bar{x} = \frac{19}{15}a$$

b Let *N* be the mid-point of *AB*. Taking moments about *N*:

$$Mg\left(\frac{3}{2}a - \bar{x}\right) = mg \times \frac{3}{2}a$$

$$\Rightarrow m = \frac{2}{3a}M\left(\frac{3}{2}a - \bar{x}\right) = \frac{2}{3a}M\left(\frac{3}{2}a - \frac{19}{15}a\right) = \frac{2}{3} \times \frac{7}{30}M = \frac{7}{45}M$$

36 a Let the distance of the centre of mass of the decoration, G, from B be $\bar{x}$ cm.



The area of the smaller circle is $\pi 10^2 = 100\pi$
The area of the larger circle is $\pi 20^2 = 400\pi$
The area of the decoration is $100\pi + 400\pi = 500\pi$

As the card is uniform, the masses of the decoration and circles are proportional to their areas.

	Decoration	Small circle	Large circle
Area	500π	100π	400π
Distance from *B* (cm)	$\bar{x}$	30	0

Taking moments about B:

$$500\pi \times \bar{x} = 100\pi \times 30 + 400\pi \times 0$$

$$\bar{x} = \frac{3000}{500} = 6\text{ cm}$$

b When the decoration is freely suspended from C, its centre of mass G is vertically below C.
Let the angle between AB and the vertical be α. Then from the diagram drawn for part **a**:

$$\tan \alpha = \frac{AC}{GA} = \frac{10}{10 + (20 - \bar{x})} = \frac{10}{24} = \frac{5}{12}$$

$$\Rightarrow \alpha = 22.6° \text{ (1 d.p.)}$$

37 a Let the distance of the centre of mass, G, of the lamina from the midpoint of AB be $\bar{x}$ cm.

The area of the semicircle is $\frac{1}{2}\pi \times 5^2 = \frac{25}{2}\pi$.

So the area of the lamina L is $100 - \frac{25}{2}\pi$.

From the question, the distance of the centre of mass of the semicircle from the midpoint of AB is $\frac{4a}{3\pi} = \frac{20}{3\pi}$ as the radius a is 5 cm in this case.

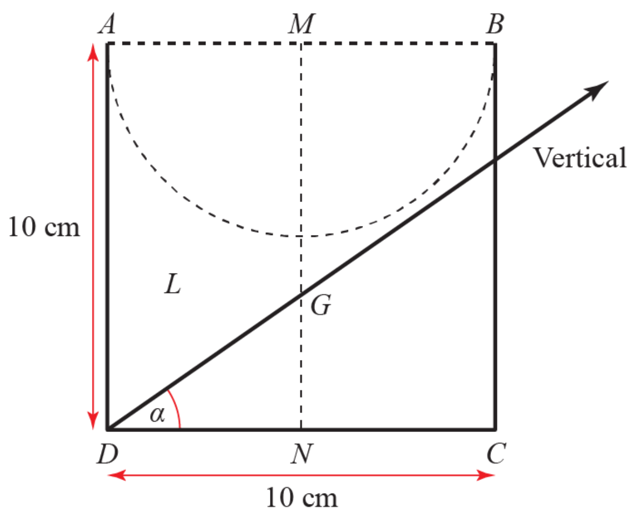
	Square	Semicircle	Lamina L
Area	100	$\frac{25}{2}\pi$	$100 - \frac{25}{2}\pi$
Distance from midpoint of AB (cm)	5	$\frac{20}{3\pi}$	$\bar{x}$

Taking moments about the midpoint of AB:

$$\left(100 - \frac{25}{2}\pi\right)\bar{x} = 100 \times 5 - \frac{20}{3\pi} \times \frac{25}{2}\pi = 500 - \frac{250}{3} = \frac{1250}{3}$$

$$\bar{x} = \frac{2500}{3(200 - 25\pi)} = 6.86 \text{ (2 d.p.)}$$

b When L is suspended freely from D, the centre of mass G hangs vertically below D.
The downward vertical is drawn in the diagram. Let the angle between CD and the vertical be α.



$$\tan\alpha = \frac{GN}{DN} = \frac{(10 - \bar{x})}{5} = \frac{10 - 6.86}{5} = 0.628$$

$$\Rightarrow \alpha = 32.1° \text{ (1 d.p.)}$$

38 a Let the distance of the centre of mass of T from A be $\bar{x}$ cm, i.e. $AG = \bar{x}$.

The area of the smaller circle is $\pi 8^2 = 64\pi$.

The area of the larger circle is $\pi 24^2 = 576\pi$.

The area of T is $576\pi - 64\pi = 512\pi$.

As the lamina is uniform, the masses of T and the large circle are proportional to their areas.

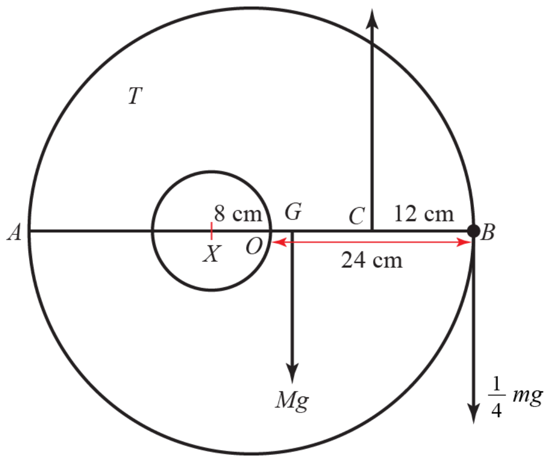
	T	**Small circle**	**Large circle**
Area	512π	64π	576π
Distance from *A* (cm)	$\bar{x}$	16	24

Taking moments about A:

$512\pi \times \bar{x} = 576\pi \times 24 - 64\pi \times 16$

$$\bar{x} = \frac{576 \times 24 - 64 \times 16}{512} = 25\,\text{cm}$$

b Let C be the midpoint of OB and the mass of T be M.



$BC = 12\,\text{cm}$, $CG = AC - AG = 36 - 25 = 11\,\text{cm}$

Taking moments about C

$$Mg \times 11 = \frac{1}{4} mg \times 12$$

$$M = \frac{3m}{11}$$

39 a Using $\sum m_i \mathbf{r}_i = \mathbf{r} \sum m_i$

$$4m\begin{pmatrix}0\\4\end{pmatrix} + 6m\begin{pmatrix}9\\0\end{pmatrix} + 2m\begin{pmatrix}0\\-4\end{pmatrix} = (4+6+2)m\begin{pmatrix}\bar{x}\\\bar{y}\end{pmatrix}$$

$$\begin{pmatrix}54\\8\end{pmatrix} = 12\begin{pmatrix}\bar{x}\\\bar{y}\end{pmatrix}$$

So $\bar{x} = \frac{54}{12} = \frac{9}{2}$ and $\bar{y} = \frac{8}{12} = \frac{2}{3}$

So the cooordinates of the centre of mass of the particles without the lamina are $\left(\frac{9}{2}, \frac{2}{3}\right)$

39 b The uniform lamina is a triangle so its centre of mass is (3, 0). Its mass is km.
The combined system of the lamina and the three particles has centre of mass $(4, \lambda)$.
Its mass $= 4m + 6m + 2m + km = (12 + k)m$

Using $\sum m_i \mathbf{r}_i = \mathbf{r} \sum m_i$ and the result from part **a** to treat the three particles as a single entity:

$$12m\begin{pmatrix} \frac{9}{2} \\ \frac{2}{3} \end{pmatrix} + km\begin{pmatrix} 3 \\ 0 \end{pmatrix} = (12+k)m\begin{pmatrix} 4 \\ \lambda \end{pmatrix}$$

$$\begin{pmatrix} 54+3k \\ 8 \end{pmatrix} = (12+k)\begin{pmatrix} 4 \\ \lambda \end{pmatrix}$$

So equating the **i** components gives:

$54 + 3k = 48 + 4k$

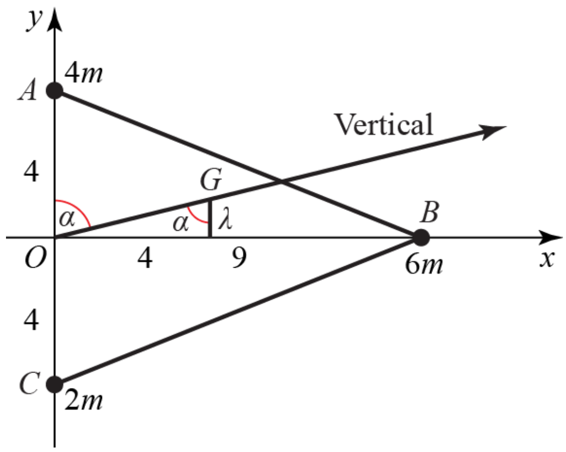
$\Rightarrow k = 6$

c Equating the **j** components from the vector equation in part **b** and substituting for k gives:

$8 = (12 + k)\lambda = 18\lambda$

$\Rightarrow \lambda = \frac{8}{18} = \frac{4}{9}$

d Let the angle between AC and the vertical be α.



$\tan\alpha = \frac{4}{\lambda} = 4 \div \frac{4}{9} = 9$

$\Rightarrow \alpha = 83.7°$ (1 d.p.)

40 a Let the distances of the centre of mass of *L*, say *G*, from *AD* and *AB* be $\bar{x}$ and $\bar{y}$ respectively.
The mass of *L* is $3m + 4m + m + 2m = 10m$.

	Plate	Rectangle	Particles		
	L	***ABCD***	***A***	***B***	***C***
Mass	$10m$	$3m$	$4m$	m	$2m$
Distance from *AD*	$\bar{x}$	$2.5a$	0	$5a$	$5a$
Distance from *AB*	$\bar{y}$	a	0	0	$2a$

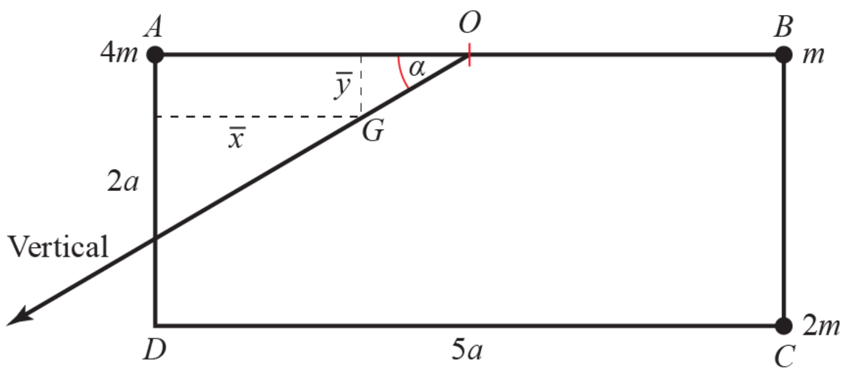
Taking moments about *AD*:
$10m \times \bar{x} = 3m \times 2.5a + m \times 5a + 2m \times 5a = 22.5ma$

$$\bar{x} = \frac{22.5a}{10} = 2.25a \quad \text{as required}$$

b Taking moments about *AB*:
$10m \times \bar{y} = 3m \times a + 2m \times 2a = 7ma$

$$\bar{y} = \frac{7a}{10} = 0.7a$$

c When *L* is freely suspended from *O*, the centre of mass *G* of the complete system hangs vertically below *O*. Let α be the angle between OA and the vertical.



$$\tan\alpha = \frac{\bar{y}}{2.5a - \bar{x}} = \frac{0.7a}{0.25a} = 2.8$$
$\Rightarrow \alpha = 70°$ (to the nearest degree)

So the angle that *AB* makes with the horizontal is $(90 - 70)° = 20°$ (to the nearest degree)

d The total weight $10mg$ of the loaded plate *L* acts vertically through the centre of mass *G*, so from part **a** the force has a perpendicular distance from *O* of $2.5a - \bar{x} = 2.5a - 2.25a = 0.25a$

The force *P* acts horizontally through *C*, so this force has a perpendicular distance from *O* of $2a$

The loaded plate is in equilibrium, so taking moments about *O*:
$P \times 2a = 10mg \times 0.25a$

$$P = \frac{2.5mg}{2} = \frac{5}{4}mg \quad \text{as required}$$

40 e The forces acting on the system are the horizontal force at C, the weight of the loaded plate through G and the reaction force at O. Let the horizontal and vertical components of the force acting on L at O be X and Y respectively.

$$\text{R}(\rightarrow)\quad X = P = \frac{5}{4}mg$$

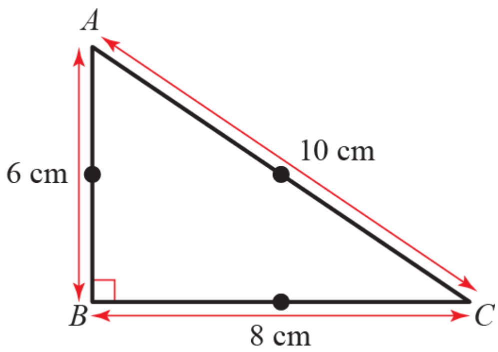
$$\text{R}(\uparrow)\quad Y = 10mg$$

Let the magnitude of the force acting on L at O be R.

$$R^2 = X^2 + Y^2 = \left(\frac{5}{4}mg\right)^2 + (10mg)^2 = \frac{1625}{16}m^2g^2$$

$$\Rightarrow R = \frac{\sqrt{1625}}{4}mg = \frac{5\sqrt{65}}{4}mg$$

41 a As the wire is uniform, each side has a mass proportional to its length and the centre of mass of each side is at the middle of the side. The triangle has sides in the ratio of 3 : 4 : 5 so it is a right-angled triangle.



Taking B as the origin and axes along AB and BC:

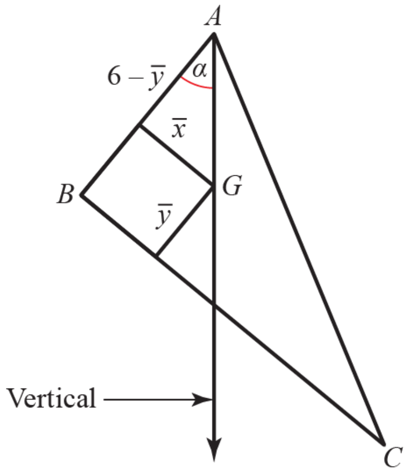
$$6\begin{pmatrix}0\\3\end{pmatrix} + 8\begin{pmatrix}4\\0\end{pmatrix} + 10\begin{pmatrix}4\\3\end{pmatrix} = (6+8+10)\begin{pmatrix}\bar{x}\\\bar{y}\end{pmatrix}$$

$$\begin{pmatrix}72\\48\end{pmatrix} = 24\begin{pmatrix}\bar{x}\\\bar{y}\end{pmatrix}$$

So distance from $AB = \bar{x} = \frac{72}{24} = 3\,\text{cm}$

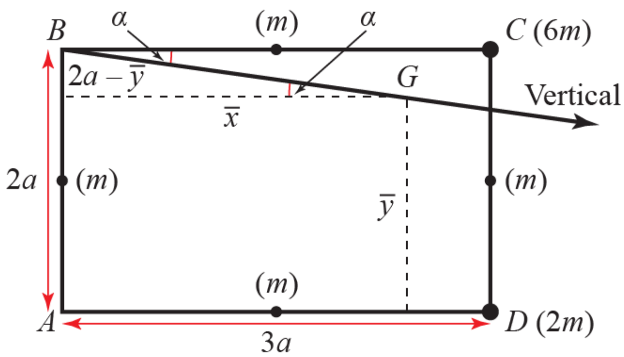
b Using vector calculation in part **a**, distance from $BC = \bar{y} = \frac{48}{24} = 2\,\text{cm}$

41 c When the frame hangs freely from A, its centre of mass G is vertically below A.
Let the acute angle made by AB with the downward vertical be α.



$$\tan\alpha = \frac{\bar{x}}{6-\bar{y}} = \frac{3}{6-2} = \frac{3}{4}$$
$$\Rightarrow \alpha = 37° \text{ (to the nearest degree)}$$

42 a Let the distance of the centre of mass of the loaded framework, say G, from AB and AD be $\bar{x}$ and $\bar{y}$ respectively. As the rods that make up the framework are uniform, the centre of mass of each rod is at its midpoint.



Taking A as the origin and axes along AB and AD:

$$m\begin{pmatrix}0\\a\end{pmatrix}+m\begin{pmatrix}1.5a\\0\end{pmatrix}+m\begin{pmatrix}1.5a\\2a\end{pmatrix}+m\begin{pmatrix}3a\\a\end{pmatrix}+6m\begin{pmatrix}3a\\2a\end{pmatrix}+2m\begin{pmatrix}3a\\a\end{pmatrix}=(1+1+1+1+6+2)m\begin{pmatrix}\bar{x}\\\bar{y}\end{pmatrix}$$

$$\begin{pmatrix}30a\\16a\end{pmatrix}=12\begin{pmatrix}\bar{x}\\\bar{y}\end{pmatrix}$$

i Distance of the centre of mass of the loaded framework from $AB = \bar{x} = \frac{30}{12}a = \frac{5}{2}a$

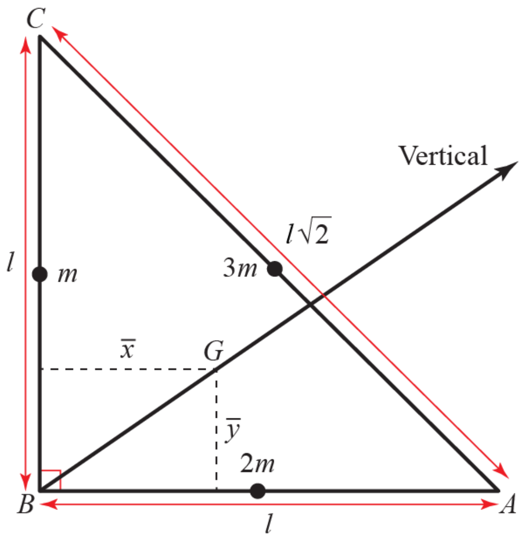
ii Distance of the centre of mass of the loaded framework from $AD = \bar{y} = \frac{16}{12}a = \frac{4}{3}a$

42 b Let α be the angle BC makes with the vertical (see diagram for part **a**).

$$\tan\alpha = \frac{2a - \bar{y}}{\bar{x}} = \frac{2a - \frac{4}{3}a}{\frac{5}{2}a} = \frac{2}{3} \times \frac{2}{5} = \frac{4}{15}$$

$\Rightarrow \alpha = 15°$ (to the nearest degree)

43 a Let the distance of the centre of mass of the loaded framework, say G, from BC and AB be $\bar{x}$ and $\bar{y}$ respectively. As $l^2 + l^2 = (l\sqrt{2})^2$, the angle at B is a right angle. As each rod is uniform, the centre of mass of each rod is at its midpoint.



Taking B as the origin and axes along AB and BC:

$$m\begin{pmatrix} 0 \\ 0.5l \end{pmatrix} + 2m\begin{pmatrix} 0.5l \\ 0 \end{pmatrix} + 3m\begin{pmatrix} 0.5l \\ 0.5l \end{pmatrix} = (1+2+3)m\begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix}$$

$$\begin{pmatrix} 2.5l \\ 2l \end{pmatrix} = 6\begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix}$$

i Distance of the centre of mass of the framework from $BC = \bar{x} = \frac{2.5l}{6} = \frac{5l}{12}$

ii Distance of the centre of mass of the loaded framework from $AB = \bar{y} = \frac{2l}{6} = \frac{l}{3}$

b When the framework changes freely from A, its centre of mass G is vertically below A. The vertical line has been drawn in the diagram (see part **a**). Let the angle that BC makes with the vertical be α.

$$\tan\alpha = \frac{\bar{y}}{\bar{x}} = \frac{\frac{l}{3}}{\frac{5l}{12}} = \frac{1}{3} \times \frac{12}{5} = \frac{4}{5} = 0.8$$

$\Rightarrow \alpha = 39°$ (to the nearest degree)

44 a Let G be the origin so that GE lies on the x-axis and AG lies on the y-axis.

The coordinates of the centre of mass of the rectangle $ABGF$ are $\left(\frac{d}{2}, \frac{3d}{2}\right)$.

The coordinates of the centre of mass of the square $CDFE$ are $\left(\frac{3d}{2}, \frac{d}{2}\right)$.

Rectangle $ABGF$ has area $3d^2$, square $CDFE$ has area d^2, but the square has density three times that of the rectangle. So the mass ratios of the rectangle, square and lamina are $3d^2$, $3 \times d^2 = 3d^2$ and $6d^2$ respectively.

$$3d^2\begin{pmatrix}\frac{d}{2}\\ \frac{3d}{2}\end{pmatrix} + 3d^2\begin{pmatrix}\frac{3d}{2}\\ \frac{d}{2}\end{pmatrix} = 6d^2\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix}$$

$$\begin{pmatrix}6d^3\\ 6d^3\end{pmatrix} = 6d^2\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix}$$

Distance of the centre of mass of the framework from $AG = \bar{x} = \frac{6d^3}{6d^2} = d$.

b Distance of the centre of mass of the framework from $GE = \bar{y} = \frac{6d^3}{6d^2} = d$.

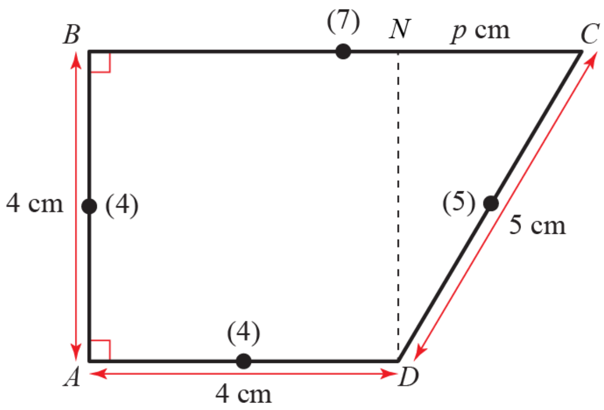
c The centre of mass of the lamina lies at (d, d), which are the coordinates of point C. If the lamina is freely suspended from point A, the centre of mass will align itself vertically below point A, so that AC will be the new vertical.

Let the angle that AB makes with the vertical be θ, then $\angle CAB = \theta$.

$$\tan\theta = \frac{CB}{AB} = \frac{2d}{d} = 2$$

$\Rightarrow \theta = 63°$ (to the nearest degree)

45 a Let N be the foot of the perpendicular from D to BC and $NC = p$ cm.



The total length of the frame is $AB + BC + CD + AD = 20 \Rightarrow 4 + BC + 5 + 4 = 20 \Rightarrow BC = 7\,\text{cm}$

Hence $NC = BC - BN = 7 - 4 = 3$

Let the distance of the centre of mass of the frame from AB be $\bar{x}$ cm

Calculating the mass of each section of wire from the densities given in the question:

	AB	***BC***	***CD***	***DA***
Mass (kg)	$0.04M$	$0.07M$	$0.075M$	$0.04M$
Distance from *AB* (cm)	0	3.5	5.5	2

Taking moments about AB:

$$(0.04 + 0.07 + 0.075 + 0.04)M\bar{x} = 0.04M \times 0 + 0.07M \times 3.5 + 0.075M \times 5.5 + 0.04M \times 2$$

$$\bar{x} = \frac{0.7375}{0.225} = 3.2777\ldots = 3.28\,\text{cm (3 s.f.)}$$

b The total mass of the framework is $0.04M + 0.07M + 0.075M + 0.04M = 0.225M$

The weight of the framework acts through the centre of mass, which is $(3.5 - \bar{x})\,\text{cm}$ from the vertical through the midpoint of BC.

Taking moments about the midpoint of BC:

$$0.225Mg(3.5 - \bar{x}) = kMg \times 3.5$$

$$\Rightarrow k = \frac{0.225(3.5 - \bar{x})}{3.5} = \frac{0.225(3.5 - 3.2777)}{3.5} = 0.0143 \text{ (3 s.f.)}$$

46 a Let the midpoint of BC be G and the midpoint of EG be F. Divide the lamina into three sections, a rectangle $ABEG$, a square $CDFG$ and a triangle EDF.

Let A be the origin so that AB lies on the x-axis and AE lies on the y-axis.
The coordinates of the centre of mass of the rectangle $ABEG$ are $(40, -20)$.
The coordinates of the centre of mass of the square $CDFG$ are $(60, -60)$.
The coordinates of the centre of mass of the triangle EDF are found by taking the average of the coordinates of its three vertices, in this case $(0, -40)$, $(40, -40)$ and $(40, -80)$, hence it is:

$$\left(\frac{0+40+40}{3}, \frac{-40-80-40}{3}\right) = \left(\frac{80}{3}, -\frac{160}{3}\right)$$

Rectangle $ABEG$ has area $3200\,\text{cm}^2$, square $CDFG$ has area $1600\,\text{cm}^2$, and triangle EDF has area $800\,\text{cm}^2$.

Using $\sum m_i \mathbf{r}_i = \mathbf{r}\sum m_i$

$$3200\begin{pmatrix}40\\-20\end{pmatrix} + 1600\begin{pmatrix}60\\-60\end{pmatrix} + 800\begin{pmatrix}\frac{80}{3}\\-\frac{160}{3}\end{pmatrix} = (3200+1600+800)\begin{pmatrix}\bar{x}\\\bar{y}\end{pmatrix}$$

$$\begin{pmatrix}\frac{736\,000}{3}\\-\frac{608\,000}{3}\end{pmatrix} = 5600\begin{pmatrix}\bar{x}\\\bar{y}\end{pmatrix}$$

The distance of the centre of mass of the lamina from $AE = \bar{x} = \dfrac{736\,000}{3\times 5600} = 43.8\,\text{cm}$ (3 s.f.)

b $R(\uparrow)$: $T_1 + T_2 = W$ **(1)**

Taking moments about A:

$W\times\bar{x} - T_2\times 80 = 0$

$\Rightarrow 80T_2 = \dfrac{736\,000}{16800}W$

$\Rightarrow T_2 = \dfrac{736}{1344}W = \dfrac{23}{42}W = 0.548W$ N (3 s.f.)

From equation **(1)** $T_1 = W - T_2 = \dfrac{19}{42}W = 0.452W$ N (3 s.f.)

46 c $R(\uparrow): T_1 + T_2 = W + kW$ **(2)**

Taking moments about *A*:

$W \times \bar{x} + kW \times 80 - T_2 \times 80 = 0$

$\Rightarrow 80T_2 = \frac{736\,000}{16\,800}W + 80kW$

$\Rightarrow T_2 = \frac{23}{42}W + kW$

So if $T_2 \leqslant 8W \Rightarrow \frac{23}{42}W + kW \leqslant 8W$

$\Rightarrow k \leqslant 8 - \frac{23}{42}$

$\Rightarrow k \leqslant 7.45$ (3 s.f.)

From equation **(2)** $T_1 = W + kW - T_2 = \frac{19}{42}W$

$\Rightarrow T_1 \leqslant 10W$

So the largest value of k satisfying $T_1 \leqslant 10W$ and $T_2 \leqslant 8W$ is $k = 7.45$ (3 s.f.)

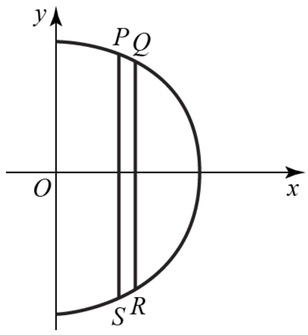
47 The centre of mass lies on the axis of symmetry, $y = 0$.

To find the x coordinate, using $\bar{x} = \frac{\int y^2 x \, dx}{\int y^2 \, dx}$, and substituting $y = \sqrt{x}$ gives:

$$\bar{x} = \frac{\int_0^4 x^2 \, dx}{\int_0^4 x \, dx} = \frac{\left[\frac{1}{3}x^3\right]_0^4}{\left[\frac{1}{2}x^2\right]_0^4} = \frac{64}{3} \div 8 = \frac{8}{3}$$

So the coordinates of the centre of mass of the solid are $\left(\frac{8}{3}, 0\right)$, a distance of $\frac{8}{3}$ from *O*.

48 a Let the straight edge lie along the y-axis. Then the centre of mass lies on the x-axis from symmetry.



If P has coordinates (x, y) and the elemental strip $PQRS$ has width δx then its area is $2y\delta x$.

The mass M of the lamina = $\frac{1}{2}\pi a^2\rho$, where ρ is the mass per unit area of the lamina.

Let $\bar{x}$ be the distance of the centre of mass from O, then $M\bar{x} = \int_0^a 2\rho xy\,\mathrm{d}x$

As the boundary of the semicircle has the equation $x^2 + y^2 = a^2$, then $y = (a^2 - x^2)^{\frac{1}{2}}$

So $M\bar{x} = \int_0^a 2\rho x(a^2 - x^2)^{\frac{1}{2}}\,\mathrm{d}x = \left[-\frac{2}{3}\rho(a^2 - x^2)^{\frac{3}{2}}\right]_0^a = \frac{2}{3}\rho a^3$

$\Rightarrow \bar{x} = \frac{\frac{2}{3}\rho a^3}{\frac{1}{2}\pi a^2\rho} = \frac{4a}{3\pi}$ as required

b Using the result from part **a** and letting $\bar{x}$ be the distance of the centre of mass of the resulting lamina from O:

Shape	Mass	Distance of centre of mass from O
Semicircle radius a	$\frac{1}{2}\pi\rho a^2$	$\frac{4a}{3\pi}$
Semicircle radius b	$\frac{1}{2}\pi\rho b^2$	$\frac{4b}{3\pi}$
Resulting	$\frac{1}{2}\pi\rho(a^2 - b^2)$	$\bar{x}$

Taking moments about O:

$\frac{1}{2}\pi\rho(a^2 - b^2)\bar{x} = \frac{1}{2}\pi\rho a^2 \times \frac{4a}{3\pi} - \frac{1}{2}\pi\rho b^2 \times \frac{4b}{3\pi}$

$\Rightarrow \bar{x} = \frac{4}{3\pi}\frac{(a^3 - b^3)}{(a^2 - b^2)} = \frac{4}{3\pi}\frac{(a-b)(a^2 + ab + b^2)}{(a-b)(a+b)} = \frac{4}{3\pi}\frac{(a^2 + ab + b^2)}{(a+b)}$ as required

c As $b \to a$, the area becomes a circular arc and from the equation found in part **b**

$\bar{x} \to \frac{4}{3\pi} \times \frac{(a^2 + a^2 + a^2)}{(a+a)} = \frac{4}{3\pi} \times \frac{3a^2}{2a} = \frac{2a}{\pi}$

49 a Let B be the origin so that BC lies on the x-axis and AB lies on the y-axis. Let the equation of the line AC be $y = c - mx$, so that the coordinates of the triangle are (0, 0), (0, c) and $\left(\frac{c}{m}, 0\right)$.

If a point on the line has coordinates (x, y), then an elemental horizontal strip of width δy at that point has area is $x\delta y$

The mass M of the triangular lamina = $\frac{1}{2}c\frac{c}{m}\rho = \frac{c^2\rho}{2m}$, where ρ is the mass per unit area

Let $\bar{y}$ be the distance of the centre of mass from BC, then $M\bar{y} = \int_0^c \rho xy\mathrm{d}y$

So given $x = \frac{(c-y)}{m}$

$$\frac{c^2\rho}{2m}\bar{y} = \int_0^c \rho xy\mathrm{d}y = \int_0^c \frac{\rho}{m}(c-y)y\mathrm{d}y = \int_0^c \frac{\rho}{m}(cy - y^2)\mathrm{d}y = \left[\frac{\rho}{m}\left(\frac{1}{2}cy^2 - \frac{1}{3}y^3\right)\right]_0^c$$

$$\Rightarrow \bar{y} = \frac{2}{c^2}\left(\frac{1}{2}c^3 - \frac{1}{3}c^3\right) = \frac{c}{3}$$

b Divide the lamina into two sections, a rectangle and a triangle. Let ρ be the mass per unit area of the lamina. Let the point P be the origin, so that PQ is the x-axis and SP is the y-axis, and the coordinates of G, the lamina's centre of mass be $(\bar{x}, \bar{y})$.

The mass of the rectangle is $2a^2\rho$ and its centre of mass is $\left(\frac{a}{2}, a\right)$.

The mass of the triangle is $a^2\rho$ and its centre of mass is $\left(\frac{4a}{3}, \frac{2a}{3}\right)$, using the result from part **a**.

Using $\sum m_i\mathbf{r}_i = \mathbf{r}\sum m_i$

$$2a^2\rho\begin{pmatrix}\frac{a}{2}\\ a\end{pmatrix} + a^2\rho\begin{pmatrix}\frac{4a}{3}\\ \frac{2a}{3}\end{pmatrix} = (2a^2\rho + a^2\rho)\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix}$$

$$\begin{pmatrix}\frac{7a}{3}\\ \frac{8a}{3}\end{pmatrix} = 3\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix}$$

The distance of the centre of mass G of the lamina from $PS = \bar{x} = \frac{7a}{9}$

c The distance of the centre of mass G of the lamina from $PS = \bar{y} = \frac{8a}{9}$

50 a Use $\bar{x} = \dfrac{\int y^2 x \, dx}{\int y^2 \, dx}$, and substitute $y = \dfrac{1}{2x^2}$

$$\bar{x} = \frac{\int_1^2 xy^2 dx}{\int_1^2 y^2 dx} = \frac{\int_1^2 x \times \frac{1}{4} x^{-4} dx}{\int_1^2 \frac{1}{4} x^{-4} dx} = \frac{\int_1^2 x^{-3} dx}{\int_1^2 x^{-4} dx}$$

$$= \frac{\left[-\frac{1}{2}x^{-2}\right]_1^2}{\left[-\frac{1}{3}x^{-3}\right]_1^2} = \frac{3\left(1^{-2} - 2^{-2}\right)}{2\left(1^{-3} - 2^{-3}\right)} = \frac{3}{2} \times \frac{3}{4} \times \frac{8}{7} = \frac{9}{7}$$

The centre of mass is $\frac{9}{7}$ m from the y-axis, hence $\left(\frac{9}{7} - 1\right) = \frac{2}{7}$ m from the larger plane face.

b Let ρ be the mass per unit volume of the hemisphere H and the solid S.

Then the mass of the S = $\int_1^2 \pi y^2 \rho dx = \int_1^2 \frac{\pi}{4} x^{-4} \rho dx = \left[-\frac{\pi\rho}{12} x^{-3}\right]_1^2 = \frac{\pi\rho}{12}\left(1 - \frac{1}{8}\right) = \frac{7\pi\rho}{96}$

The mass of the hemisphere and its distance of its centre of mass from its plane face can be found from the standard formulae, using $r = 0.5$.

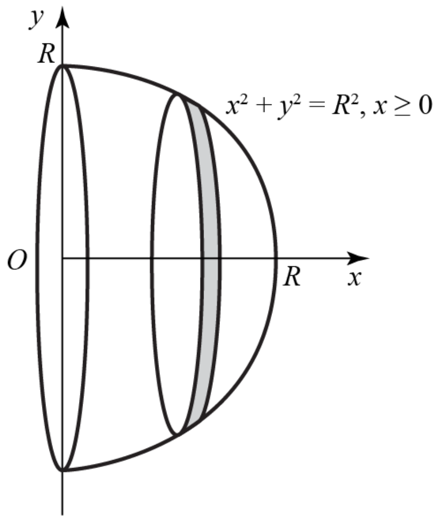
Shape	Mass	Distance of centre of mass from trophy's plane face
Solid, S	$\frac{7\pi\rho}{96}$	$1 - \frac{2}{7} = \frac{5}{7}$
Hemisphere, H	$\frac{2\pi\rho}{3}\left(\frac{1}{2}\right)^3 = \frac{\pi\rho}{12}$	$1 + \frac{3}{8} \times \frac{1}{2} = \frac{19}{16}$
Trophy, T	$\pi\rho\left(\frac{7}{96} + \frac{1}{12}\right) = \frac{5\pi\rho}{32}$	$\bar{x}$

Taking moments around the trophy's plane face (the bottom of the trophy as shown):

$$\frac{5\pi\rho}{32}\bar{x} = \frac{\pi\rho}{12} \times \frac{19}{16} + \frac{7\pi\rho}{96} \times \frac{5}{7}$$

$$\bar{x} = \frac{32}{5}\left(\frac{19}{192} + \frac{5}{96}\right) = \frac{32}{5} \times \frac{29}{192} = \frac{29}{30} = 0.967 \text{ m (3 s.f.)}$$

51 a A hemisphere is generated when a semicircle is notated through 180° about the x-axis.



Divide the hemisphere into circular discs, with each disc having mass $\rho\pi y^2 \delta x$ and centre of mass at a distance x from O.

So $\bar{x} = \dfrac{\int_0^R \rho\pi x y^2 \, dx}{\int_0^R \rho\pi y^2 \, dx} = \dfrac{\int_0^R x(R^2 - x^2)\, dx}{\int_0^R (R^2 - x^2)\, dx}$

$$= \frac{\left[\frac{1}{2}R^2x^2 - \frac{1}{4}x^4\right]_0^R}{\left[R^2x - \frac{1}{3}x^3\right]_0^R} = \frac{\frac{1}{2}R^4 - \frac{1}{4}R^4}{R^3 - \frac{1}{3}R^3} = \frac{3}{2} \times \frac{1}{4}R = \frac{3}{8}R$$

b Using the standard formulae for a cone:

Shape	Mass	Distance of centre of mass from V
Cone	$\frac{1}{3}\pi a^2 ka\rho$	$\frac{3}{4}ka$
Hemisphere	$\frac{2}{3}\pi a^3 \rho$	$ka + \frac{3}{8}a$
Top	$\frac{1}{3}\pi a^3 \rho(k+2)$	$\bar{x}$

Taking moments about V:

$$\frac{1}{3}\pi a^3 \rho(k+2)\bar{x} = \frac{1}{3}\pi a^3 \rho k\left(\frac{3}{4}ka\right) + \frac{2}{3}\pi a^3 \rho\left(ka + \frac{3a}{8}\right)$$

$$(k+2)\bar{x} = \frac{3}{4}k^2 a + 2ka + \frac{3a}{4}$$

$$\bar{x} = \frac{(3k^2 + 8k + 3)a}{4(k+2)}$$

51 c The manufacturer's requirement is that $\bar{x} = ka$

Hence from part **b** $\frac{3k^2+8k+3}{4(k+2)}=k$

$\Rightarrow 3k^2+8k+3=4k^2+8k$

$\Rightarrow k^2=3$, so $k=\sqrt{3}$

52 a Using the standard results for a hemisphere:

Shape	Mass	Ratio of masses	Distance of centre of mass from *O*
Large hemisphere	$\frac{2}{3}\pi a^3\rho$	8	$\frac{3}{8}a$
Small hemisphere	$\frac{2}{3}\pi\left(\frac{a}{2}\right)^3\rho$	1	$\frac{3}{16}a$
Remainder	$\frac{2}{3}\pi\frac{7a^3}{8}\rho$	7	$\bar{x}$

Taking moments about *O*:

$$7\bar{x}=8\times\frac{3}{8}a-1\times\frac{3}{16}a$$

$$\bar{x}=\frac{1}{7}\times\frac{45}{16}a=\frac{45a}{112}$$

b Using the result from part **a**:

	Mass ratios	Distance of centre of mass from *O*
Bowl	M	$\frac{45}{112}a$
Liquid	kM	$\frac{3}{16}a$
Bowl + liquid	$(k+1)M$	$\frac{17}{48}a$

Taking moments about *O*:

$$(k+1)M\times\frac{17}{48}a=M\times\frac{45}{112}a+kM\times\frac{3}{16}a$$

$$k\left(\frac{17}{48}-\frac{3}{16}\right)=\frac{45}{112}-\frac{17}{48}$$

$$\frac{8}{48}k=\frac{45}{112}-\frac{17}{48}$$

$$k=6\times\left(\frac{45}{112}-\frac{17}{48}\right)=\frac{6(2160-1906)}{5376}=\frac{254}{896}=\frac{2}{7}$$

53 a $V=\int_0^1 \pi y^2 \mathrm{d}x=\int_0^1 \frac{\pi}{4}(x-2)^4 \mathrm{d}x$ substituting $y=\frac{1}{2}(x-2)^2$

$\Rightarrow V=\left[\frac{\pi}{20}(x-2)^5\right]_0^2=-\frac{\pi}{20}(-2)^5=\frac{32\pi}{20}=\frac{8\pi}{5}\text{cm}^3$

b Let $\bar{x}$ be the distance of the centre of mass of S from its plane face

To find, using $M\bar{x}=\int \rho\pi y^2 x\,\mathrm{d}x$, and substituting $y^2=\frac{1}{4}(x-2)^2$ and $M=\frac{8\pi\rho}{5}$ from part **a** gives:

$\bar{x}=\frac{5}{32}\int_0^2 (x-2)^4 x\mathrm{d}x$

Integrate using the substitution $u=x-2$

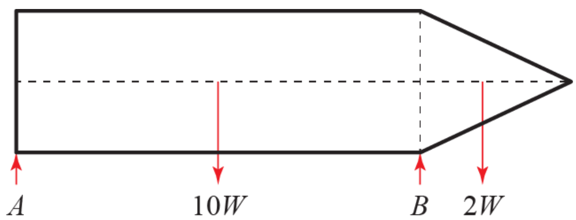
$\bar{x}=\frac{5}{32}\int_{-2}^0 u^4(u+2)\mathrm{d}u=\frac{5}{32}\int_{-2}^0 u^5+2u^4\mathrm{d}u$

$=\frac{5}{32}\left[\frac{1}{6}u^6+\frac{2}{5}u^5\right]_{-2}^0=\frac{5}{32}\left(\frac{2}{5}\times 2^5-\frac{1}{6}2^6\right)=\frac{5}{32}\left(\frac{64}{5}-\frac{64}{6}\right)$

$=10\left(\frac{1}{5}-\frac{1}{6}\right)=\frac{10}{30}=\frac{1}{3}$

The centre of mass lies on the axis of symmetry at a distance of $\frac{1}{3}$cm from the plane base.

c Let the reaction force at A be A, and at B be B.



Taking moments about B:

$A\times 8+2W\times\frac{1}{3}=10W\times 4$

$A=\frac{\left(40-\frac{2}{3}\right)W}{8}=\frac{118W}{24}=\frac{59W}{12}$

54 a Using the formulae for standard uniform bodies:

Shape	Mass	Mass ratios	Distance of centre of mass from base
Cylinder	$\pi r^2 h\rho$	1	$\frac{h}{2}$
Cone	$\frac{1}{3}\pi r^2 \frac{h}{2}\rho$	$\frac{1}{6}$	$h-\frac{1}{4}\left(\frac{h}{2}\right)=\frac{7h}{8}$
Ornament	$\frac{5}{6}\pi r^2 h\rho$	$\frac{5}{6}$	$\bar{x}$

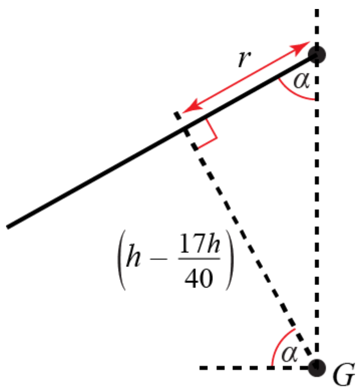
Take moments about O, the centre of plane base:

$$\frac{5}{6}\bar{x}=1\times\frac{h}{2}-\frac{1}{6}\times\frac{7h}{8}$$

$$\frac{5}{6}\bar{x}=\frac{h}{2}-\frac{7h}{48}=\frac{17h}{48}$$

$$\bar{x}=\frac{17h}{48}\times\frac{6}{5}=\frac{17h}{40}$$

b The centre of mass G of the ornament will be directly below the point of suspension.

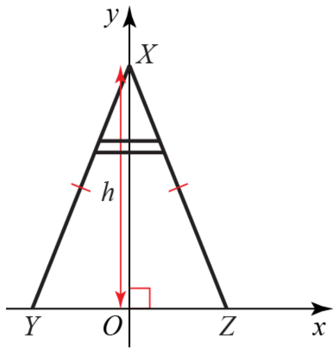


$$\tan\alpha=\frac{h-\frac{17h}{40}}{r}=\frac{23h}{40r}$$

As $h=4r$, this gives $\tan\alpha=\frac{23r}{10r}=2.3$

$\Rightarrow \alpha=66.5°$ (1 d.p.)

55 a Let O be the midpoint of YZ. Let OZ be the x-axis and OX the y-axis. The triangle is isosceles and the y-axis is the line of symmetry.



The equation of the line XZ is $y = h - mx \Rightarrow x = \dfrac{h-y}{m}$

The area of the elemental strip of width δy is $2x\delta y$

Let $\bar{y}$ be the distance of the centre of mass from O, then:

$$\bar{y} = \frac{\int_0^h 2yx\,\mathrm{d}y}{\int_0^h 2x\,\mathrm{d}y} = \frac{\int_0^h y\left(\frac{h-y}{m}\right)\mathrm{d}y}{\int_0^h \left(\frac{h-y}{m}\right)\mathrm{d}y} = \frac{\int_0^h yh - y^2\,\mathrm{d}y}{\int_0^h h - y\mathrm{d}y}$$

$$= \frac{\left[\frac{h}{2}y^2 - \frac{1}{3}y^3\right]_0^h}{\left[hy - \frac{1}{2}y^2\right]_0^h} = \frac{\frac{1}{6}h^3}{\frac{1}{2}h^2} = \frac{1}{3}h$$

So distance from $X = h - \dfrac{h}{3} = \dfrac{2h}{3}$

b Let E be the midpoint of BC, then BE = 4a and AE = 3a, as ΔABE is a 3 : 4 : 5 triangle, and $DE = a$, from part **a**.

Let ρ be the mass per unit area of the lamina.

Shape	Mass	Distance of centre of mass from *A*
ΔABC	$12\rho a^2$	$2a$
ΔBCD	$4\rho a^2$	$2a + \dfrac{2a}{3}$
Plate *ABDC*	$8\rho a^2$	$\bar{x}$

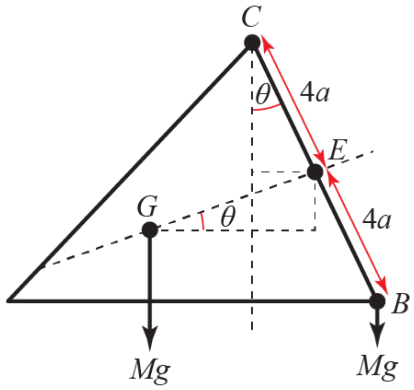
Taking moments about A:

$$8\rho a^2\bar{x} = 12\rho a^2 \times 2a - 4\rho a^2\left(\frac{8a}{3}\right)$$

$$= 24\rho a^2 - \frac{32\rho a^3}{3} = \frac{40\rho a^3}{3}$$

So $\bar{x} = \dfrac{40a}{8\times 3} = \dfrac{5a}{3}$

55 c Let θ be the angle between CB and the vertical.



From part **b**, the distance *GE* is $3a - \frac{5a}{3} = \frac{4a}{3}$

Taking moments about *C*:

$$Mg \times 8a\sin\theta = Mg\left(\frac{4a}{3}\cos\theta - 4a\sin\theta\right)$$

$$\Rightarrow 12Mga\sin\theta = \frac{4}{3}Mga\cos\theta$$

$$\Rightarrow \frac{\sin\theta}{\cos\theta} = \tan\theta = \frac{4}{36} = \frac{1}{9}$$

So $\theta = \arctan\frac{1}{9}$

So *CB* makes an angle $\arctan\left(\frac{1}{9}\right)$ with the vertical.

56 a Let ρ be the mass per unit area of the material and $\bar{x}$ the distance of the centre of mass of the closed container from *O*. Using the formulae for standard uniform bodies:

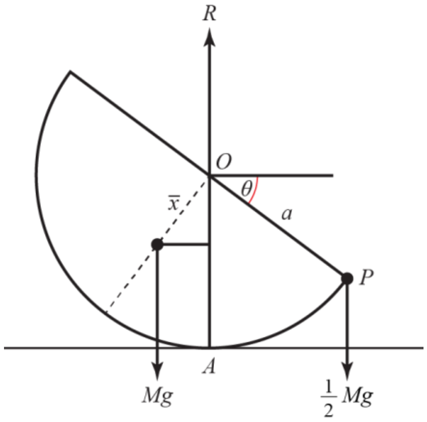
Shape	Mass	Distance of centre of mass from *O*
Circular disc	$\pi a^2 \rho$	0
Hemispherical bowl	$2\pi a^2 \rho$	$\frac{1}{2}a$
Closed container	$3\pi a^2 \rho$	$\bar{x}$

Taking moments about *O*:

$$3\pi a^2 \rho \bar{x} = 0 + 2\pi a^2 \rho \times \frac{a}{2}$$

$$\bar{x} = \frac{a}{3}$$

56 b The container is resting in equilibrium, so the weight of P acts to one side and the weight of C balances on the other side.



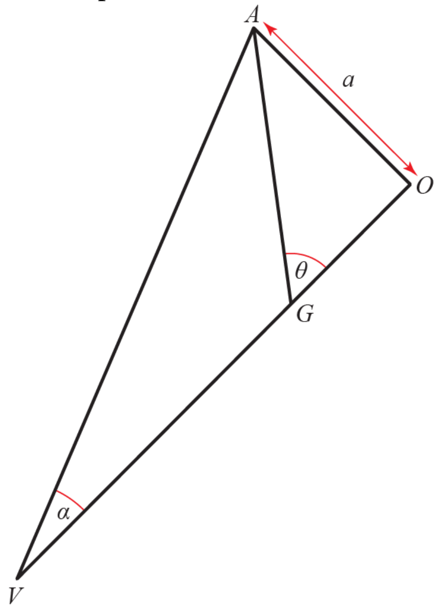
Taking moments about O:

$$Mg \times \frac{a}{3}\sin\theta = \frac{1}{2}Mg \times a\cos\theta$$

So $\tan\theta = \frac{3}{2}$

$\Rightarrow \theta = 56°$ (to the nearest degree)

57 Let V be the vertex of the cone and O be the centre of its base.
Let G be the position of its centre of mass.



From ΔVAO, $\tan\alpha = \frac{OA}{OV} = \frac{a}{h}$, where h is the height of the cone.

So $\tan\alpha = \frac{1}{3} = \frac{a}{h} \Rightarrow h = 3a$

So from the standard result for a uniform solid right circular cone $OG = \frac{1}{4}h = \frac{3a}{4}$

Then from ΔGAO

$$\tan\theta = \frac{AO}{GO} = \frac{a}{\frac{3a}{4}} = \frac{4}{3}$$

$\Rightarrow \theta = 53.1°$ (1 d.p.)

58 a The hemisphere K has mass M. The hemisphere H has double the radius, so it has 8 times the volume. Its mass is $8M$. Therefore the composite body S has mass $M + 8M = 9M$.

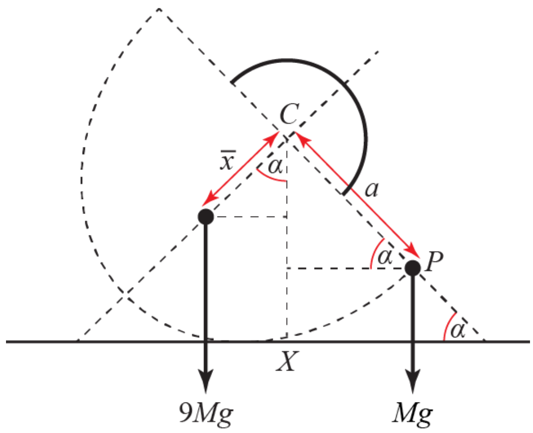
Shape	Mass	Distance of centre of mass from C
H	$8M$	$\frac{3a}{8}$
K	M	$\frac{-3a}{16}$
S	$M + 8M$	$\bar{x}$

Taking moments about C:

$$9M\bar{x} = 8M \times \frac{3a}{8} - M \times \frac{3a}{16} = M \times \frac{45a}{16}$$

$$\bar{x} = \frac{45a}{9\times 16} = \frac{5a}{16}$$

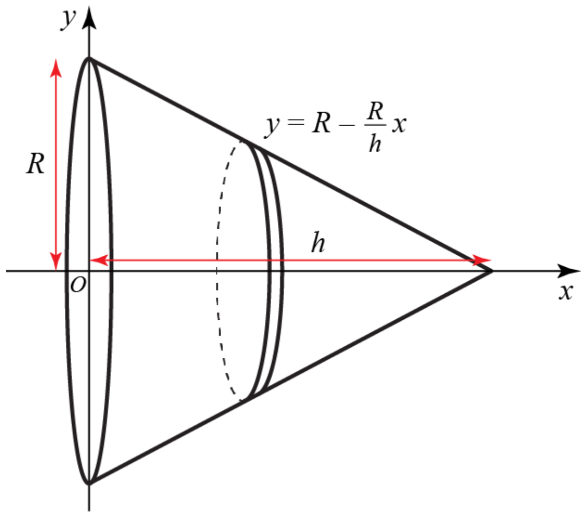
b The composite body with particle P attached rests in equilibrium, with C above the point of contact with the plane, X.



$Mg \times a\cos\alpha = 9Mg\bar{x}\sin\alpha = 9 \times \frac{5}{16}Mg\sin\alpha$ substituting result for $\bar{x}$ from part **a**

$$\Rightarrow \tan\alpha = \frac{16}{45}$$

59 a A line $y = R - \frac{R}{h}x$ rotated around the x-axis with generate a solid cone with its base on the y-axis.



The volume of an elemental strip of width δx is $\pi x^2\,\delta x$.
Let $\bar{x}$ be the distance of the centre of mass of the solid come from its base, then:

$$\bar{x} = \frac{\int_0^h \pi x y^2 \mathrm{d}x}{\int_0^h \pi y^2 \mathrm{d}x} = \frac{\pi\int_0^h x\left(R - \frac{R}{h}x\right)^2 \mathrm{d}x}{\pi\int_0^h \left(R - \frac{R}{h}x\right)^2 \mathrm{d}x} = \frac{\int_0^h R^2x - 2\frac{R^2}{h}x^2 + \frac{R^2}{h^2}x^3\,\mathrm{d}x}{\int_0^h R^2 - 2\frac{R^2}{h}x + \frac{R^2}{h^2}x^2\mathrm{d}x}$$

$$= \frac{\left[\frac{R^2x^2}{2} - \frac{2R^2x^3}{3h} + \frac{R^2x^4}{4h^2}\right]_0^h}{\left[R^2x - \frac{R^2x^2}{h} + \frac{R^2x^3}{3h^2}\right]_0^h} = \frac{R^2h^2\left(\frac{1}{2} - \frac{2}{3} + \frac{1}{4}\right)}{R^2h\left(1 - 1 + \frac{1}{3}\right)}$$

$$= 3h \times \frac{1}{12} = \frac{h}{4}$$

59 b Using the result from part **a**:

Shape	Mass	Mass ratio	Distance from base of centre of mass
Large cone	$\frac{1}{3}\pi a^2 H\rho$	H	$\frac{H}{4}$
Small cone	$\frac{1}{3}\pi a^2 h\rho$	h	$\frac{h}{4}$
Solid S	$\frac{1}{3}\pi a^2\rho(H-h)$	$H-h$	$\bar{x}$

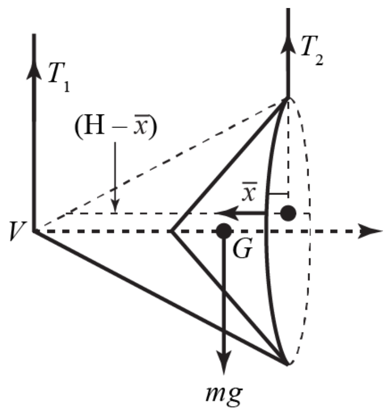
Taking moments about the base:

$$(H-h)\bar{x} = H\times\frac{H}{4} - h\times\frac{h}{4}$$

$$\bar{x} = \frac{1}{4}\frac{(H^2-h^2)}{(H-h)} = \frac{1}{4}\frac{(H-h)(H+h)}{(H-h)} = \frac{1}{4}(H+h)$$

So the distance from the vertex is $H-\frac{1}{4}(H+h) = \frac{3}{4}H - \frac{1}{4}h = \frac{1}{4}(3H-h)$ as required

c Let the tension in the string attached to the vertex be T_1 and tension in the other string be T_2.



Taking moments about the centre of mass G:

$T_2\bar{x} = T_1(H-\bar{x})$ where $H-\bar{x} = \frac{1}{4}(3H-h)$ from part **b**

$$\Rightarrow \frac{T_1}{T_2} = \frac{\bar{x}}{H-\bar{x}} = \frac{\frac{1}{4}(H+h)}{\frac{1}{4}(3H-h)} = \frac{H+h}{3H-h}$$

60 a Let ρ be the mass per unit volume of the material of the cylinder and $\bar{x}$ the distance of the centre of mass of the toy from O. Using the formulae for standard uniform bodies:

Shape	Mass	Mass ratio	Distance of centre of mass from O
Hemisphere	$\frac{2}{3}\pi r^3 6\rho$	$4r$	$\frac{5r}{8}$
Cylinder	$\pi r^2 h\rho$	h	$\frac{h}{2}+r$
Toy	$\pi r^2\rho(4r+h)$	$4r+h$	$\bar{x}$

Taking moments about O:

$$(4r+h)\bar{x} = 4r\times\frac{5r}{8}+h\left(\frac{h}{2}+r\right)$$

$$\bar{x} = \frac{5r^2+h^2+2rh}{2(4r+h)} = \frac{h^2+2hr+5r^2}{2(h+4r)} \qquad \text{as required}$$

b If the toy remains in equilibrium when resting on any point of the curved surface, its centre of mass must be in the centre of the hemisphere's flat surface, so $\bar{x} = r$.

Hence from part **a** $\frac{h^2+2hr+5r^2}{2(h+4r)} = r$

$\Rightarrow h^2+2hr+5r^2 = 2rh+8r^2$

$\Rightarrow h^2 = 3r^2$

$\Rightarrow h = \sqrt{3}r$

61 a Let $\bar{x}$ be distance of the centre of mass from AB on the axis of symmetry in the direction away from O. Using the formulae for standard uniform bodies:

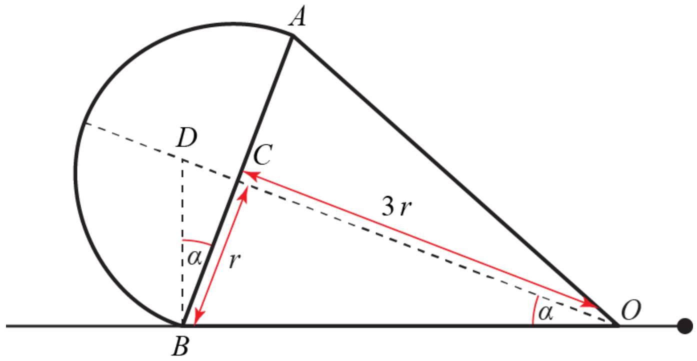
Shape	Mass	Distance of centre of mass from AB
Hemisphere	M	$\frac{3}{8}r$
Cone	m	$-\frac{3}{4}r$
Toy	$m+M$	$\bar{x}$

Taking moments about AB:

$$(m+M)\bar{x} = \frac{3}{8}Mr-\frac{3}{4}mr = \frac{3r}{8}(M-2m)$$

$$\bar{x} = \frac{3(M-2m)}{8(M+m)}r \qquad \text{as required}$$

61 b Let D be the point on the axis of symmetry vertically above B when the toy is placed on a horizontal surface. The toy will not remain in equilibrium if its centre of mass is not on the line segment OD, i.e if $\bar{x} > CD$.



From the diagram: $\tan\alpha = \frac{r}{3r} = \frac{CD}{r} \Rightarrow CD = \frac{1}{3}r$

So $\bar{x} > CD \Rightarrow \frac{3(M-2m)}{8(M+m)}r > \frac{1}{3}r$

$\Rightarrow 9(M-2m) > 8(M+m)$

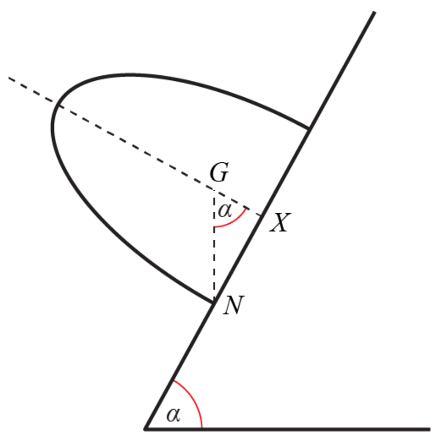
$\Rightarrow M > 26m$ as required

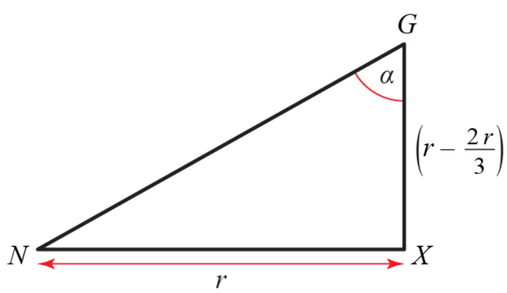
62 a The centre of mass lies on the axis of symmetry OX. Let the distance of the centre of mass of S from O be $\bar{x}$ and let it be at point G.

Using $\bar{x} = \frac{\int \pi y^2 x \,dx}{\int \pi y^2 \,dx}$ and substituting $y^2 = rx$ gives:

$$\bar{x} = \frac{\int_0^y rx^2 \,dx}{\int_0^y rx \,dx} = \frac{\left[\frac{1}{3}rx^3\right]_0^y}{\left[\frac{1}{2}rx^2\right]_0^y} = \frac{1}{3}r^4 \div \frac{1}{2}r^3 = \frac{2}{3}r$$

b The solid will not topple providing G is vertically above a point on the plane face in contact with the inclined plane. The maximum value of α is when G is directly above N, the edge of the solid.





$\tan\alpha = \frac{r}{r - \frac{2}{3r}} = r \div \frac{1}{3}r = 3$

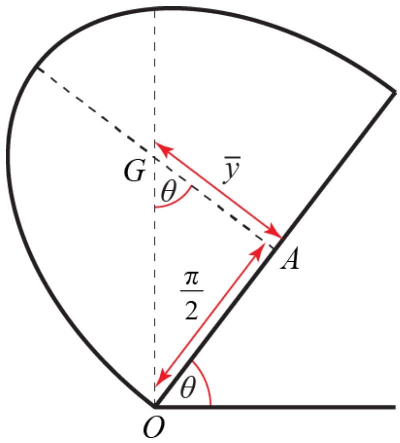
$\Rightarrow \alpha = 72°$ (to the nearest degree)

63 a Using $\bar{y} = \frac{\int \frac{1}{2} y^2 \, dx}{\int y \, dx}$ with $y = \sin x$ gives:

$$\bar{y} = \frac{\int_0^\pi \frac{1}{2}\sin^2 x \, dx}{\int_0^\pi \sin x \, dx} = \frac{\frac{1}{4}\int_0^\pi 1 - \cos 2x \, dx}{\int_0^\pi \sin x \, dx}$$

$$= \frac{1}{4}\frac{\left[x - \frac{1}{2}\sin 2x\right]_0^\pi}{\left[-\cos x\right]_0^\pi} = \frac{1}{4}\frac{\pi}{(1+1)} = \frac{\pi}{8}$$

b When S is on the point of toppling, G is above O. Let A be the point midway along the base of S.



$$\tan\theta = \frac{\frac{\pi}{2}}{\bar{y}} = \frac{\frac{\pi}{2}}{\frac{\pi}{8}} = 4$$

$\Rightarrow \theta = 76°$ (to the nearest degree)

64 a Let ρ be the mass per unit volume of the material and $\bar{x}$ be the distance of the centre of mass of the solid S from O. Using the formulae for standard uniform bodies:

Shape	Mass	Mass ratios	Distance of centre of mass from O
Cylinder	$\pi\rho(2a)^2\left(\frac{3}{2}a\right)$	6	$\frac{3}{4}a$
Hemisphere	$\frac{2}{3}\pi\rho a^3$	$\frac{2}{3}$	$\frac{3}{8}a$
Solid, S	$\pi\rho\left(6a^3-\frac{2}{3}a^3\right)$	$\frac{16}{3}$	$\bar{x}$

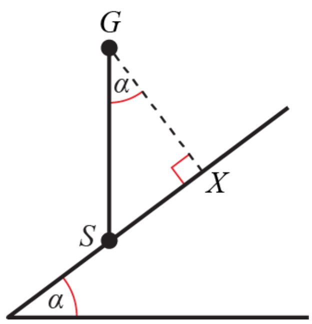
Taking moments about O:

$$\frac{16}{3}\bar{x}=6\times\frac{3}{4}a-\frac{2}{3}\times\frac{3}{8}a$$

$$=\frac{9}{2}a-\frac{1}{4}a=\frac{17}{4}a$$

$$\bar{x}=\frac{51a}{64}=0.797a\text{ (3 s.f.)}$$

b On the point of toppling: G is above S – the lowest point on the bottom circular face.

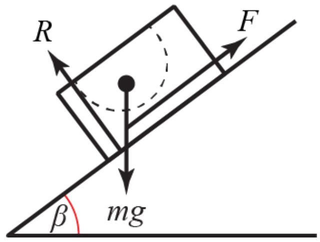


Let X be the centre of the base of the cylinder.

$$\tan\alpha=\frac{SX}{XG}=\frac{2a}{\frac{3}{2}a-\bar{x}}=\frac{64\times 2}{96-51}=\frac{128}{45}$$ substituting value for $\bar{x}$ from part **a**

$\Rightarrow \alpha=70.6°$ (3 s.f.)

64 c There are three forces acting on *S*, its weight, the reaction force and friction.



When S is on the point of sliding $F = \mu R = 0.8R$

Resolving perpendicular to the plane

R(↖): $R - mg\cos\beta = 0 \Rightarrow R = mg\cos\beta$

Resolving along the plane

R(↗): $F - mg\sin\beta = 0 \Rightarrow F = mg\sin\beta$

Using $F = 0.8R$ gives

$mg\sin\beta = 0.8mg\cos\beta \Rightarrow \tan\beta = 0.8$

So $\beta = 38.7°$ (3 s.f.)

65 a Let ρ be the mass per unit volume of the material and $\bar{x}$ be distance of the centre of mass of the bowl *B* from *O*. Using the formulae for standard uniform bodies:

Shape	Mass	Mass ratio	Distance of centre of mass from *O*
Large hemisphere	$\frac{2}{3}\pi\rho(6a)^3 = 144\pi\rho a^3$	27	$\frac{3}{8}\times 6a$
Small hemisphere	$\frac{2}{3}\pi\rho(2a)^3 = \frac{16}{3}\pi\rho a^3$	1	$\frac{3}{8}\times 2a$
Bowl *B*	$\frac{2}{3}\pi\rho(6^3 - 2^3)a^3 = \frac{416}{3}\pi\rho a^3$	26	$\bar{x}$

Taking moments about *O*:

$$26\bar{x} = 27\times\frac{3}{8}\times 6a - 1\times\frac{3}{8}\times 2a$$

$$= \frac{243a}{4} - \frac{3a}{4} = 60a$$

$$\bar{x} = \frac{30a}{13}$$

65 b Let $\bar{y}$ be distance of the centre of mass of the bowl *B* from *O*.

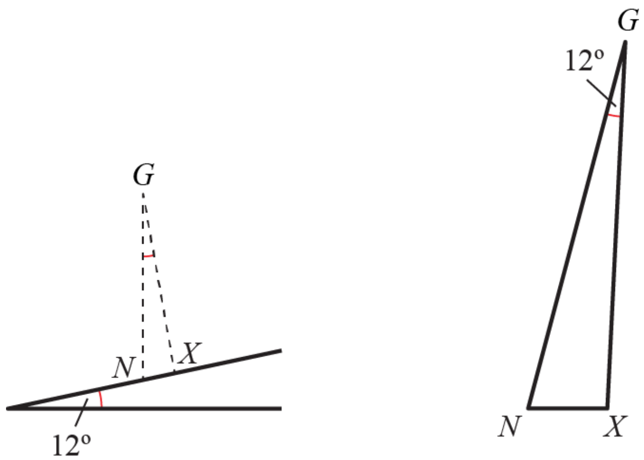
Shape	Mass	Mass ratio	Distance of centre of mass from *O*
Bowl, *B*	$\frac{416}{3}\pi\rho a^3$	52	$\frac{30a}{13}$
Cylinder	$24\pi\rho a^3$	9	$6a + 3a$
Combined solid, *S*	$\frac{488}{3}\pi\rho a^3$	61	$\bar{y}$

Taking moments about *O*:

$$61\bar{y} = 52 \times \frac{30a}{13} + 9 \times 9a = 120a + 81a$$

$$\bar{y} = \frac{201}{61}a$$

c Let *N* be the point on the plane vertically below the centre of mass of the combined solid *G*, and let *X* be the centre of the cylindrical base of *S*.



$$NX = GX\tan 12° = \left(12a - \frac{201}{61}a\right)\tan 12° = \frac{531}{61}a\tan 12° = 1.85a \text{ (3 s.f.)}$$

The solid will topple when point *N* is outside the base circle; as *NX* < 2a, *S* will not topple.

66 a Let ρ be the mass per unit volume of the material and $\bar{x}$ be the distance of the centre of mass of the solid from O. Using the formulae for standard uniform bodies:

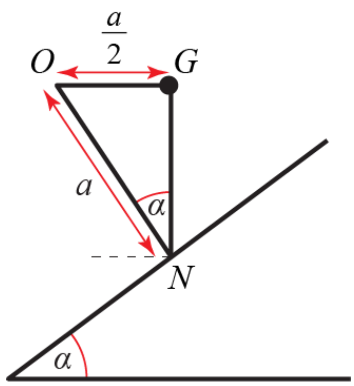
Shape	Mass	Mass ratio	Distance from *O* of centre of mass
Hemisphere	$\frac{2}{3}\pi\rho a^3$	2	$\frac{3a}{8}$
Cone	$\frac{1}{3}\pi\rho a^3$	1	$\frac{a}{4}$
Solid	$\frac{1}{3}\pi\rho a^3$	1	$\bar{x}$

Taking moments about O:

$$1\times\bar{x} = 2\times\frac{3}{8}a - 1\times\frac{a}{4}$$

$$\bar{x} = \frac{a}{2}$$

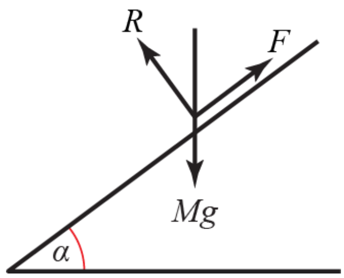
b As the solid is resting in equilibrium, its centre of mass G is above the point of contact N between the solid and the plane.



From ΔOGN, and using the result from part **a**:

$$\sin\alpha = \frac{\frac{a}{2}}{a} = \frac{1}{2} \Rightarrow \alpha = \frac{\pi}{6}$$

66 c There are three forces acting on the solid, its weight, the reaction force and friction.
For limiting equilibrium, when the solid is about to slip $F = \mu R$.



Resolving the forces perpendicular and parallel to the inclined plane:

R(↗): $F = mg\sin\alpha$

R(↖)$R = mg\cos\alpha$

As $F \le \mu R$, $mg\sin\alpha \le \mu mg\cos\alpha$

$\Rightarrow \mu \geqslant \dfrac{\sin\alpha}{\cos\alpha} \Rightarrow \mu \geqslant \tan\alpha$

$\alpha = \dfrac{\pi}{6} \Rightarrow \tan\alpha = \dfrac{1}{\sqrt{3}}$

So $\mu \geqslant \dfrac{1}{\sqrt{3}}$, hence $\dfrac{1}{\sqrt{3}}$ is the smallest value of μ

67 a Let ρ be the mass per unit volume of the material and $\bar{x}$ be distance of the centre of mass of the bollard from O. Using the formulae for standard uniform bodies:

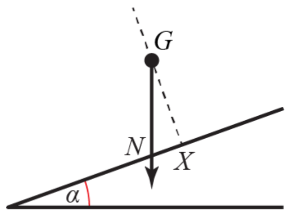
Shape	Mass	Mass ratio	Distance from *O* of centre of mass
Cone	$\frac{1}{3}\pi\rho(3r)^2h$	$3h$	$\frac{3}{4}h$
Cylinder	$\pi\rho(4r)^2r$	$16r$	$h+\frac{r}{2}$
Bollard	$\pi\rho(16r+3h)r^2$	$16r+3h$	$\bar{x}$

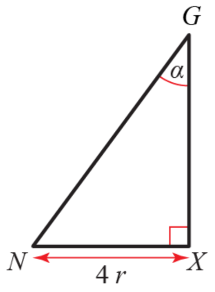
Taking moments about O:

$$(16r+3h)\bar{x} = 3h\times\frac{3h}{4} + 16r\left(h+\frac{r}{2}\right) = \frac{9h^2}{4} + 16rh + 8r^2$$

$$\bar{x} = \frac{32r^2+64rh+9h^2}{4(16r+3h)}$$

67 b When the bollard is on the point of toppling, its centre of mass G lies above N a point on the plane which is at the lowest point on its base.





Let X be the centre point of the base, so $GX = OX - \bar{x}$

As $h = 4r, OX = h + r = 5r$

And from part **a**: $\bar{x} = \dfrac{32r^2 + 256r^2 + 144r^2}{4(28r)} = \dfrac{432r}{112} = \dfrac{27r}{7}$

So $GX = 5r - \dfrac{27r}{7} = \dfrac{8r}{7}$

From ΔGNX: $\tan\alpha = \dfrac{NX}{GX} = \dfrac{4r}{\frac{8r}{7}} = \dfrac{28}{8} = 3.5$

$\Rightarrow \alpha = 74°$ (to the nearest degree)

68 a Let ρ be the mass per unit volume of the material and $\bar{x}$ be the distance of the centre of mass of the tree from O. Using the formulae for standard uniform bodies:

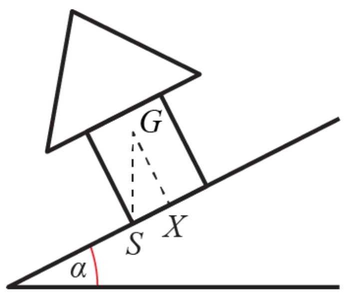
Shape	**Mass**	**Mass ratio**	**Position of centre of mass from *O***
Cylinder	$\rho\pi r^2 h$	3	$\dfrac{h}{2}$
Cone	$\dfrac{1}{3}\rho\pi(2r)^2 h$	4	$-\dfrac{h}{4}$
Tree	$\rho\pi r^2 h\left(1+\dfrac{4}{3}\right)$	7	$\bar{x}$

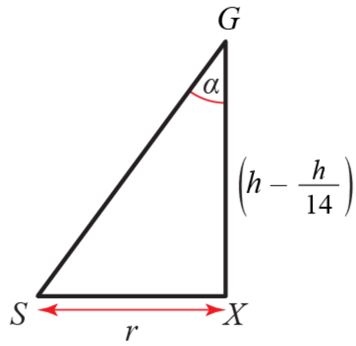
Taking moments about O:

$7\bar{x} = 3\times\dfrac{h}{2} - 4\times\dfrac{h}{4} = \dfrac{h}{2}$

$\bar{x} = \dfrac{h}{14}$

68 b When the tree is on the point of toppling, its centre of mass G lies above S, a point on the desk which is at the lowest point on its cylindrical base. GX can be found using the result in part **a**.





$$\tan\alpha = \frac{r}{h - \frac{h}{14}} = \frac{7}{26}$$

$$\Rightarrow r = \frac{7}{26}\left(\frac{13h}{14}\right) = \frac{1}{4}h$$

69 a Using the formulae for standard uniform bodies:

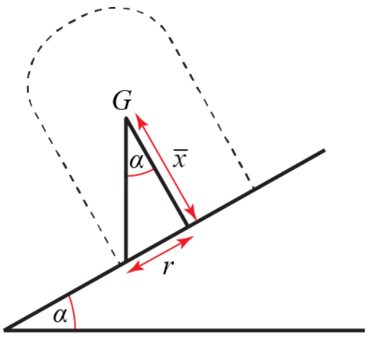
Shape	Mass	Distance of centre of mass from O
Hemisphere, H	$2M$	$h + \frac{3}{8}r$
Cylinder, C	$3M$	$\frac{h}{2}$
Body	$5M$	$\bar{x}$

Taking moments about O:

$$5M\bar{x} = 2M\left(h + \frac{3}{8}r\right) + 3M \times \frac{h}{2} = 2h + \frac{3}{4}r + \frac{3h}{2} = \frac{7h}{2} + \frac{3r}{4}$$

$$\bar{x} = \frac{14h + 3r}{20}$$

69 b When the body is on the point of toppling, its centre of mass G lies directly above a point on the plane which is at the lowest point on its cylindrical base.



From the diagram: $\tan\alpha = \frac{r}{\bar{x}} = \frac{20r}{14h+3r}$

As $\tan\alpha = \frac{4}{3}$, this gives $\frac{20r}{14h+3r} = \frac{4}{3}$

$\Rightarrow 60r = 56h + 12r$

$\Rightarrow 48r = 56h$

$\Rightarrow h = \frac{48}{56}r = \frac{6}{7}r$

70 a Let ρ be the mass per unit volume of the material in the cylinder and $\bar{x}$ be the distance of the centre of mass of the top from O. Using the formulae for standard uniform bodies:

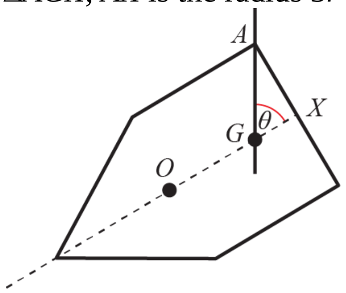
Shape	Mass	Mass ratio	Position of centre of mass from O
Cylinder	$36\pi\rho r^3$	1	$2r$
Cone	$36\pi\rho r^3$	1	$-r$
Toy	$72\pi\rho r^3$	2	$\bar{x}$

Taking moments about O:

$2\bar{x} = 1\times 2r + 1\times(-r) = r$

$\bar{x} = \frac{r}{2}$

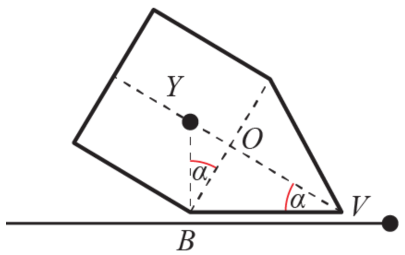
b In ΔAGX, AX is the radius $3r$ of the cylinder and X is the centre of the base of the cylinder.

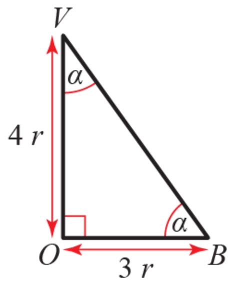


$\tan\theta = \frac{AX}{GX} = \frac{AX}{OX - OG} = \frac{3r}{4r - \bar{x}} = \frac{3r}{4r - \frac{r}{2}} = \frac{6}{7}$

$\Rightarrow \theta = 47.5°$ (1 d.p.)

70 c The toy will not topple if G (the centre of mass) is vertically above a point between B and V.
Let Y be the point directly above B on the axis of symmetry of the toy.





$\tan\alpha = \dfrac{3r}{4r}$ from ΔVOB and $\tan\alpha = \dfrac{OY}{OB} = \dfrac{OY}{3r}$ from ΔBOY

So $\dfrac{OY}{3r} = \dfrac{3}{4} \Rightarrow OY = \dfrac{9r}{4}$

But $OG = \dfrac{r}{2}$ so $OG < OY$

This means that the centre of mass is above the face of the body in contact with the place, so the toy will not topple.

71 a Total mass $= \displaystyle\int_0^{1.5} \frac{3}{(1+h)(2+h)}\,\mathrm{d}h$

Rewrite $\dfrac{3}{(1+h)(2+h)}$ as $\left(\dfrac{1}{1+h} - \dfrac{1}{2+h}\right)$ so:

$$m = 3\int_0^{1.5}\left(\frac{1}{1+h} - \frac{1}{2+h}\right)\mathrm{d}h = 3\left[\ln(1+h) - \ln(2+h)\right]_0^{1.5}$$

$$= 3(\ln 2.5 - \ln 3.5 + \ln 2) = 3\ln\frac{5}{3.5} = 1.07\,\text{kg (2 d.p.)}$$

b Let $\bar{h}$ be the distance of the centre of mass of the rod from its base.
Taking moments about the base of the rod:

$$3\ln\frac{5}{3.5}\,\bar{h} = \int_0^{1.5}\frac{3h}{(1+h)(2+h)}\,\mathrm{d}h = 3\int_0^{1.5}\left(\frac{2}{2+h} - \frac{1}{1+h}\right)\mathrm{d}h$$

$$= 3\left[2\ln(2+h) - \ln(1+h)\right]_0^{1.5} = 3\left[\ln\frac{(2+h)^2}{1+h}\right]_0^{1.5}$$

$$= 3(\ln 4.9 - \ln 4) = 3\ln\frac{49}{40}$$

So $\bar{h} = \dfrac{\ln\frac{49}{40}}{\ln\frac{5}{3.5}} = 0.57\,\text{m (2 d.p.)}$

72 a Total mass $m = \int_0^{2.4} \frac{5}{1+4x^2}\,dx = 5\int_0^{2.4} \frac{1}{1+4x^2}\,dx = 5\left[\frac{1}{2}\arctan 2x\right]_0^{2.4}$

So $m = 2.5\arctan 4.8 = 3.41\,\text{kg}$ (2 d.p.)

b Let $\bar{x}$ be the distance of the centre of mass of the oar from its end.
Taking moments about the end of the oar:

$$2.5\arctan 4.8\,\bar{x} = \int_0^{2.4} \frac{5x}{1+4x^2}\,dx = \left[\frac{5}{8}\ln(1+4x^2)\right]_0^{2.4} = \frac{5}{8}\ln 24.04$$

So $\bar{x} = \frac{2}{8}\frac{\ln 24.04}{\arctan 4.8} = 0.58\,\text{m}$ (2 d.p.)

73 a Since the mass per unit length of the rod increases as the distance from A increases, the centre of mass of the rod will be closer to B than to A.

b Total mass $m = \int_0^{20} (10+kx)\,dx = \left[10x + \frac{k}{2}x^2\right]_0^{20} = 200 + 200k$

As m = 750, this gives $200 + 200k = 750$

$\Rightarrow k = \frac{11}{4}$

c Let $\bar{x}$ be the distance of the centre of mass of the oar from A.
Taking moments about A:

$$750\bar{x} = \int_0^{20}\left(10 + \frac{11}{4}x\right)x\,dx = \left[5x^2 + \frac{11}{12}x^3\right]_0^{20} = 2000 + \frac{22\,000}{3} = \frac{28\,000}{3}$$

$$\bar{x} = \frac{28\,000}{750\times 3} = \frac{112}{3\times 3} = \frac{112}{9}\,\text{m}$$

74 Let the length of the rod be L and its mass be M.
Let $\bar{x}$ be the distance of the centre of mass of the oar from A.

$$M = \int_0^L (8+x^2)\mathrm{d}x = \left[8x + \frac{x^3}{3}\right]_0^L = 8L + \frac{L^3}{3}$$

$$M\bar{x} = \int_0^L (8+x^2)x\mathrm{d}x = \left[4x^2 + \frac{x^4}{4}\right]_0^L = 4L^2 + \frac{L^4}{4}$$

If the tension in the string at A is T then the tension in the string at B is $2T$.

$$\mathrm{R}(\uparrow):\ T + 2T = Mg = \left(8L + \frac{L^3}{3}\right)g$$

$$\Rightarrow T = \left(\frac{8L}{3} + \frac{L^3}{9}\right)g$$

Taking moments about A:

$$Mg\bar{x} = 2LT$$

$$\Rightarrow \left(4L^2 + \frac{L^4}{4}\right)g = 2LT$$

$$\Rightarrow T = \left(2L + \frac{L^3}{8}\right)g$$

Equating the two expressions for T gives:

$$\left(\frac{8L}{3} + \frac{L^3}{9}\right)g = \left(2L + \frac{L^3}{8}\right)g$$

$$\Rightarrow \frac{2}{3} = \frac{L^2}{72} \Rightarrow L^2 = 48$$

$$\Rightarrow L = \sqrt{48} = 4\sqrt{3}\ \mathrm{m}$$

Challenge

1 The two particles have equal but opposite velocities just before collision at point A. Let their velocities at point A be u and –u respectively.

After collision, P_1 travels three times the distance travelled by P_2 before colliding at point B. Therefore, the velocities after collision are $-3v$ and v respectively.

Using conservation of momentum:

$$mu + 2m(-u) = m(-3v) + 2mv$$

$$\Rightarrow u = v$$

Coefficient of restitution: $e = \dfrac{u-(-u)}{u-(-3u)} = \dfrac{1}{2}$

2 a Let the instantaneous speed at any point be w and the normal reaction between the ball and the ring is N. Then resolving towards the centre of the circle:

$N = \dfrac{mw^2}{R}$ using Newton's second law

So the frictional force opposite to the direction of motion is given by:

$$F = \mu \frac{mw^2}{R}$$

Resolving in the direction of motion:

$$m\frac{\mathrm{d}w}{\mathrm{d}t} = -\mu \frac{mw^2}{R}$$

Let v be the velocity at time t, so integrating:

$$\int_u^v \frac{1}{w^2}\,\mathrm{d}w = -\frac{\mu}{R}\int_0^t \mathrm{d}t$$

$$\Rightarrow \left[-\frac{1}{w}\right]_u^v = -\frac{\mu}{R}[t]_0^t$$

$$\Rightarrow -\frac{1}{v} + \frac{1}{u} = -\frac{\mu}{R}t$$

$$\Rightarrow \frac{1}{v} = \frac{1}{u} + \frac{\mu}{R}t = \frac{R + u\mu t}{uR}$$

$$\Rightarrow v = \frac{uR}{R + u\mu t}$$

Challenge

2 b From part a $v = \frac{\mathrm{d}x}{\mathrm{d}t} = \frac{uR}{R + u\mu t}$

The ball completes one revolution in time t when $x = 2\pi R = \pi$

$$\int_0^{\pi} 1\mathrm{d}x = \int_0^t \frac{uR}{R + u\mu t}\mathrm{d}t == \int_0^t \frac{20}{0.5 + 10t}\mathrm{d}t$$

$$\Rightarrow \pi = \left[2\ln(0.5 + 10t)\right]_0^t = 2(\ln(0.5 + 10t) - \ln 0.5)$$

$$\Rightarrow \pi = 2\ln(1 + 20t)$$

$$\Rightarrow 20t = e^{\frac{\pi}{2}} - 1$$

$$\Rightarrow t = \frac{e^{\frac{\pi}{2}} - 1}{20} = 0.191\text{s (3 s.f.)}$$

3 a $\mathrm{R}(\uparrow):\ T_1 + T_2 + T_3 \sin 45° = 4Mg$

$\mathrm{R}(\rightarrow): T_3 \cos 45° = 0$

Therefore $T_3 = 0$ and:

$T_1 + T_2 = 4Mg$ (**1**)

Taking moments about B:

$5T_2 + 10T_1 = 20Mg$

$T_2 + 2T_1 = 4Mg$ (**2**)

Subtracting (**1**) from (**2**) gives

$T_1 = 0$ and $T_2 = 4Mg$

So $T_1 = T_3 = 0$ and $T_2 = 4Mg$

Challenge

3 b First find the centre of mass of the lamina. Let *A* be the origin.
Using geometry, the coordinates of the points on the lamina are:

$$A(0,0),\ B(10,0),\ C(5,0),\ D\left(\frac{15}{2},-5\sqrt{2}\right),\ E\left(\frac{5}{2},-5\sqrt{2}\right)$$

The *x*-coordinate of the centre of mass of the lamina is $\bar{x}=5$ by symmetry.

The lamina is composed of three equivalent triangles *ACE*, *CED*, *BCD*, each with height $5\sqrt{2}$ and triangle *CED* has twice the density as the other two triangles.

So the *y*-coordinate of the centre of mass of the lamina is given by:

$$4M\,\bar{y}=M\left(-\frac{5\sqrt{2}}{3}\right)+2M\left(-\frac{10\sqrt{2}}{3}\right)+M\left(-\frac{5\sqrt{2}}{3}\right)$$

$$\bar{y}=-\frac{30\sqrt{2}}{12}=-\frac{5\sqrt{2}}{2}$$

If a mass of 10M is attached to the lamina at B, while *AB* remains horizontal the position of the new centre of mass is:

$$14M\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix}=4M\begin{pmatrix}5\\ -\frac{5\sqrt{2}}{2}\end{pmatrix}+10M\begin{pmatrix}10\\ 0\end{pmatrix}$$

$$\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix}=\begin{pmatrix}\frac{60}{7}\\ -\frac{5\sqrt{2}}{7}\end{pmatrix}$$

The new centre of mass will align itself so that it is vertically below point A.

So angle made by *AB* with the vertical in this new position is $\theta=\arctan\left(\frac{5\sqrt{2}}{60}\right)=6.72°$ (3 s.f.)

Challenge

4 Consider a cylindrical disc at a distance x cm from the base of the hemisphere. The thickness of the disc is δx and the radius of the disc is $(r^2 - x^2)^{\frac{1}{2}}$ so that the volume of the disc is $\pi(r^2 - x^2)\delta x$.

Let m be the total mass of the hemisphere:

$$m = \int_0^r \pi(r^2 - x^2)(5x+2)\mathrm{d}x = \pi\left[2r^2x + \frac{5}{2}r^2x^2 - \frac{2}{3}x^3 - \frac{5}{4}x^4\right]_0^r$$

$$= \frac{4\pi}{3}r^3 + \frac{5\pi}{4}r^4$$

Let $\bar{x}$ be the distance of the centre of mass of the hemisphere from its plane face.
Taking moments about the face:

$$\left(\frac{4\pi}{3}r^3 + \frac{5\pi}{4}r^4\right)\bar{x} = \int_0^r \pi(r^2 - x^2)(5x+2)x\mathrm{d}x = \pi\left[r^2x^2 + \frac{5}{3}r^2x^3 - \frac{1}{2}x^4 - x^5\right]_0^r$$

$$= \frac{\pi}{2}r^4 + \frac{2\pi}{3}r^5$$

$$\bar{x} = \frac{\frac{r}{2} + \frac{2r^2}{3}}{\frac{4}{3} + \frac{5r}{4}} = \frac{6r + 8r^2}{16 + 15r}$$

When the hemisphere is on the point of tipping, $\tan(\arctan 2) = \frac{r}{\bar{x}}$

So $2\left(\frac{6r + 8r^2}{16 + 15r}\right) = r$

$\Rightarrow 12 + 16r = 16 + 15r$

$\Rightarrow r = 4\,\text{cm}$