Discrete random variables 1D

1 a Rearrange the formula $Y = 4X - 6$ to get an expression for X in terms of Y:

$Y = 4X - 6$ gives

$$X = \frac{Y+6}{4}$$

$$X = \frac{Y}{4} + \frac{3}{2}$$

$$\mathrm{E}(X) = \mathrm{E}\left(\frac{Y}{4} + \frac{3}{2}\right) = \frac{1}{4}\mathrm{E}(Y) + \frac{3}{2}$$

$$= \frac{1}{4} \times 2 + \frac{3}{2} = \frac{1}{2} + \frac{3}{2} = 2$$

b $$\mathrm{Var}(X) = \mathrm{Var}\left(\frac{Y}{4} + \frac{3}{2}\right) = \left(\frac{1}{4}\right)^2 \mathrm{Var}(Y)$$

$$= \frac{1}{16} \times 32 = 2$$

c Standard deviation $= \sqrt{\mathrm{Var}(X)} = \sqrt{2} = 1.4142$ (4 d.p.)

2 a Rearrange the formula get an expression for X in terms of Y:

$$2Y = 4 - 3X$$

$$3X = 4 - 2Y$$

$$X = \frac{4-2Y}{3}$$

$$X = \frac{4}{3} - \frac{2}{3}Y$$

$$\mathrm{E}(X) = \mathrm{E}\left(\frac{4}{3} - \frac{2}{3}Y\right) = \frac{4}{3} - \frac{2}{3}\mathrm{E}(Y)$$

$$= \frac{4}{3} - \frac{2}{3}(-1) = 2$$

b $$\mathrm{Var}(X) = \mathrm{Var}\left(\frac{4}{3} - \frac{2Y}{3}\right) = \left(-\frac{2}{3}\right)^2 \mathrm{Var}(Y)$$

$$= \frac{4}{9} \times 9 = 4$$

c $$\mathrm{Var}(X) = \mathrm{E}(X^2) - (\mathrm{E}(X))^2$$

So $$\mathrm{E}(X^2) = \mathrm{Var}(X) + (\mathrm{E}(X))^2$$

$$= 4 + 2^2 = 8$$

3 Rearranging the formula for Y to get an expression for X gives:

$$X = \frac{Y}{2} - \frac{3}{2}$$

$$\mathrm{E}(X) = \mathrm{E}\left(\frac{Y}{2} - \frac{3}{2}\right) = \frac{1}{2}\mathrm{E}(Y) - \frac{3}{2}$$

$$= \frac{1}{2} \times 8 - \frac{3}{2} = 4 - \frac{3}{2} = \frac{5}{4} = 2.5$$

$\mathrm{E}(X) = \sum x\mathrm{P}(X = x) = 2.5$

$0.3 + 2a + 3b + 4 \times 0.2 = 2.5$

$2a + 3b + 1.1 = 2.5$

$2a + 3b = 1.4$ **(1)**

$\sum \mathrm{P}(X = x) = 1$

So $0.3 + a + b + 0.2 = 1$

$a + b = 0.5$ **(2)**

$2 \times$ **(2)** $\Rightarrow$ $2a + 2b = 1$ **(3)**

(1) $-$ **(3)** $\Rightarrow$ $b = 0.4$

From **(2)** $a + 0.4 = 0.5 \Rightarrow a = 0.1$

Solution:

$a = 0.1, \quad b = 0.4$

4 a The probability distribution of Y is:

y	1	0	–1
$\mathbf{P(Y = y)}$	a	b	0.3

$\mathrm{E}(Y) = a + 0 \times b - 1 \times 0.3 = a - 0.3$

As $a + b + 0.3 = 1, \ 0 \le a \le 0.7$

$\max a = 0.7 \Rightarrow \mathrm{E}(Y) = 0.4$

$\min a = 0 \Rightarrow \mathrm{E}(Y) = -0.3$

So range of possible values for $\mathrm{E}(Y)$ is $-0.3 \le \mathrm{E}(Y) \le 0.4$

b $\mathrm{E}(Y) = 0.2$ gives $0.2 = a - 0.3 \Rightarrow a = 0.5$

As probabilities sum to 1:

$a + b + 0.3 = 1 \Rightarrow b = 1 - 0.3 - 0.5 = 0.2$

5 a The probability distribution of Y is:

y	1	0	1	4	9
$\mathbf{P}(Y = y)$	a	b	c	b	a

The sum of probabilities sum to 1, so:

$a + b + c + b + a = 1$

$\Rightarrow 2a + 2b + c = 1$ **(1)**

E(Y) = 2.4, so:

$1 \times a + 0 \times b + 1 \times c + 4b + 9a = 2.4$

$\Rightarrow a + c + 4b + 9a = 2.4$

$\Rightarrow 10a + 4b + c = 2.4$ **(2)**

$\mathrm{P}(Y > 2) = \mathrm{P}(Y = 4) + \mathrm{P}(Y = 9)$, and as $\mathrm{P}(Y > 2) = 0.4$, this gives:

$a + b = 0.4$ **(3)**

b Multiply equation (**3**) by 2:

$2a + 2b = 0.8$

Subtract this equation from (**1**), gives $c = 0.2$

Substitute c into equation (**2**):

$10a + 4b + 0.2 = 2.4$

$10a + 4b = 2.2$ **(4)**

Multiply equation (**3**) by 2:

$10a + 10b = 4$

Subtract equation (**4**) from this equation:

$6b = 4 - 2.2 \Rightarrow b = 0.3$

Substitute b and c into equation (**1**):

$2a + 2 \times 0.3 + 0.2 = 1 \Rightarrow a = 0.1$

Solution: $a = 0.1$, $b = 0.3$, $c = 0.2$

c Use the values found in part **b** to find the probability distribution for $2X + 3$ and Y:

x	–2	–1	0	1	2
$2x + 3$	–1	1	3	5	7
y	1	0	1	4	9
$\mathbf{P}(Y = y)$	0.1	0.3	0.2	0.3	0.1

From the table $\mathrm{P}(2X + 3 \leq Y) = \mathrm{P}(X = -2) + \mathrm{P}(X = 2) = 0.1 + 0.1 = 0.2$

Alternatively note that $2X + 3 \leq Y \Rightarrow 2X + 3 \leq (X + 1)^2$

So $2X + 3 \leq X^2 + 2X + 1$, which simplifies to $X^2 \geq 2$

So $\mathrm{P}(2X + 3 \leq Y) = \mathrm{P}(X^2 \geq 2) = 2a = 0.2$

6 a $E(Y) = E(1-2X) = 1-2E(X)$

So $-5.6 = 1 - 2E(X)$

$2E(X) = 6.6$

$E(X) = 3.3$

Alternatively:

$$Y = 1-2X \Rightarrow X = \frac{1}{2} - \frac{Y}{2}$$

$$E(X) = E\left(\frac{1}{2} - \frac{Y}{2}\right) = \frac{1}{2} - \frac{1}{2}E(Y) = 0.5 - (-2.8) = 3.3$$

b The sum of probabilities sum to 1, so:

$a + a + a + b + b + c = 1$

$\Rightarrow 3a + 2b + c = 1$ **(1)**

$E(X) = 3.3$, from part a, so:

$1 \times a + 2 \times a + 3 \times a + 4 \times b + 5 \times b + 6 \times c = 3.3$

$\Rightarrow 6a + 9b + 6c = 3.3$

$\Rightarrow 2a + 3b + 2c = 1.1$ **(2)**

$P(Y \leq -5) = 0.6$ and as $P(Y \leq -5) = P(X \geq 3)$, this gives:

$a + 2b + c = 0.6$ **(3)**

c **(1)** $-$ **(3)** $\Rightarrow$ $2a = 0.4$, $a = 0.2$

$2 \times$ **(3)** $\Rightarrow$ $2a + 4b + 2c = 1.2$ **(4)**

(4) $-$ **(2)** $\Rightarrow$ $b = 0.1$

Substitute a and b into equation **(3)**:

$0.2 + 0.2 + c = 0.6 \Rightarrow c = 0.2$

Solution:

$a = 0.2$, $b = 0.1$, $c = 0.2$

d $X > 5 + Y \Rightarrow X > 5 + 1 - 2X$ as $Y = 1 - 2X$

$\Rightarrow 3X > 6 \Rightarrow X > 2$

So $P(X > 5 + Y) = P(X > 2)$

$= a + 2b + c = 0.2 + 2 \times 0.1 + 0.2 = 0.6$