Continuous distributions 3B

1 ***Method 1***

If $x < 0$, $F(x) = 0$ so $F(0) = 0$

If $0 \leqslant x \leqslant 2$

$$F(x) = F(0) + \int_0^x \frac{3t^2}{8}\, dt = \left[\frac{3t^3}{24}\right]_0^x = \frac{3x^3}{24} - 0 = \frac{3x^3}{24}$$

So the full solution is:

$$F(x) = \begin{cases} 0 & x < 0 \\ \dfrac{3x^2}{24} & 0 \leqslant x \leqslant 2 \\ 1 & x > 2 \end{cases}$$

Method 2

If $0 \leqslant x \leqslant 2$

$$F(x) = \int \frac{3x^2}{8} dx = \frac{3x^3}{24} + c$$

$$F(2) = 1 \Rightarrow 1 + c = 1 \Rightarrow c = 0$$

So the full solution is:

$$F(x) = \begin{cases} 0 & x < 0 \\ \dfrac{3x^2}{24} & 0 \leqslant x \leqslant 2 \\ 1 & x > 2 \end{cases}$$

2 ***Method 1***

If $x < 1$, $F(x) = 0$ so $F(1) = 0$

If $1 \leqslant x \leqslant 3$

$$F(x) = F(1) + \int_1^x \frac{1}{4}(4-t)\, dt = \left[t - \frac{t^2}{8}\right]_1^x = \left(x - \frac{x^2}{8}\right) - \left(1 - \frac{1}{8}\right) = x - \frac{x^2}{8} - \frac{7}{8}$$

So the full solution is:

$$F(x) = \begin{cases} 0 & x < 1 \\ x - \dfrac{x^2}{8} - \dfrac{7}{8} & 1 \leqslant x \leqslant 3 \\ 1 & x > 3 \end{cases}$$

Method 2

If $1 \leqslant x \leqslant 3$

$$F(x) = \int \frac{1}{4}(4-x) dx = x - \frac{x^2}{8} + c$$

$$F(3) = 1 \Rightarrow 3 - \frac{9}{8} + c = 1 \Rightarrow c = -\frac{7}{8}$$

So $F(x) = x - \dfrac{x^2}{8} - \dfrac{7}{8}$, which leads to the full solution given for Method 1.

3 If $x \leqslant 0$, F(x) = 0 so F(0) = 0

If $0 < x < 3$

$$\mathrm{F}(x) = \int \frac{1}{9} x \, \mathrm{d}x = \frac{1}{18}x^2 + c$$

As $\mathrm{F}(0) = 0 \Rightarrow c = 0$

If $3 \leqslant x \leqslant 6$

$$\mathrm{F}(x) = \int \frac{1}{9}(6 - x)\,\mathrm{d}x = \frac{2x}{3} - \frac{x^2}{18} + d$$

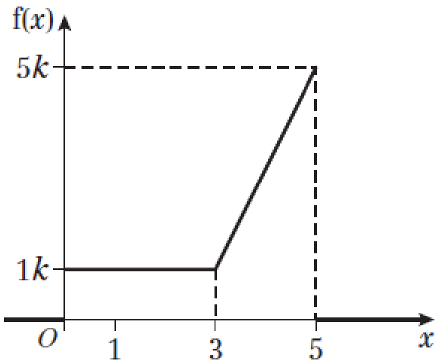
As $\mathrm{F}(6) = 1 \Rightarrow 4 - 2 + d = 1 \Rightarrow d = -1$

So the full solution is:

$$\mathrm{F}(x) = \begin{cases} 0 & x \leqslant 0 \\ \dfrac{x^2}{18} & 0 < x < 3 \\ \dfrac{2x}{3} - \dfrac{x^2}{18} - 1 & 3 \leqslant x \leqslant 6 \\ 1 & x > 6 \end{cases}$$

This shows the solution using Method 2; the problem can also be solved using Method 1.

4 **a** The graph is a horizontal line from (0, k) to (3, k), and a straight line from (3, k) to (3, 5k). Otherwise f(x) is 0.



b The area under the curve must equal 1, so:

$$\int_0^3 k\,\mathrm{d}x + \int_3^5 k(2x - 5)\,\mathrm{d}x = 1$$

$$k\Big[x\Big]_0^3 + k\Big[(x^2 - 5x)\Big]_3^5 = 1$$

$$3k + k\big((25 - 25) - (9 - 15)\big) = 1$$

$$9k = 1$$

$$k = \frac{1}{9}$$

4 c If $x < 0$, $F(x) = 0$ so $F(0) = 0$

If $0 \leqslant x < 3$

$$F(x) = \int \frac{1}{9}\,dx = \frac{x}{9} + c$$

As $F(0) = 0 \Rightarrow c = 0$

If $3 \leqslant x \leqslant 5$

$$F(x) = \int \frac{1}{9}(2x-5)\,dx = \frac{x^2}{9} - \frac{5x}{9} + d$$

As $F(5) = 1 \Rightarrow \frac{25}{9} - \frac{25}{9} + d = 1 \Rightarrow d = 1$

So the full solution is:

$$F(x) = \begin{cases} 0 & x < 0 \\ \frac{x}{9} & 0 \leqslant x < 3 \\ \frac{x^2}{9} - \frac{5x}{9} + 1 & 3 \leqslant x \leqslant 5 \\ 1 & x > 5 \end{cases}$$

5 $f(x) = \frac{d}{dx}F(x)$

So where $F(x)$ is constant, $f(x) = 0$

For $2 \leqslant x \leqslant 3$, $f(x) = \frac{d}{dx}\frac{1}{5}(x^2 - 4) = \frac{2x}{5}$

So the probability density function is:

$$f(x) = \begin{cases} \frac{2x}{5} & 2 \leqslant x \leqslant 3 \\ 0 & \text{otherwise} \end{cases}$$

6 a $P(X \leqslant 2.5) = F(2.5) = \frac{1}{2}(2.5 - 1) = 0.75$

b $P(X > 1.5) = 1 - F(1.5) = 1 - \frac{1}{2}(1.5 - 1) = 0.75$

c $P(1.5 \leqslant X \leqslant 2.5) = F(2.5) - F(1.5) = 0.75 - 0.25 = 0.5$

7 As F(x) is a cumulative distribution function, F(2) = 0 and F(4) = 1

$\mathrm{F}(2) = 0 \Rightarrow \frac{2^p}{6} + q = 0$ **(1)**

$\mathrm{F}(4) = 0 \Rightarrow \frac{4^p}{6} + q = 1$ **(2)**

Subtracting equation (**1**) from equation (**2**) gives:

$$\frac{4^p}{6} - \frac{2^p}{6} = 1 \Rightarrow 4^p - 2^p - 6 = 0$$

Let $y = 2^p$, then $y^2 = 2^p \cdot 2^p = 4^p$ and the equation can be written as:

$$y^2 - y - 6 = 0 \Rightarrow (y-3)(y+2) = 0$$

Taking the positive root, $y = 3 \Rightarrow 2^p = 3$

So taking logs of both sides, $\ln 2^p = \ln 3 \Rightarrow p \ln 2 = \ln 3 \Rightarrow p = \frac{\ln 3}{\ln 2}$

Substituting $2^p = 3$ into equation (**1**) gives, $q = -\frac{1}{2}$

8 **a** $\mathrm{f}(x) = \frac{\mathrm{d}}{\mathrm{d}x}\mathrm{F}(x)$

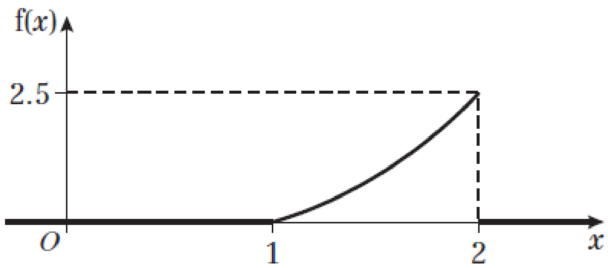
So where F(x) is constant, f(x) =0

For $1 \leqslant x \leqslant 2$, $\mathrm{f}(x) = \frac{\mathrm{d}}{\mathrm{d}x}\frac{1}{2}(x^3 - 2x^2 + x) = \frac{3x^2}{2} - 2x + \frac{1}{2}$

So the probability density function is:

$$\mathrm{f}(x) = \begin{cases} \frac{3x^2}{2} - 2x + \frac{1}{2} & 1 \leqslant x \leqslant 2 \\ 0 & \text{otherwise} \end{cases}$$

b Between (1, 0) and (2, 2.5) is an arc of a positive quadratic, otherwise the function lies on the x-axis:



c $\mathrm{P}(X < 1.5) = \mathrm{F}(1.5) = \frac{1}{2}(1.5^3 - 2(1.5^2) + 1.5)$

$$= \frac{1}{2}\left(\frac{27}{8} - \frac{9}{2} + \frac{3}{2}\right) = \frac{1}{2} \times \frac{3}{8} = \frac{3}{16} = 0.1875$$

9 a As area under curve must be 1, $\int_0^2 k(4-x^2)\,\mathrm{d}x = \left[k\left(4x-\frac{x^3}{3}\right)\right]_0^2 = 1$

$$\Rightarrow k\left(8-\frac{8}{3}\right) = \frac{16k}{3} = 1$$

$$\Rightarrow k = \frac{3}{16}$$

b ***Method 1***

If $x < 0$, $F(x) = 0$ so $F(0) = 0$

If $0 \leqslant x \leqslant 2$

$$F(x) = \int_0^x \frac{3}{16}(4-t^2)\,\mathrm{d}t = \left[\frac{3}{16}\left(4t-\frac{t^3}{3}\right)\right]_0^x = \frac{3}{16}\left(4x-\frac{x^3}{3}\right)$$

So the full solution is:

$$F(x) = \begin{cases} 0 & x < 0 \\ \frac{3}{16}\left(4x-\frac{x^3}{3}\right) & 0 \leqslant x \leqslant 2 \\ 1 & x > 2 \end{cases}$$

Method 2

If $0 \leqslant x \leqslant 2$

$$F(x) = \int \frac{3}{16}(4-x^2)\,\mathrm{d}x = \frac{3}{16}\left(4x-\frac{x^3}{3}\right)+c$$

$$F(2) = 1 \Rightarrow \frac{3}{16}\left(8-\frac{8}{3}\right)+c = 1 \Rightarrow c = 0$$

This leads to the same full solution as given for Method 1.

c $P(0.69 < X < 0.70) = F(0.70) - F(0.69) = \frac{3}{16}\left(2.8-\frac{0.343}{3}\right) - \frac{3}{16}\left(2.76-\frac{0.328509}{3}\right)$

$$= 0.50356 - 0.49697 = 0.00659 = 0.007 \text{ (1 s.f.)}$$

10 If $F(x)$ is a valid cumulative distribution function then for any value x the function $F(x) \leqslant 1$. For this function $F(x) > 1$ for $x > e$ so it is not a valid cumulative distribution function.

11 a As $F(x)$ is a cumulative distribution function, $F(0) = 0$ and $F(3) = 1$. So from $F(3) = 1$:

$$\frac{1}{120}(3k-27) = 1 \Rightarrow 3k = 120+27 = 147 \Rightarrow k = 49$$

b $P(X > 2) = 1 - P(X \leqslant 2) = 1 - \frac{1}{120}(49\times 2 - 2^3) = \frac{98-8}{120} = \frac{90}{120} = 0.25$

12 If $x < 1$, $F(x) = 0$ so $F(0) = 0$

If $1 \leqslant x < 7$

$$F(x) = \int \frac{1}{x \ln 7} dx = \frac{\ln x}{\ln 7} + c$$

As $F(7) = 1$, $c = 0$

So the full solution is:

$$F(x) = \begin{cases} 0 & x < 1 \\ \dfrac{\ln x}{\ln 7} & 1 \leqslant x \leqslant 7 \\ 1 & x > 7 \end{cases}$$

13 If $x < 0$, $F(x) = 0$ so $F(0) = 0$

If $0 \leqslant x < 0.5$

$$F(x) = \int \pi \cos(\pi x) dx = \sin(\pi x) + c$$

As $F(0) = 0$, $c = 0$

So the full solution is:

$$F(x) = \begin{cases} 0 & x < 0 \\ \sin(\pi x) & 0 \leqslant x \leqslant 0.5 \\ 1 & x > 0.5 \end{cases}$$

14 a As $F(x)$ is a cumulative distribution function, $F(0) = 0$ and $F(3) = 1$. So from $F(3) = 1$:

$$k(2 + \ln 3) = 1 \Rightarrow k = \frac{1}{2 + \ln 3}$$

b $f(x) = \dfrac{d}{dx} F(x)$

So where $F(x)$ is constant, $f(x) = 0$

For $1 \leqslant x \leqslant 3$, $f(x) = \dfrac{d}{dx} \dfrac{1}{2 + \ln 3}(x - 1 + \ln x) = \dfrac{1}{2 + \ln 3}\left(1 + \dfrac{1}{x}\right)$

So the probability density function is:

$$f(x) = \begin{cases} \dfrac{1}{2 + \ln 3}\left(1 + \dfrac{1}{x}\right) & 1 \leqslant x \leqslant 3 \\ 0 & \text{otherwise} \end{cases}$$

Challenge

a $F(t) = \int 1.25e^{-1.25t}\,dt = -e^{-1.25t} + c$

$F(0) = 0 \Rightarrow -e^{0} + c = 0 \Rightarrow c = 1$

So the full solution is:

$$F(t) = \begin{cases} 0 & t < 0 \\ 1 - e^{-1.25t} & t \geq 0 \end{cases}$$

b $P(1 < T < 2) = P(T < 2) - P(T < 1) = 1 - e^{-2.5} - (1 - e^{-1.25}) = e^{-1.25} - e^{-2.5} = 0.2044$ (4 d.p.)

c $P(T > 3) = 1 - P(T \leqslant 3) = 1 - (1 - e^{-3.75}) = e^{-3.75} = 0.0235$ (4 d.p.) $e^{-3.75} = 0.023518... \approx 0.0235$