Quality of tests 8D

1 a $H_0 : \lambda = 6.5 \quad H_1 : \lambda < 6.5$

Assume H_0, so that $X \sim \text{Po}(6.5)$

Significance level 5%, so require c such that $P(X \leqslant c) < 0.05$

From the Poisson cumulative distribution tables

$P(X \leqslant 3) = 0.1118$ and $P(X \leqslant 2) = 0.0430$

$P(X \leqslant 3) > 0.05$ and $P(X \leqslant 2) < 0.05$ so the critical value is 2

Hence the critical region is $X \leqslant 2$

Size $= P(X \leqslant 2 \mid \lambda = 6.5) = 0.0430$

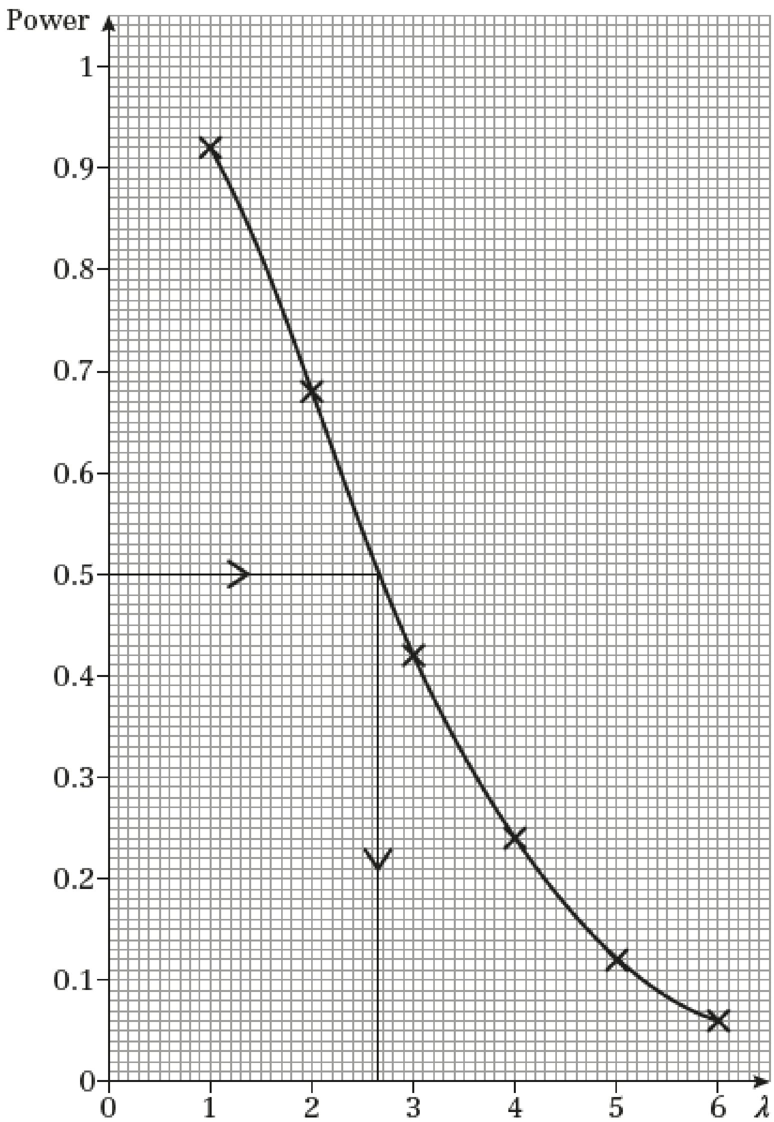
b Power function $= P(X \leqslant 2 \mid X \sim \text{Po}(\lambda))$

$$= e^{-\lambda} + \frac{e^{-\lambda}\lambda^1}{1!} + \frac{e^{-\lambda}\lambda^2}{2!} = e^{-\lambda}\left(1 + \lambda + \frac{1}{2}\lambda^2\right)$$

c $\lambda = 2 \Rightarrow s = 5e^{-2} = 0.68$ (2 d.p.)

$\lambda = 5 \Rightarrow t = \dfrac{37}{2}e^{-5} = 0.12$ (2 d.p.)

d



1 e When $\lambda = 6.5$, the correct conclusion is to accept H_0. So since size is 0.0430, the probability of accepting $\lambda = 6.5$ is 0.957, which is greater than 0.5. The test is very likely to come to correct conclusion.

When $\lambda < 6.5$, the correct conclusion is to reject H_0. So require the power of the test > 0.5, and this can be found by reading from the graph: so the test is more likely than not to come to the correct conclusion for $\lambda < 2.65$

2 a $H_0 : p = 0.45 \quad H_1 : p < 0.45$

Critical region $X \leqslant 2$, where $X \sim B(12, 0.45)$

From the binomial cumulative distribution function tables:

Size $= P(X \leqslant 2) = 0.0421$ (4 d.p.)

b Power function $= P(X \leqslant 2 \mid X \sim B(12, p))$

$$= P(X = 0 \mid X \sim B(12, p)) + P(X = 1 \mid X \sim B(12, p)) + P(X = 2 \mid X \sim B(12, p))$$

$$= \binom{12}{0} p^0 (1-p)^{12} + \binom{12}{1} p^1 (1-p)^{11} + \binom{12}{2} p^2 (1-p)^{10}$$

$$= (1-p)^{12} + 12p(1-p)^{11} + \frac{12 \times 11}{2} p^2 (1-p)^{10}$$

$$= (1-p)^{12} + 12p(1-p)^{11} + 66p^2(1-p)^{10}$$

c Power $= P(X \leqslant 2 \mid X \sim B(12,\ 0.3)) = 0.2528$ (from the tables)

Alternatively use the power function, Power $= 0.7^{12} + 3.6 \times 0.7^{11} + 5.94 \times 0.7^{10} = 0.2528$ (4 d.p.)

3 a $H_0 : p = 0.4 \quad H_1 : p > 0.4$

Critical region $X \geqslant 8$

Power $= P(X \geqslant 8 \mid X \sim B(10,\ 0.5))$

$= 1 - P(X \leqslant 7)$

$= 1 - 0.9453 = 0.0547$

b Power $= P(X \geqslant 8 \mid X \sim B(10,\ 0.8))$

Let $Y \sim B(10, 0.2)$ then

Power $= P(X \geqslant 8 \mid X \sim B(10,\ 0.8)) = P(X \leqslant 2 \mid X \sim B(10,\ 0.2))$

$= 0.6778$

c The test is more powerful for values of p further away from 0.4

4 a $H_0 : p = \frac{1}{2} \quad H_1 : p < \frac{1}{2}$

Test A: critical region $X \leqslant 2$ where $X \sim B(10, p)$

Size $= P(X \leqslant 2 \mid X \sim B(10, 0.5)) = 0.0547$

4 b Power function $= \mathrm{P}(X \leqslant 2 \mid X \sim \mathrm{B}(10, p))$

$$= \binom{10}{0} p^0 (1-p)^{10} + \binom{10}{1} p(1-p)^9 + \binom{10}{2} p^2 (1-p)^8$$

$$= (1-p)^{10} + 10p(1-p)^9 + \frac{10 \times 9}{2} p^2 (1-p)^8$$

$$= (1-p)^{10} + 10p(1-p)^9 + 45p^2(1-p)^8$$

c Let the random variable Y denote the number of heads recorded in 5 spins of the coin, then $Y \sim \mathrm{B}(5, p)$

Test B: $\mathrm{H}_0 : p = \frac{1}{2}$ $\quad \mathrm{H}_1 : p < \frac{1}{2}$

Size $= \mathrm{P}(\text{Type I error}) = \mathrm{P}(\mathrm{H}_0 \text{ rejected} \mid X \sim \mathrm{B}(10, 0.5))$

$= \mathrm{P}(\text{fails test 1}) + \mathrm{P}(\text{passes test 1 then fails test 2})$

$= \mathrm{P}(Y = 0 \mid X \sim \mathrm{B}(10, 0.5)) + (1 - \mathrm{P}(Y = 0 \mid X \sim \mathrm{B}(10, 0.5)))\mathrm{P}(Y = 0 \mid X \sim \mathrm{B}(10, 0.5))$

$= 0.03125 + (1 - 0.03125)0.03125$

$= 0.03125 + 0.03027 = 0.0615$ (4 d.p.)

d Power function $= \mathrm{P}(Y = 0 \mid X \sim \mathrm{B}(5, p)) + \left((1 - \mathrm{P}(Y = 0 \mid X \sim \mathrm{B}(5, p))\mathrm{P}(Y = 0 \mid X \sim \mathrm{B}(5, p))\right)$

$= (1-p)^5 + (1 - (1-p)^5)(1-p)^5$

$= (1-p)^5(2 - (1-p)^5)$

e From the tables for the binomial cumulative distribution function

Power $= \mathrm{P}(X \leqslant 2 \mid X \sim \mathrm{B}(10, 0.25)) = 0.5256$

Power $= \mathrm{P}(X \leqslant 2 \mid X \sim \mathrm{B}(10, 0.35)) = 0.2616$

f Use test A as this is more powerful – the table shows test A has a higher power within the likely range of the parameter ($p < 0.5$).

5 a $\mathrm{H}_0 : p = 0.15$ $\quad \mathrm{H}_1 : p < 0.15$

Assume H_0, so that $X \sim \mathrm{Geo}(0.15)$

Significance level 1%

Require $\mathrm{P}(X \geqslant c) < 0.01$

So $(1 - 0.15)^{c-1} < 0.01$

$(c-1)\log 0.85 < \log 0.01$

$c - 1 > \frac{\log 0.01}{\log 0.85}$

$c > 29.336$

So the critical value is 30 and the critical region is $X \geqslant 30$

Size $= \mathrm{P}(\mathrm{H}_0 \text{ rejected} \mid \mathrm{H}_0 \text{ true}) = \mathrm{P}(X \geqslant 30 \mid X \sim \mathrm{Geo}(0.15))$

$= (1 - 0.15)^{30-1} = 0.85^{29} = 0.0090$ (4 d.p.)

5 b Power function $= \text{P}(\text{H}_0 \text{ rejected} \mid X \sim \text{Geo}(p))$
$= \text{P}(X \geqslant 30 \mid X \sim \text{Geo}(p)) = (1-p)^{29}$

6 a $H_0 : p = 0.7$
$H_1 : p \geqslant 0.7$
If 10 trials are done then under B(10,0.7)
$P(X \geqslant 9) = 0.1493...$
$P(X \geqslant 10) = 0.02824...$
So the critical number of trials without a flat tyre is 10
Size of the test
$= P(\text{reject } H_0 \text{ when it is true})$
$= \text{P}(X \geqslant 10 | X \sim B(10, 0.7))$
$= 0.02824...$
≈ 0.028

b Power function of the test
$= P(\text{reject } H_0 \text{ when it is false}) = \lambda^{10}$

c $H_0 : p = 0.7$
$H_1 : p \geqslant 0.7$
If 12 trials are done then under B(12,0.7)
$P(X \geqslant 11) = 0.085...$
$P(X \geqslant 12) = 0.013...$
So the critical number of trials without a flat tyre is 12

Power function of the test

$= P(\text{reject } H_0 \text{ when it is false}) = \lambda^{12}$

d Because $0.95^{10} > 0.95^{12}$
the test is more powerful when 10 trials are done.