Integration techniques 6A

1 a Integrating by parts $u = x^n$ and $\frac{dv}{dx} = e^{\frac{x}{2}}$

so $\frac{du}{dx} = nx^{n-1}, v = 2e^{\frac{x}{2}}$

$$\text{So } I_n = 2x^n e^{\frac{x}{2}} - \int 2nx^{n-1}e^{\frac{x}{2}}\, dx$$
$$= 2x^n e^{\frac{x}{2}} - 2n\int x^{n-1}e^{\frac{x}{2}}\, dx$$
$$= 2x^n e^{\frac{x}{2}} - 2n\, I_{n-1} \quad *$$

b $I_3 = 2x^3 e^{\frac{x}{2}} - 6I_2$

$$= 2x^3 e^{\frac{x}{2}} - 6\left(2x^2 e^{\frac{x}{2}} - 4I_1\right)$$

Substituting $n = 3, 2$ and 1 respectively in *

$$= 2x^3 e^{\frac{x}{2}} - 12x^2 e^{\frac{x}{2}} + 24\left(2xe^{\frac{x}{2}} - 2I_0\right), \text{where } I_0 = \int e^{\frac{x}{2}}\, dx = 2e^{\frac{x}{2}} + C$$
$$= 2x^3 e^{\frac{x}{2}} - 12x^2 e^{\frac{x}{2}} + 48xe^{\frac{x}{2}} - 48I_0$$
$$= \text{So } \int x^3 e^{\frac{x}{2}}\, dx = 2x^3 e^{\frac{x}{2}} - 12x^2 e^{\frac{x}{2}} + 48xe^{\frac{x}{2}} - 96e^{\frac{x}{2}} + C$$

2 a Let $u = (\ln x)^n$ and $\frac{dv}{dx} = x$, so $\frac{du}{dx} = n\frac{(\ln x)^{n-1}}{x}, v = \frac{x^2}{2}$

Integration by parts:

$$\int_1^e x(\ln x)^n\, dx = \left[\frac{x^2(\ln x)^n}{2}\right]_1^e - \int_1^e \frac{nx^2(\ln x)^{n-1}}{2x}\, dx$$
$$= \left[\frac{e^2}{2} - 0\right] - \frac{n}{2}\int_1^e x(\ln x)^{n-1}\, dx$$

$$\text{So } I_n = \frac{e^2}{2} - \frac{n}{2}I_{n-1} \quad *$$

2 b $\int_1^e x(\ln x)^4 \, dx = I_4$

Substituting $n = 4, 3, 2$ and 1 respectively in the reduction formula *

$$I_4 = \frac{e^2}{2} - \frac{4}{2} I_3$$
$$= \frac{e^2}{2} - 2\left(\frac{e^2}{2} - \frac{3}{2} I_2\right)$$

$$= \frac{e^2}{2} - e^2 + 3\left(\frac{e^2}{2} - \frac{2}{2} I_1\right)$$
$$= \frac{e^2}{2} - e^2 + \frac{3e^2}{2} - 3\left(\frac{e^2}{2} - \frac{1}{2} I_0\right), \text{where } I_0 = \int_1^e x dx = \left[\frac{x^2}{2}\right]_1^e = \frac{e^2}{2} - \frac{1}{2}$$

So $\int_1^e x(\ln x)^4 \, dx = \frac{e^2}{2} - e^2 + \frac{3e^2}{2} - \frac{3e^2}{2} + \frac{3}{2}\left(\frac{e^2}{2} - \frac{1}{2}\right)$

$$= \frac{e^2}{4} - \frac{3}{4} = \frac{e^2 - 3}{4}$$

3 $\int_0^1 \left[(x+1)(x+2)\sqrt{1-x}\right] dx = \int_0^1 \left[(x^2+3x+2)\sqrt{1-x}\right] dx$

$$= \int_0^1 \left[x^2\sqrt{1-x}\right] dx + \int_0^1 \left[3x\sqrt{1-x}\right] dx + \int_0^1 \left[2\sqrt{1-x}\right] dx$$
$$= I_2 + 3I_1 + 2I_0$$

Now $I_0 = \int_0^1 \sqrt{1-x} \, dx = \left[-\frac{2}{3}(1-x)^{\frac{3}{2}}\right]_0^1 = 0 - \left(-\frac{2}{3}\right) = \frac{2}{3}$

$I_1 = \frac{2}{5} I_0 = \left(\frac{2}{5}\right)\left(\frac{2}{3}\right) = \frac{4}{15}$ ← Using the given formula with $n = 1$

$I_2 = \frac{4}{7} I_1 = \left(\frac{4}{7}\right)\left(\frac{4}{15}\right) = \frac{16}{105}$ ← Using the given formula with $n = 2$

So $\int_0^1 \left[(x+1)(x+2)\sqrt{1-x}\right] dx = \frac{16}{105} + 3\left(\frac{4}{15}\right) + 2\left(\frac{2}{3}\right)$

$$= \frac{16 + 12(7) + 4(35)}{105}$$
$$= \frac{240}{105} = \frac{16}{7}$$

4 a Using integration by parts with $u = x^n$ and $\frac{dv}{dx} = e^{-x}$

so $\frac{du}{dx} = nx^{n-1}$ and $v = -e^{-x}$

$\int x^n e^{-x} \, dx = -x^n e^{-x} - \int -nx^{n-1} e^{-x} \, dx$, so $I_n = -x^n e^{-x} + nI_{n-1}$

4 b Repeatedly using the reduction formula to find I_3

$$I_3 = -x^3e^{-x} + 3I_2$$
$$= -x^3e^{-x} + 3\left(x^2e^{-x} + 2I_1\right)$$
$$= -x^3e^{-x} - 3x^2e^{-x} + 6I_1$$
$$= -x^3e^{-x} - 3x^2e^{-x} + 6\left(-xe^{-x} + I_0\right)$$

But $I_0 = \int e^{-x}\ dx = -e^{-x} + C$

So

$$I_3 = -x^3e^{-x} - 3x^2e^{-x} - 6xe^{-x} - 6e^{-x} + K$$
$$= -e^{-x}(x^3 + 3x^2 + 6x + 6) + K$$

c $I_4 = -x^4e^{-x} + 4I_3$

$$= -x^4e^{-x} + 4(-x^3e^{-x} - 3x^2e^{-x} - 6xe^{-x} - 6e^{-x} + C)$$

Using the result from **b**

So $\int_0^1 x^4e^{-x}\ dx = \left[-x^4e^{-x} - 4x^3e^{-x} - 12x^2e^{-x} - 24xe^{-x} - 24e^{-x}\right]_0^1$

$$= \left[-65e^{-1}\right] - \left[-24\right]$$
$$= 24 - 65e^{-1} \quad \text{or} \quad \frac{24e - 65}{e}$$

5 a $I_n = \int \tanh^n x\,dx = \int \tanh^{n-2} x \tanh^2 x\,dx$

$$= \int \tanh^{n-2} x\left(1 - \operatorname{sech}^2 x\right)dx$$

Using $1 - \tanh^2 x = \operatorname{sech}^2 x$

$$= \int \tanh^{n-2} x\,dx - \int \tanh^{n-2} \operatorname{sech}^2 x\,dx$$

So $I_n = I_{n-2} - \dfrac{1}{n-1}\tanh^{n-1} x,\quad n \neq 1$

b $\int \tanh^5 x\,dx = I_5 = I_3 - \dfrac{1}{4}\tanh^4 x$

$$= \left(I_1 - \frac{1}{2}\tanh^2 x\right) - \frac{1}{4}\tanh^4 x$$
$$= \int \tanh x\,dx - \frac{1}{2}\tanh^2 x - \frac{1}{4}\tanh^4 x$$
$$= \ln\cosh x - \frac{1}{2}\tanh^2 x - \frac{1}{4}\tanh^4 x + C$$

5 c As $\int \tanh^n x\,dx = \int \tanh^{n-2} x dx = \frac{1}{n-1}\tanh^{n-1} x$, it follows that

$$\int_0^{\ln 2} \tanh^n x\,dx = \int_0^{\ln 2} \tanh^{n-2} x\,dx - \left[\frac{1}{n-1}\tanh^{n-1} x\right]_0^{\ln 2} \quad *$$

Now $\tanh(\ln 2) = \frac{e^{\ln 2} - e^{-\ln 2}}{e^{\ln 2} + e^{-\ln 2}} = \frac{2-\frac{1}{2}}{2+\frac{1}{2}} = \frac{3}{5}$

Reminder: $e^{-\ln a} = e^{-\ln a^{-1}} = a^{-1}$

So $\int_0^{\ln 2} \tanh^4 x\,dx = \int_0^{\ln 2} \tanh^2 x\,dx - \frac{1}{3}\times\left(\frac{3}{5}\right)^3$

Using * with $n = 4$ and $\tanh(\ln 2) = \frac{3}{5}$

$$= \left[\int_0^{\ln 2} \tanh^0 x\,dx - 1\times\left(\frac{3}{5}\right)\right] - \frac{1}{3}\times\frac{27}{125}$$

Using * with $n = 2$ and $\tanh(\ln 2) = \frac{3}{5}$

$$= \ln 2 - \frac{3}{5} - \frac{9}{125}$$

$$= \ln 2 - \frac{84}{125}$$

6 a $\int \tan^4 x\,dx = \frac{1}{3}\tan^3 x - \int \tan^2 x\,dx$

$$= \frac{1}{3}\tan^3 x - \left(\tan x - \int \tan^0 x dx\right)$$

$$= \frac{1}{3}\tan^3 x - \tan x + \int 1\,dx$$

$$= \frac{1}{3}\tan^3 x - \tan x + x + C$$

b $\int_0^{\frac{\pi}{4}} \tan^n x\,dx = \left[\frac{1}{n-1}\tan^{n-1} x\right]_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} \tan^{n-2} x\,dx = \frac{1}{n-1} - \int_0^{\frac{\pi}{4}} \tan^{n-2} x\,dx$

Let $I_n = \int_0^{\frac{\pi}{4}} \tan^n x dx$, then $I_n = \frac{1}{n-1} - I_{n-2}$

$$I_5 = \frac{1}{4} - I_3 = \frac{1}{4} - \left(\frac{1}{2} - I_1\right) = \frac{1}{4} - \frac{1}{2} + \int_0^{\frac{\pi}{4}} \tan x\,dx = \frac{1}{4} - \frac{1}{2} + [\ln \sec x]_0^{\frac{\pi}{4}}$$

$$= -\frac{1}{4} + \left(\ln\sqrt{2} - \ln 1\right)$$

So $\int_0^{\frac{\pi}{4}} \tan^5 x\,dx = \ln\sqrt{2} - \frac{1}{4}$

6 c Defining $J_n = \int_0^{\frac{\pi}{3}} \tan^n x \, dx$,

$$J_n = \left[\frac{1}{n-1}\tan^{n-1} x\right]_0^{\frac{\pi}{3}} - J_{n-2} = \frac{\left(\sqrt{3}\right)^{n-1}}{n-1} - J_{n-2}$$

So $J_6 = \frac{\left(\sqrt{3}\right)^5}{5} - J_4 = \frac{\left(\sqrt{3}\right)^5}{5} - \left(\frac{\left(\sqrt{3}\right)^3}{3} - J_2\right) = \frac{\left(\sqrt{3}\right)^5}{5} - \frac{\left(\sqrt{3}\right)^5}{3} + \left(\frac{\sqrt{3}}{1} - J_0\right)$

As $J_0 = \int_0^{\frac{\pi}{3}} 1 \, dx = \frac{\pi}{3}, \int_0^{\frac{\pi}{3}} \tan^6 x \, dx = \frac{9\sqrt{3}}{5} - \frac{3\sqrt{3}}{3} + \sqrt{3} - \frac{\pi}{3} = \frac{9\sqrt{3}}{5} - \frac{\pi}{3}$

7 a $I_n = \int_1^a (\ln x)^n \, dx = \int_1^a 1(\ln x)^n \, dx$

Let $u = (\ln x)^n$ and $\frac{dv}{dx} = 1$, so $\frac{du}{dx} = n\frac{(\ln x)^{n-1}}{x}, v = x$

Integration by parts:

$$\int_1^a (\ln x)^n \, dx = \left[x(\ln x)^n\right]_1^a - \int_1^a \frac{n(\ln x)^{n-1}}{x} x \, dx$$

$$= \left[a(\ln a)^n - 0\right] - n\int_1^a (\ln x)^{n-1} \, dx$$

So $I_n = a(\ln a)^n - nI_{n-1}$

b Putting $a = 2, I_n = \int_1^2 (\ln x)^n \, dx = 2(\ln 2)^n - nI_{n-1}$

$$I_3 = \int_1^2 (\ln x)^3 \, dx = 2(\ln 2)^3 - 3I_2$$

$$= 2(\ln 2)^3 - 3\{2(\ln 2)^2 - 2I_1\}$$

$$= 2(\ln 2)^3 - 6(\ln 2)^2 + 6\{2(\ln 2) - I_0\}$$

$$= 2(\ln 2)^3 - 6(\ln 2)^2 + 12(\ln 2) - 6I_0$$

As $I_0 = \int_1^2 1 \, dx = [x]_1^2 = 1$,

$$\int_1^2 (\ln x)^3 \, dx = 2(\ln 2)^3 - 6(\ln 2)^3 + 12(\ln 2) - 6$$

7 c Putting $a = \mathrm{e}, I_n = \int_1^{\mathrm{e}} (\ln x)^n \, \mathrm{d}x = \mathrm{e}(\ln \mathrm{e})^n - nI_{n-1} = \mathrm{e} - nI_{n-1}$

$$I_6 = \int_1^{\mathrm{e}} (\ln x)^6 \, \mathrm{d}x = \mathrm{e} - 6I_5$$
$$= \mathrm{e} - 6(\mathrm{e} - 5I_4)$$
$$= \mathrm{e} - 6\mathrm{e} + 30(\mathrm{e} - 4I_3)$$
$$= \mathrm{e} - 6\mathrm{e} + 30\mathrm{e} - 120(\mathrm{e} - 3I_2)$$
$$= \mathrm{e} - 6\mathrm{e} + 30\mathrm{e} - 120\mathrm{e} + 360(\mathrm{e} - 2I_1)$$
$$= \mathrm{e} - 6\mathrm{e} + 30\mathrm{e} - 120\mathrm{e} + 360\mathrm{e} - 720(\mathrm{e} - I_0)$$

As $I_0 = \int_1^{\mathrm{e}} 1 \, \mathrm{d}x = [x]_1^{\mathrm{e}} = \mathrm{e} - 1,$

$$\int_1^{\mathrm{e}} (\ln x)^6 \, \mathrm{d}x = \mathrm{e} - 6\mathrm{e} + 30\mathrm{e} - 120\mathrm{e} + 360\mathrm{e} - 720\mathrm{e} + 720(\mathrm{e} - 1)$$
$$= 265\mathrm{e} - 720$$
$$= 5(53\mathrm{e} - 144)$$

8 a $I_7 = \frac{6}{7} \times \frac{4}{5} \times \frac{2}{3} \times 1 = \frac{16}{35}$

b $\int_0^{\frac{\sigma}{2}} \sin^2 x \cos^4 x \mathrm{d}x = \int_0^{\frac{\sigma}{2}} \sin^2 x (1 - \sin^2 x)^2 \, \mathrm{d}x = \int_0^{\frac{\sigma}{2}} (\sin^2 x - 2\sin^4 x + \sin^6 x) \mathrm{d}x$

$$= I_2 - 2I_4 + I_6$$

$I_2 = \frac{1}{2} \times \frac{\pi}{2} = \frac{\pi}{4}; \quad I_4 = \frac{3}{4} I_2 = \frac{3\pi}{16}; \quad I_6 = \frac{5}{6} I_4 = \frac{5\pi}{32}$

So $\int_0^{\frac{\sigma}{2}} \sin^2 x \cos^4 x \mathrm{d}x = \frac{\pi}{4} - \frac{3\pi}{8} + \frac{5\pi}{32} = \frac{\pi}{32}$

c Using $x = \sin\theta, \int_0^1 x^5 \sqrt{1 - x^2} \, \mathrm{d}x = \int_0^{\frac{\sigma}{2}} \sin^5 \theta \cos\theta (\cos\theta \, \mathrm{d}\theta)$

$$= \int_0^{\frac{\sigma}{2}} \sin^5 x (1 - \sin^2 x) \mathrm{d}x = I_5 - I_7$$

$I_5 = \frac{4}{5} \times \frac{2}{3} \times 1 = \frac{8}{15}$ and $I_7 = \frac{16}{35}$ from **a**

So $\int_0^1 x^5 \sqrt{1 - x^2} \, \mathrm{d}x = \frac{8}{15} - \frac{16}{35} = \frac{56 - 48}{105} = \frac{8}{105}$

d Using $x = 3t, \int_0^{\frac{\sigma}{6}} \sin^8 3t \, \mathrm{d}t = \int_0^{\frac{\sigma}{2}} \sin^8 x \left(\frac{1}{3} \mathrm{d}x \right) = \frac{1}{3} I_8$

$$= \frac{1}{3} \times \frac{7}{8} \times \frac{5}{6} \times \frac{3}{4} \times \frac{1}{2} \times \frac{\pi}{2} = \frac{35\pi}{768}$$

9 a $I_{n+1} = \int \frac{\sin^{2n+2} x}{\cos x}\,dx$

So $I_n - I_{n+1} = \int \frac{\sin^{2n} x - \sin^{2n+2} x}{\cos x}\,dx$

$= \int \frac{\sin^{2n} x(1-\sin^2 x)}{\cos x}\,dx$

$= \int \sin^{2n} x \cos x\,dx$ as $1-\sin^2 x = \cos^2 x$

So $I_n - I_{n+1} = \frac{\sin^{2n+1} x}{2n+1}$

or $I_{n+1} = I_n - \frac{\sin^{2n+1} x}{2n+1}$ $*$ [$+C$ not necessary at this stage]

b $\int \frac{\sin^4 x}{\cos x}\,dx = I_2$

Substituting $n=1$ in * gives $I_2 = I_1 - \frac{\sin^3 x}{3}$

$= \left(I_0 - \frac{\sin x}{1}\right) - \frac{\sin^3 x}{3}$ using $n=0$ in *

$I_0 = \int \frac{1}{\cos x}\,dx = \int \sec x\,dx = \ln|(\sec x + \tan x)| + C$

So $\int \frac{\sin^4 x}{\cos x}\,dx = \ln|(\sec x + \tan x)| - \sin x - \frac{\sin^3 x}{3} + C$

Applying the given limits gives

$$\int_0^{\frac{\sigma}{4}} \frac{\sin^4 x}{\cos x}\,dx = \left[\ln|\sec x + \tan x]| - \sin x - \frac{\sin^3 x}{3}\right]_0^{\frac{\sigma}{4}}$$

$$= \ln\left(1+\sqrt{2}\right) - \frac{\sqrt{2}}{2} - \frac{\left(\frac{\sqrt{2}}{2}\right)^3}{3}$$

$$= \ln\left(1+\sqrt{2}\right) - \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{12}$$

$$= \ln\left(1+\sqrt{2}\right) - \frac{7\sqrt{2}}{12}$$

10 a Let $u=(1-x^3)^n$ and $\frac{dv}{dx}=x$, so $\frac{du}{dx}=n\left(1-x^3\right)^{n-1}\left(-3x^2\right), v=\frac{x^2}{2}$

Integration by parts gives

$$\int_0^1 x\left(1-x^3\right)^n dx=\left[\frac{x^2}{2}\left(1-x^3\right)^n\right]_0^1-\int_0^1 -3nx^2\left(1-x^3\right)^{n-1}\frac{x^2}{2}dx$$

$$=[0-0]+\frac{3n}{2}\int_0^1 x^4\left(1-x^3\right)^{n-1}dx \text{ providing } n\geq 0$$

Writing $x^4=x.x^3=x\{1-(1-x^3)\}$ and $I_n=\int_0^1 x(1-x^3)^n\,dx$

we have $I_n=\frac{3n}{2}\int_0^1 x\{1-(1-x^3)\}(1-x^3)^{n-1}dx$

$$=\frac{3n}{2}\int_0^1 x(1-x^3)^{n-1}dx-\frac{3n}{2}\int_0^1 x(1-x^3)^n dx$$

$$=\frac{3n}{2}I_{n-1}-\frac{3n}{2}I_n$$

$$\Rightarrow (3n+2)I_n=3nI_{n-1}, \text{so } I_n=\frac{3n}{3n+2}I_{n-1}, n\geqslant 1$$

b $I_4=\frac{12}{14}I_3=\frac{12}{14}\times\frac{9}{11}I_2=\frac{12}{14}\times\frac{9}{11}\times\frac{6}{8}I_1=\frac{12}{14}\times\frac{9}{11}\times\frac{6}{8}\times\frac{3}{5}I_0=\frac{12}{14}\times\frac{9}{11}\times\frac{6}{8}\times\frac{3}{5}\int_0^1 x\,dx$

$$=\frac{12}{14}\times\frac{9}{11}\times\frac{6}{8}\times\frac{3}{5}\times\frac{1}{2}=\frac{243}{1540}$$

11 a Integrating by parts with $u=(a^2-x^2)^n$ and $\frac{dv}{dx}=1$

$\frac{du}{dx}=-2nx\left(a^2-x^2\right)^{n-1} \quad v=x$

So $\int_0^a\left(a^2-x^2\right)^n dx=\left[x\left(a^2-x^2\right)^n\right]_0^a-\int_0^a x\left\{-2nx\left(a^2-x^2\right)^{n-1}\right\}dx$

$$=[0-0]+2n\int_0^a x^2(a^2-x^2)^{n-1}dx=2n\int_0^a x^2(a^2-x^2)^{n-1}dx \text{ (if } n>0)$$

Writing x^2 as $\left\{a^2-\left(a^2-x^2\right)\right\}$ and defining $I_n=\int_0^a\left(a^2-x^2\right)^n dx$,

we have

$$I_n=2n\int_0^a\left\{a^2\left(a^2-x^2\right)^{n-1}-\left(a^2-x^2\right)^n\right\}dx$$

$$=2na^2I_{n-1}=2nI_n$$

So $(2n+1)I_n=2na^2I_{n-1}$

$$I_n=\frac{2na^2I_{n-1}}{(2n+1)}$$

b i With $a=1, I_n=\int_0^1\left(1-x^2\right)^n dx$ and $I_n=\frac{2n}{2n+1}I_{n-1}$

So $I_4=\frac{8}{9}I_3=\frac{8}{9}\times\frac{6}{7}I_2=\frac{8}{9}\times\frac{6}{7}\times\frac{4}{5}I_1=\frac{8}{9}\times\frac{6}{7}\times\frac{4}{5}\times\frac{2}{3}\times 1=\frac{128}{315}$

$I_0=\int_0^a dx=a$

11 b ii With $a = 3, I_n = \int_0^3 \left(9 - x^2\right)^n \,\mathrm{d}x$ and $I_n = \dfrac{18n}{2n+1} I_{n-1}$

So $I_3 = \dfrac{54}{7} I_2 = \dfrac{54}{7} \times \dfrac{36}{5} I_1 = \dfrac{54}{7} \times \dfrac{36}{5} \times \dfrac{18}{3} I_0 = \dfrac{54}{7} \times \dfrac{36}{5} \times \dfrac{18}{3} \times 3 = \dfrac{34\,992}{35}$

iii With $a = 2, I_n = \int_0^2 \left(4 - x^2\right)^n \,\mathrm{d}x$ and $I_n = \dfrac{8n}{2n+1} I_{n-1}$

So $I_{\frac{1}{2}} = \dfrac{4}{2} I_{0\frac{1}{2}} = 2\int_0^2 \dfrac{\mathrm{d}x}{\sqrt{4 - x^2}} = 2\left[\arcsin\left(\dfrac{x}{2}\right)\right]_0^2 = 2\arcsin 1 = 2 \times \dfrac{\pi}{2} = \pi$

c Using the substitution $x = 2\sin\theta$,

$$\int_0^2 \left(4 - x^2\right)^{\frac{1}{2}} \mathrm{d}x = \int_0^{\frac{\sigma}{2}} (2\cos\theta)(2\cos\theta \,\mathrm{d}\theta)$$

$$= 2\int_0^{\frac{\sigma}{2}} (1 + \cos 2\theta)\,\mathrm{d}\theta$$

$$= \left[2\theta + \sin 2\theta\right]_0^{\frac{\sigma}{2}} = \pi$$

12 a Integrating by parts with $u = x^n$ and $\dfrac{\mathrm{d}v}{\mathrm{d}x} = \sqrt{4 - x}$

$\dfrac{\mathrm{d}u}{\mathrm{d}x} = nx^{n-1}, \quad v = -\dfrac{2}{3}(4 - x)^{\frac{3}{2}}$

So $\displaystyle\int_0^4 x^n\sqrt{4 - x}\,\mathrm{d}x = \left[-\frac{2}{3}x^n(4 - x)^{\frac{3}{2}}\right]_0^4 + \frac{2}{3}n\int_0^4 x^{n-1}(4 - x)^{\frac{3}{2}}\,\mathrm{d}x$

$$= [0 - 0] + \frac{2}{3}n\int_0^4 x^{n-1}(4 - x)^{\frac{3}{2}}\,\mathrm{d}x \ (n > 0)$$

$$= \frac{2}{3}n\int_0^4 x^{n-1}\left\{(4 - x)\sqrt{4 - x}\right\}\mathrm{d}x$$

$$= \frac{2}{3}n\int_0^4 x^{n-1}4\sqrt{4 - x}\,\mathrm{d}x + \frac{2}{3}n\int_0^4 x^{n-1}\left\{-x\sqrt{4 - x}\right\}\mathrm{d}x$$

$$= \frac{8}{3}n\int_0^4 x^{n-1}\sqrt{4 - x}\,\mathrm{d}x - \frac{2}{3}n\int_0^4 x^n\sqrt{4 - x}\,\mathrm{d}x$$

> You need to write $(4 - x)^{\frac{3}{2}}$ as $(4 - x)\sqrt{4 - x}$

So $I_n = \dfrac{8}{3}nI_{n-1} - \dfrac{2}{3}nI_n$

$\Rightarrow (2n + 3)I_n = 8nI_{n-1} \Rightarrow I_n = \dfrac{8n}{2n+3} I_{n-1}, n \geqslant 1$

b $\displaystyle\int_0^4 x^3\sqrt{4 - x}\,\mathrm{d}x = I_3 = \frac{24}{9}I_2 = \frac{24}{9} \times \frac{16}{7}I_1 = \frac{24}{9} \times \frac{16}{7} \times \frac{8}{5}I_0 = \frac{1024}{105}I_0$

As $\displaystyle I_0 = \int_0^4 \sqrt{4 - x}\,\mathrm{d}x = \left[-\frac{2}{3}(4 - x)^{\frac{3}{2}}\right]_0^4 = \left[0 - \left\{-\frac{2}{3}(4)^{\frac{3}{2}}\right\}\right] = \frac{16}{3}.$

$\displaystyle\int_0^4 x^3\sqrt{4 - x}\,\mathrm{d}x = \frac{1024}{105} \times \frac{16}{3} = 52.0\,(3\text{ s.f.})$

13 a $I_n = \int \cos^n x\,\mathrm{d}x = \int \cos^{n-1} x \cos x\,\mathrm{d}x$

Integrating by parts with $u = \cos^{n-1} x$ and $\dfrac{\mathrm{d}v}{\mathrm{d}x} = \cos x$

$\dfrac{\mathrm{d}u}{\mathrm{d}x} = (n-1)\cos^{n-2} x(-\sin x), \quad v = \sin x$

So $I_n = \int \cos^n x\,\mathrm{d}x = \cos^{n-1} x \sin x - \int -(n-1)\cos^{n-2} x \sin^2 x\,\mathrm{d}x$

$= \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x(1-\cos^2 x)\,\mathrm{d}x$

$= \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x\mathrm{d}x - (n-1)\int \cos^n x\,\mathrm{d}x$

Giving $I_n = \cos^{n-1} x \sin x + (n-1)I_{n-2} - (n-1)I_n$

So $nI_n = \cos^{n-1} x \sin x + (n-1)I_{n-2}$

b It follows that $n\int_0^{2\sigma} \cos^n x\mathrm{d}x = \left[\cos^{n-1} x \sin x\right]_0^{2\sigma} + (n-1)\int_0^{2\sigma} \cos^{n-2} x\,\mathrm{d}x$

So $nJ_n = (n-1)J_{n-2}$, as $\left[\cos^{n-1} x \sin x\right]_0^{2\sigma} = 0$

c i $J_4 = \int_0^{2\sigma} \cos^4 x\mathrm{d}x = \dfrac{3}{4}J_2 = \dfrac{3}{4}\times\dfrac{1}{2}J_0 = \dfrac{3}{8}\int_0^{2\sigma} 1\,\mathrm{d}x = \dfrac{3}{8}\times 2\pi = \dfrac{3\pi}{4}$

ii $J_8 = \int_0^{2\sigma} \cos^8 x\mathrm{d}x = \dfrac{7}{8}J_6 = \dfrac{7}{8}\times\dfrac{5}{6}J_4 = \dfrac{35}{48}J_4 = \dfrac{35}{48}\times\dfrac{3\pi}{4} = \dfrac{35\pi}{64}$ Using **c i**

d If n is odd, J_n always reduces to a multiple if J_1,

but $J_1 = \int_0^{2\sigma} \cos x\mathrm{d}x = [\sin x]_0^{2\sigma} = 0.$

(You could also consider the graphical representation.)

14 a Integrating by parts with $u = x^{n-1}$ and $\frac{dv}{dx} = x\sqrt{1-x^2}$

Using the hint.

$\frac{du}{dx} = (n-1)x^{n-2}, \quad v = -\frac{1}{3}(1-x^2)^{\frac{3}{2}}$

$$\text{So } I_n = \int_0^1 x^{n-1}\left\{x\sqrt{1-x^2}\right\}dx = \left[-\frac{1}{3}x^{n-1}(1-x^2)^{\frac{3}{2}}\right]_0^1 + \frac{(n-1)}{3}\int_0^{\frac{\pi}{2}} x^{n-2}(1-x^2)^{\frac{3}{2}}dx$$

$$= \frac{(n-1)}{3}\int_0^{\frac{\pi}{2}} x^{n-2}(1-x^2)^{\frac{3}{2}}dx \text{ as } \left[-\frac{1}{3}x^{n-1}(1-x^2)^{\frac{3}{2}}\right]_0^1 = 0$$

$$= \frac{(n-1)}{3}\int_0^{\frac{\pi}{2}}\left\{x^{n-2}(1-x^2)\sqrt{1-x^2}\,dx\right\}$$

$$= \frac{(n-1)}{3}\int_0^{\frac{\pi}{2}}\left\{x^{n-2}\sqrt{1-x^2} - x^n\sqrt{1-x^2}\right\}dx$$

$$\text{So } I_n = \frac{(n-1)}{3}I_{n-2} - \frac{(n-2)}{3}I_n$$

$$\Rightarrow \{3+(n-1)\}I_n = (n-1)I_{n-2}$$

$$\Rightarrow (n+2)I_n = (n-1)I_{n-2} \quad *$$

b Using * $I_7 = \frac{6}{9}I_5 = \frac{6}{9}\times\frac{4}{7}I_3 = \frac{6}{9}\times\frac{4}{7}\times\frac{2}{5}I_1 = \frac{48}{315}\int_0^1 x\sqrt{1-x^2}\,dx$

$$= \frac{48}{315}\left[-\frac{1}{3}(1-x^2)^{\frac{3}{2}}\right]_0^1$$

$$= \frac{48}{315}\left[\frac{1}{3}\right] = \frac{16}{315}$$

15 a Integrating by parts with $u = x^n$ and $\frac{dv}{dx} = \cosh x$

$\frac{du}{dx} = nx^{n-1}, \quad v = \sinh x$

So $\int x^n \cosh x\,dx = x^n \sinh x - \int nx^{n-1}\sinh x\,dx$

Integrating by parts again with $u = x^{n-1}$ and $\frac{dv}{dx} = \sinh x$

$\frac{du}{dx} = (n-1)x^{n-2}, \quad v = \cosh x$

$$\text{So } I_n = x^n \sinh x - n\left\{x^{n-1}\cosh x - \int (n-1)x^{n-2}\cosh x\,dx\right\}$$

$$= x^n \sinh x - nx^{n-1}\cosh x + n(n-1)I_{n-2}, \quad n \geqslant 2 \quad *$$

15 b $I_4 = x^4 \sinh x - 4x^3 \cosh x + 12I_2,$ ← Substituting $n = 4$ in *

$= x^4 \sinh x - 4x^3 \cosh x + 12\left\{x^2 \sinh x - 2x\cosh x + 2I_0\right\}$

$= x^4 \sinh x - 4x^3 \cosh x + 12\left\{x^2 \sinh x - 2x\cosh x\right\} + 24\int \cosh x\mathrm{d}x$

$= x^4 \sinh x - 4x^3 \cosh x + 12\left\{x^2 \sinh x - 2x\cosh x\right\} + 24\sinh x + C$

$= (x^4 + 12x^2 + 24)\sinh x - \left(4x^3 + 24x\right)\cosh x + C$

c $\int_0^1 x^3 \cosh x\mathrm{d}x = \left[x^3 \sinh x - 3x^2 \cosh x\right]_0^1 + 6\int_0^1 x\cosh x\mathrm{d}x$ — Using **a**

$= \{\sinh 1 - 3\cosh 1\} + 6\left\{\left[x\sinh x\right]_0^1 - \int_0^1 1\sinh x\mathrm{d}x\right\}$ — Integrating by parts

$= \{\sinh 1 - 3\cosh 1\} + 6\{\sinh 1 - [\cosh 1 - 1]\}$

$= 7\sinh 1 - 9\cosh 1 + 6$

$= 7\left(\dfrac{\mathrm{e}^1 - \mathrm{e}^{-1}}{2}\right) - 9\left(\dfrac{\mathrm{e}^1 + \mathrm{e}^{-1}}{2}\right) + 6$

$= 6 - \mathrm{e} - 8\mathrm{e}^{-1}$ or $\dfrac{6\mathrm{e} - \mathrm{e}^2 - 8}{\mathrm{e}}$

16 a $I_{n-2} = \int \dfrac{\sin(n-2)x}{\sin x}\,\mathrm{d}x$

So $I_n - I_{n-2} = \int \dfrac{\sin nx - \sin(n-2)x}{\sin x}\,\mathrm{d}x$

$= \int \dfrac{2\cos\left\{\frac{n+(n-2)}{2}\right\}x\sin\left\{\frac{n-(n-2)}{2}\right\}x}{\sin x}\,\mathrm{d}x$ — Using Edexcel formula booklet

$= \int \dfrac{2\cos(n-1)x\sin x}{\sin x}\,\mathrm{d}x$

$= \int 2\cos(n-1)x\,\mathrm{d}x$

$= \dfrac{2\sin(n-1)x}{n-1}, n \geqslant 2$ — It is not necessary to have $+C$.

b i $\int \dfrac{\sin 4x}{\sin x}\,\mathrm{d}x = I_4$

Using **a** with $n = 4: I_4 = I_2 + \dfrac{2\sin 3x}{3}$ ← $I_2 = \int \dfrac{\sin 2x}{\sin x}\,\mathrm{d}x = \int \dfrac{2\sin x\cos x}{\sin x}\,\mathrm{d}x$

$= \int 2\cos x\,\mathrm{d}x + \dfrac{2\sin 3x}{3}$

$= 2\sin x + \dfrac{2}{3}\sin 3x + C$

16 b ii Using **a** with $n=5: I_5 = I_3 + \frac{2\sin 4x}{4}$

$$= \left\{ I_1 + \frac{2\sin 2x}{2} \right\} + \frac{2\sin 4x}{4}$$

$$= \int 1 \, dx + \sin 2x + \frac{\sin 4x}{2}$$

$$= x + \sin 2x + \frac{\sin 4x}{2}$$

It follows that $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sin 5x}{\sin x} \, dx = \left[x + \sin 2x + \frac{\sin 4x}{2} \right]_{\frac{\pi}{6}}^{\frac{\pi}{3}}$

$$= \left[\frac{\pi}{3} + \frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{4} \right] - \left[\frac{\pi}{6} + \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{4} \right]$$

$$= \frac{\pi}{6} - \frac{\sqrt{3}}{2}$$

$$= \frac{\pi - 3\sqrt{3}}{6}$$

17 a $I_n = \int \sinh^n x dx = \int \sinh^{n-1} x \sinh x dx$

Integrating by parts with $u = \sinh^{n-1} x$ and $\frac{dv}{dx} = \sinh x$

$\frac{du}{dx} = (n-1)\sinh^{n-2} x \cosh x, \ v = \cosh x$

So $I_n = \int \sinh^n x dx = \sinh^{n-1} x \cosh x - \int (n-1)\sinh^{n-2} x \cosh^2 x dx$

$$= \sinh^{n-1} x \cosh x - (n-1)\int \sinh^{n-2} x(1+\sinh^2 x) \, dx$$

$$= \sinh^{n-1} x \cosh x - (n-1)\int \sinh^{n-2} x dx - (n-1)\int \sinh^n x dx$$

Giving $I_n = \sinh^{n-1} x \cosh x - (n-1)I_{n-2} - (n-1)I_n$

So $nI_n = \sinh^{n-1} x \cosh x - (n-1)I_{n-2}, \quad n \geqslant 2$ *

17 b i $I_5 = \frac{1}{5}\sinh^4 x\cosh x - \frac{4}{5}I_3$ ← Using * with $n = 5$

$= \frac{1}{5}\sinh^4 x\cosh x - \frac{4}{5}\left\{\frac{1}{3}\sinh^2 x\cosh x - \frac{2}{3}I_1\right\}$

$= \frac{1}{5}\sinh^4 x\cosh x - \frac{4}{15}\sinh^2 x\cosh x + \frac{8}{15}\int \sinh x\,dx$

$= \frac{1}{5}\sinh^4 x\cosh x - \frac{4}{15}\sinh^2 x\cosh x + \frac{8}{15}\cosh x + C$

When $x = \ln 3, \sinh x = \frac{e^{\ln 3} - e^{-\ln 3}}{2} = \frac{3 - \frac{1}{3}}{2} = \frac{4}{3}, \cosh x = \frac{e^{\ln 3} + e^{-\ln 3}}{2} = \frac{3 + \frac{1}{3}}{2} = \frac{5}{3}$

When $x = 0, \sinh x = 0, \cosh x = 1$

Applying the limits 0 and ln 3 to the result in **b**

$\int_0^{\ln 3} \sinh^5 x\,dx = \left[\frac{1}{5}\left(\frac{4}{5}\right)^4\left(\frac{5}{3}\right) - \frac{4}{15}\left(\frac{4}{3}\right)^2\left(\frac{5}{3}\right) + \frac{8}{15}\left(\frac{5}{3}\right)\right] - \left[0 + 0 + \frac{8}{15}\right]$

$= \frac{752}{1215} = 0.619\,(3\text{ s.f.})$

ii $\int \sinh^4 x\,dx = I_4 = \frac{1}{4}\sinh^3 x\cosh x - \frac{3}{4}I_2$ ← Using * with $n = 4$

$= \frac{1}{4}\sinh^3 x\cosh x - \frac{3}{4}\left\{\frac{1}{2}\sinh x\cosh x - \frac{1}{3}I_0\right\}$

$= \frac{1}{4}\sinh^3 x\cosh x - \frac{3}{8}\sinh x\cosh x + \frac{3}{8}\int 1\,dx$

$= \frac{1}{4}\sinh^3 x\cosh x - \frac{3}{8}\sinh x\cosh x + \frac{3}{8}x + C$

When $x = \text{arsinh}\,1 \quad \sinh x = 1, \cosh x = \sqrt{1 + \sinh^2 x} = \sqrt{2}$

When $x = 0 \quad \sinh x = 0 \quad \cosh x = 1$

Applying the limits 0 and arsinh 1 gives

$\int_0^{\text{arsinh}\,1} \sinh^4 x\,dx = \frac{1}{4}(1)^3(\sqrt{2}) - \frac{3}{8}(1)(\sqrt{2}) + \frac{3}{8}\text{arsinh}\,1$

$= \frac{\sqrt{2}}{4} - \frac{3\sqrt{2}}{8} + \frac{3}{8}\ln\left(1 + \sqrt{1^2 + 1}\right)$

$= -\frac{\sqrt{2}}{8} + \frac{3}{8}\ln\left(1 + \sqrt{2}\right)$

$= \frac{1}{8}\left\{3\ln\left(1 + \sqrt{2}\right) - \sqrt{2}\right\}$

18 a $I_n = \int_0^{\ln\sqrt{3}} \tanh^n x\,\mathrm{d}x$

$= \int_0^{\ln\sqrt{3}} \tanh^{n-2} x \tanh^2 x\,\mathrm{d}x$

$= \int_0^{\ln\sqrt{3}} \tanh^{n-2} x(1-\operatorname{sech}^2 x)\,\mathrm{d}x$

$= \int_0^{\ln\sqrt{3}} \tanh^{n-2} x\,\mathrm{d}x - \int_0^{\ln\sqrt{3}} \tanh^{n-2} x\operatorname{sech}^2 x\,\mathrm{d}x$

$= I_{n-2} - \int_0^{\ln\sqrt{3}} \tanh^{n-2} x\operatorname{sech}^2 x\,\mathrm{d}x$

Now if we use the substitution

$u = \tanh x$,

$\frac{\mathrm{d}u}{\mathrm{d}x} = \operatorname{sech}^2 x$,

we get

$\int_0^{\ln\sqrt{3}} \tanh^{n-2} x\operatorname{sech}^2 x\,\mathrm{d}x$

$= \int_0^{0.5} u^{n-2}\,\mathrm{d}u$

$= \left[\frac{u^{n-1}}{n-1}\right]_0^{0.5}$

$= \frac{0.5^{n-1}}{n-1}$.

So, $I_n = I_{n-2} - \frac{1}{n-1}\left(\frac{1}{2}\right)^{n-1}$

18 b By letting $n = 2r + 1$ in the previous result, we may write $I_{2r+1} = I_{2r-1} - \frac{1}{2r}\left(\frac{1}{2}\right)^{2r}$

This can be expressed as $\frac{1}{2r}\left(\frac{1}{2}\right)^{2r} = I_{2r-1} - I_{2r+1}$.

So in order to find

$$\sum_{r=1}^{m} \frac{1}{2r}\left(\frac{1}{2}\right)^{2r} = \sum_{r=1}^{m}\left(I_{2r-1} - I_{2r+1}\right)$$
$$= \left(I_1 - I_3\right) + \left(I_3 - I_5\right) + \cdots + \left(I_{2m-1} - I_{2m+1}\right)$$

we notice that all middle terms cancel out and so we are left with

$$\sum_{r=1}^{m} \frac{1}{2r}\left(\frac{1}{2}\right)^{2r} = I_1 - I_{2m+1}.$$

As $m \to \infty$, we have $\lim_{m\to\infty}\left(\sum_{r=1}^{m} \frac{1}{2r}\left(\frac{1}{2}\right)^{2r}\right) = I_1 - \lim_{m\to\infty}\left(I_{2m+1}\right)$.

We have assumed that $\lim_{n\to\infty}\left(\int_0^{\ln\sqrt{3}} \tanh^n x dx\right) = 0$ and so $\lim_{m\to\infty}\left(I_{2m+1}\right) = \lim_{m\to\infty}\left(\int_0^{\ln 3} \tanh^{2m+1} x dx\right) = 0.$

Thus we have $\sum_{r=1}^{\infty} \frac{1}{2r}\left(\frac{1}{2}\right)^{2r} = I_1 - 0.$

Where

$$I_1 = \int_0^{\ln\sqrt{3}} \tanh x dx$$
$$= \left[\ln\left(\cosh x\right)\right]_0^{\ln\sqrt{3}}$$
$$= \ln\left(\cosh\left(\frac{\ln 3}{2}\right)\right)$$
$$= \ln\left(\frac{2}{\sqrt{3}}\right)$$

19 a First set $\frac{du}{dx} = x^2$, $v = \left(\ln x\right)^n$ which gives $u = \frac{x^3}{3}$ and $\frac{dv}{dx} = \frac{n}{x}\left(\ln x\right)^{n-1}$.

Now apply integration by parts to get

$$I_n = \int_0^e x^2\left(\ln x\right)^n dx$$
$$= \left[\frac{x^3}{3}\left(\ln x\right)^n\right]_0^e - \int_0^e \frac{x^3 n}{3x}\left(\ln x\right)^{n-1} dx$$
$$= \frac{e^3}{3} - \frac{n}{3}\int_0^e x^2\left(\ln x\right)^{n-1} dx$$
$$= \frac{e^3}{3} - \frac{n}{3} I_{n-1}$$

19 b This volume of revolution is calculated by

$$V = \int_1^e \pi\left(3x(\ln x)^2\right)^2 \mathrm{d}x$$

$$= 9\pi\int_1^e \left(x^2(\ln x)^4\right)\mathrm{d}x$$

Fortunately, the same relation holds in this case even though we have different limits. This can be seen by

$$J_m = \int_1^e x^2(\ln x)^m \,\mathrm{d}x$$

$$= \left[\frac{x^3}{3}(\ln x)^m\right]_1^e - \int_1^e \frac{x^3 m}{3x}(\ln x)^{m-1}\,\mathrm{d}x$$

$$= \frac{e^3}{3} - \frac{m}{3}\int_1^e x^2(\ln x)^{m-1}\,\mathrm{d}x$$

$$= \frac{e^3}{3} - \frac{m}{3}J_{m-1}$$

In this case we have $V = 9\pi J_4$ and so we calculate

$$J_4 = \frac{e^3}{3} - \frac{4}{3}J_3$$

$$= \frac{e^3}{3} - \frac{4}{3}\left(\frac{e^3}{3} - \frac{3}{3}J_2\right)$$

$$= \frac{e^3}{3} - \frac{4}{3}\left(\frac{e^3}{3} - \frac{3}{3}\left(\frac{e^3}{3} - \frac{2}{3}J_1\right)\right)$$

$$= \frac{e^3}{3} - \frac{4}{3}\left(\frac{e^3}{3} - \frac{3}{3}\left(\frac{e^3}{3} - \frac{2}{3}\left(\frac{e^3}{3} - \frac{1}{3}J_0\right)\right)\right)$$

this certainly looks horrible, but it'll be ok. Using the above equation for J_m, we calculate

$$J_0 = \int_1^e x^2\mathrm{d}x$$

$$= \left[\frac{x^3}{3}\right]_1^e$$

$$= \frac{e^3}{3} - \frac{1}{3}$$

Thus we have

$$V = 9\pi J_4$$

$$= 9\pi\left(\frac{e^3}{3} - \frac{4}{3}\left(\frac{e^3}{3} - \frac{3}{3}\left(\frac{e^3}{3} - \frac{2}{3}\left(\frac{e^3}{3} - \frac{1}{3}\left(\frac{e^3}{3} - \frac{1}{3}\right)\right)\right)\right)\right)$$

$$= \frac{\pi}{9}\left(11e^3 - 8\right)$$

Challenge

a Using integration by parts, we find that

$$I_n = \int x^a (\ln x)^n \, dx$$

$$= \frac{x^{a+1}}{a+1}(\ln x)^n - \int \frac{x^{a+1} n}{(a+1)x}(\ln x)^{n-1} \, dx$$

$$= \frac{x^{a+1}}{a+1}(\ln x)^n - \frac{n}{a+1}\int x^a (\ln x)^{n-1} \, dx$$

Thus $(a+1)I_n = x^{a+1}(\ln x)^n - nI_{n-1}$.

b $I_3 = \int \sqrt{x}(\ln x)^3 \, dx$ and $a = \frac{1}{2}$

So by using the above result we calculate

$$(a+1)I_n = x^{a+1}(\ln x)^n - nI_{n-1}$$

$$(0.5+1)I_n = x^{0.5+1}(\ln x)^n - nI_{n-1}$$

$$I_n = \frac{2}{3}\left(x^{1.5}(\ln x)^n - nI_{n-1}\right)$$

$$I_3 = \frac{2}{3}\left(x^{1.5}(\ln x)^3 - 3I_2\right)$$

and so $= \frac{2}{3}\left(x^{1.5}(\ln x)^3 - 3\left(\frac{2}{3}\left(x^{1.5}(\ln x)^2 - 2I_1\right)\right)\right)$

$$= \frac{2}{3}\left(\begin{array}{l} x^{1.5}(\ln x)^3 \\ -3\left(\frac{2}{3}\left(x^{1.5}(\ln x)^2 - 2\left(\frac{2}{3}\left(x^{1.5}\ln x - I_0\right)\right)\right)\right) \end{array}\right)$$

and I_0 can be easily calculated as $I_0 = \int \sqrt{x}\,dx = \frac{2x^{1.5}}{3}$

Combining the above results we get

$$\frac{2}{3}\left(\begin{array}{l} x^{1.5}(\ln x)^3 \\ -3\left(\frac{2}{3}\left(x^{1.5}(\ln x)^2 - 2\left(\frac{2}{3}\left(x^{1.5}\ln x - 1\left(\frac{2x^{1.5}}{3}\right)\right)\right)\right)\right) \end{array}\right)$$

$$= \frac{2}{3}x^{1.5}\left((\ln x)^3 - \left(2\left((\ln x)^2 - \left(\frac{4}{3}\left(\ln x - \left(\frac{2}{3}\right)\right)\right)\right)\right)\right)$$

$$= \frac{2}{27}x^{1.5}\left(9(\ln x)^3 - 18(\ln x)^2 + 24\ln x - 16\right)$$