Continuous distributions 3C

1 a The area under the curve must equal 1, so:

$$\int_0^2 kx^2 = 1$$

$$\left[\frac{kx^3}{3}\right]_0^2 = 1$$

$$\frac{8k}{3} = 1$$

$$k = \frac{3}{8}$$

b $\mathrm{E}(X) = \int x\,\mathrm{f}(x)\,\mathrm{d}x = \int_0^2 \frac{3x^3}{8}\,\mathrm{d}x = \left[\frac{3x^4}{32}\right]_0^2 = \frac{48}{32} = \frac{3}{2} = 1.5$

c $\mathrm{Var}(X) = \int x^2\,\mathrm{f}(x)\mathrm{d}x - \mu^2 = \int_0^2 \frac{3x^4}{8}\mathrm{d}x - 1.5^2$

$$= \left[\frac{3x^5}{40}\right]_0^2 - 1.5^2 = \frac{96}{40} - 2.25 = \frac{48}{20} - \frac{45}{20} = \frac{3}{20} = 0.15$$

2 a $\mathrm{E}(Y) = \int_0^3 \frac{y^3}{9}\,\mathrm{d}y = \left[\frac{y^4}{36}\right]_0^3 = \frac{81}{36} = \frac{9}{4} = 2.25$

b Use $\mathrm{Var}(Y) = \int x^2\,\mathrm{f}(x)\mathrm{d}x - \mu^2 = \int x^2\,\mathrm{f}(x)\mathrm{d}x - (\mathrm{E}(Y))^2$

$$\mathrm{Var}(Y) = \int_0^3 \frac{y^4}{9}\mathrm{d}y - 2.25^2 = \left[\frac{y^5}{45}\right]_0^3 - \frac{81}{16}$$

$$= \frac{243}{45} - \frac{81}{16} = \frac{27}{5} - \frac{81}{16} = \frac{432-405}{80} = \frac{27}{80} = 0.3375$$

c $\sigma = \sqrt{0.3375} = 0.581$ (3 s.f.)

3 a $\mathrm{E}(Y) = \int_0^4 \frac{y^2}{8}\,\mathrm{d}y = \left[\frac{y^3}{24}\right]_0^4 = \frac{64}{24} = \frac{8}{3}$

b $\mathrm{Var}(Y) = \int_0^4 \frac{y^3}{8}\mathrm{d}y - \left(\frac{8}{3}\right)^2 = \left[\frac{y^4}{32}\right]_0^4 - \frac{64}{9}$

$$= \frac{256}{32} - \frac{64}{9} = 8 - \frac{64}{9} = \frac{72-64}{9} = \frac{8}{9}$$

c $\sigma = \sqrt{\frac{8}{9}} = 0.943$ (3 s.f.)

3 d From part **a**, $\mu = \frac{8}{3}$, so

$$P(Y > \mu) = P\left(Y > \frac{8}{3}\right) = \int_{\frac{8}{3}}^{4} \frac{y}{8}\,dy$$

$$= \left[\frac{y^2}{16}\right]_{\frac{8}{3}}^{4} = 1 - 0.4444\ldots = 0.556 \text{ (3 s.f.)}$$

e $\text{Var}(3Y+2) = 3^2\,\text{Var}(Y) = 9 \times \frac{8}{9} = 8$

f $E(Y+2) = E(Y) + 2 = \frac{8}{3} + 2 = \frac{14}{3}$

4 a The area under the curve must equal 1, so:

$$\int_0^1 k(1-x)\,dx = 1$$

$$\left[kx - \frac{kx^2}{2}\right]_0^1 = 1$$

$$k - \frac{1}{2}k = 1$$

$$k = 2$$

b $E(X) = \int_0^1 2(x - x^2)\,dx = \left[x^2 - \frac{2x^3}{3}\right]_0^1 = 1 - \frac{2}{3} = \frac{1}{3}$

c Use $\text{Var}(X) = \int x^2\,f(x)\,dx - \mu^2 == \int x^2\,f(x)\,dx - (E(X))^2$

$$\text{Var}(X) = \int_0^1 2(x^2 - x^3)\,dx - \left(\frac{1}{3}\right)^2 = \left[\frac{2x^3}{3} - \frac{x^4}{2}\right]_0^1 - \frac{1}{9}$$

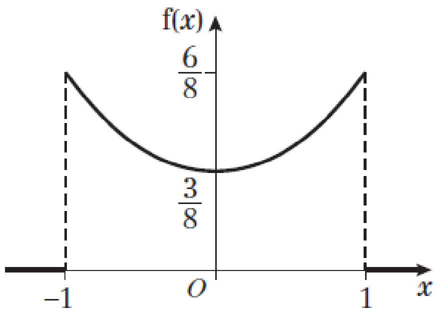
$$= \left(\frac{2}{3} - \frac{1}{2}\right) - \frac{1}{9} = \frac{1}{6} - \frac{1}{9} = \frac{3-2}{18} == \frac{1}{18}$$

d $P\left(X > \frac{1}{3}\right) = \int_{\frac{1}{3}}^{1} 2(1-x)\,dx = \left[2x - x^2\right]_{\frac{1}{3}}^{1} = (2-1) - \left(\frac{2}{3} - \frac{1}{9}\right) = \frac{1}{3} + \frac{1}{9} = \frac{4}{9}$

5 a $P(X < 0.5) = \int_0^{0.5} 12x^2 - 12x^3\,dx = \left[4x^3 - 3x^4\right]_0^{0.5} = \frac{1}{2} - \frac{3}{16} = \frac{5}{16} = 0.3125$

b $E(X) = \int_0^1 x\,f(x)\,dx = \int_0^1 12x^3 - 12x^4\,dx = \left[3x^4 - \frac{12x^5}{5}\right]_0^1 = 3 - \frac{12}{5} = \frac{3}{5} = 0.6$

6 a Between $\left(-1, \frac{3}{4}\right)$ and $\left(1, \frac{3}{4}\right)$, the curve is a positive quadratic with a minimum at $\left(0, \frac{3}{8}\right)$. Otherwise f($x$) is 0.



b As the probability density function is symmetrical about $x = 0$, E(X) = 0

Alternatively, $\text{E}(X) = \int_{-1}^{1} \frac{3}{8}(x + x^3)\text{d}x = \left[\frac{3}{16}x^2 + \frac{3}{32}x^4\right]_{-1}^{1} = \frac{3}{16} + \frac{3}{32} - \left(\frac{3}{16} + \frac{3}{32}\right) = 0$

c $\sigma^2 = \text{Var}(X) = \text{E}(X^2) - (\text{E}(X))^2 = \int_{-1}^{1} \frac{3x^2}{8} + \frac{3x^4}{8}\text{d}x - 0^2$

$= \left[\frac{3x^3}{24} + \frac{3x^5}{40}\right]_{-1}^{1} = \left(\frac{3}{24} + \frac{3}{40}\right) - \left(-\frac{3}{24} - \frac{3}{40}\right)$

$= \frac{6}{24} + \frac{6}{40} = \frac{10}{40} + \frac{6}{40} = \frac{16}{40} = \frac{2}{5} = 0.4$

d $\text{P}(-\sqrt{0.4} < X < \sqrt{0.4}) = \int_{-\sqrt{0.4}}^{\sqrt{0.4}} \frac{3}{8} + \frac{3x^2}{8}\text{d}x = \left[\frac{3x}{8} + \frac{3x^3}{24}\right]_{-\sqrt{0.4}}^{\sqrt{0.4}}$

$= \left(\frac{3}{8} \times \sqrt{0.4} + \frac{3}{24} \times (\sqrt{0.4})^3\right) - \left(\frac{3}{8} \times (-\sqrt{0.4}) + \frac{3}{24} \times (\sqrt{0.4})^3\right)$

$= \frac{3}{4} \times \sqrt{0.4} + \frac{1}{4}(\sqrt{0.4})^3 = 0.4743 + 0.0632 = 0.538$ (3 s.f.)

7 a The area under the curve must equal 1, so:

$\int_0^2 kt^3\,\text{d}t = 1$

$\left[\frac{kt^4}{4}\right]_0^2 = 1$

$4k = 1$

$k = \frac{1}{4}$

b $\text{E}(T) = \int_0^2 \frac{t^4}{4}\text{d}t = \left[\frac{t^5}{20}\right]_0^2 = \frac{32}{20} = 1.6$

7 c $E(2T+3) = 2E(T)+3 = 2\times 1.6+3 = 6.2$

d $\text{Var}(T) = \int_0^2 \frac{t^5}{4}\text{d}t - \left(\frac{8}{5}\right)^2 = \left[\frac{t^6}{24}\right]_0^2 - \left(\frac{8}{5}\right)^2$

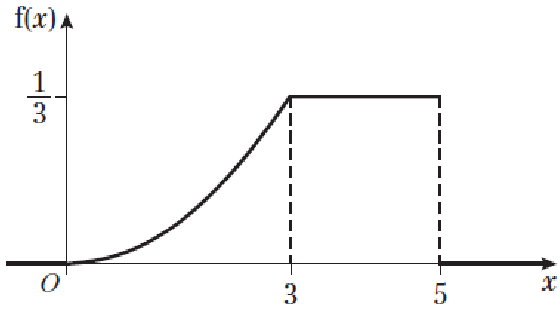
$= \frac{64}{24} - \frac{64}{25} = \frac{8}{3} - \frac{64}{25} = \frac{200}{75} - \frac{192}{75} = \frac{8}{75}$

e $\text{Var}(2T+3) = 2^2\,\text{Var}(\text{T}) = 4\times\frac{8}{75} = \frac{32}{75}$

f $\text{P}(\text{T}<1) = \int_0^1 \frac{t^3}{4}\text{d}t = \left[\frac{t^4}{16}\right]_0^1 = \frac{1}{16}$

8 a Between $(0,0)$ and $\left(3,\frac{1}{3}\right)$, the curve is a positive quadratic with a minimum at $(0,0)$.

Between $\left(3,\frac{1}{3}\right)$ and $\left(5,\frac{1}{3}\right)$, the graph is a horizontal straight line. Otherwise f(x) is 0.



b $E(X) = \int_0^3 \frac{x^3}{27}\text{d}x + \int_3^5 \frac{x}{3}\text{d}x = \left[\frac{x^4}{108}\right]_0^3 + \left[\frac{x^2}{6}\right]_3^5$

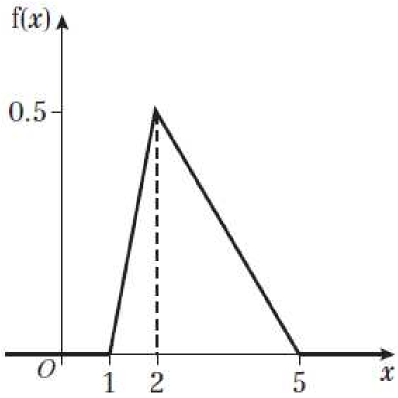
$= \frac{81}{108} + \left(\frac{25}{6} - \frac{9}{6}\right) = \frac{9}{12} + \frac{32}{12} = \frac{41}{12} = 3.417 \text{ (3 d.p.)}$

c $\text{Var}(X) = \int_0^3 \frac{x^4}{27}\text{d}x + \int_3^5 \frac{x^2}{3}\text{d}x - \left(\frac{41}{12}\right)^2 = \left[\frac{x^5}{135}\right]_0^3 + \left[\frac{x^3}{9}\right]_3^5 - \frac{1681}{144}$

$= \frac{243}{135} + \left(\frac{125}{9} - \frac{27}{9}\right) - \frac{1681}{144} = \frac{243}{135} + \frac{98}{9} - \frac{1681}{144} = 1.015 \text{ (3 d.p.)}$

d $\sigma = \sqrt{1.015} = 1.01 \text{ (2 d.p.)}$

9 a Between $(1,0)$ and $(2,0.5)$, the graph is a straight line with a positive gradient, and between $(2,0.5)$ and $(5,0)$, the graph is a straight line with a negative gradient. Otherwise f(x) is 0.



b $$\mathrm{E}(X)=\int_1^2\left(\frac{x^2}{2}-\frac{x}{2}\right)\mathrm{d}x+\int_2^5\left(\frac{5x}{6}-\frac{x^2}{6}\right)\mathrm{d}x=\left[\frac{x^3}{6}-\frac{x^2}{4}\right]_1^2+\left[\frac{5x^2}{12}-\frac{x^3}{18}\right]_2^5$$
$$=\left(\frac{8}{6}-1\right)-\left(\frac{1}{6}-\frac{1}{4}\right)+\left(\frac{125}{12}-\frac{125}{18}\right)-\left(\frac{20}{12}-\frac{8}{18}\right)=\frac{7}{6}-\frac{3}{4}+\frac{105}{12}-\frac{117}{18}$$
$$=\frac{7}{6}-\frac{3}{4}+\frac{35}{4}-\frac{13}{2}=\frac{7}{6}+\frac{48}{6}-\frac{39}{6}=\frac{16}{6}=\frac{8}{3}$$

c $$\mathrm{Var}(X)=\int_1^2\left(\frac{x^3}{2}-\frac{x^2}{2}\right)\mathrm{d}x+\int_2^5\left(\frac{5x^2}{6}-\frac{x^3}{6}\right)\mathrm{d}x-\left(\frac{8}{3}\right)^2=\left[\frac{x^4}{8}-\frac{x^3}{6}\right]_1^2+\left[\frac{5x^3}{18}-\frac{x^4}{24}\right]_2^5-\frac{64}{9}$$
$$=\left(\frac{16}{8}-\frac{8}{6}\right)-\left(\frac{1}{8}-\frac{1}{6}\right)+\left(\frac{625}{18}-\frac{625}{24}\right)-\left(\frac{40}{18}-\frac{16}{24}\right)-\frac{64}{9}$$
$$=\frac{15}{8}-\frac{7}{6}+\frac{585}{18}-\frac{609}{24}-\frac{64}{9}=\frac{15}{8}-\frac{21}{18}+\frac{585}{18}-\frac{203}{8}-\frac{128}{18}=\frac{436}{18}-\frac{188}{8}$$
$$=\frac{436}{18}-\frac{47}{2}=\frac{436}{18}-\frac{423}{18}=\frac{13}{18}$$

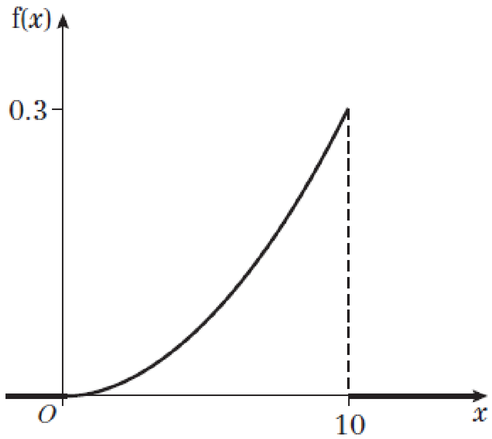
10 a The area under the curve must equal 1, so:
$$\int_0^{10}kt^2\ \mathrm{d}t=1$$
$$\left[\frac{kt^3}{3}\right]_0^{10}=1$$
$$\frac{1000k}{3}=1$$
$$k=0.003$$

b $$\mathrm{E}(T)=\int_0^{10}0.003t^3\,\mathrm{d}t=\left[\frac{0.003t^4}{4}\right]_0^{10}=\frac{30}{4}=7.5$$

10 c $\text{Var}(x)=\int_0^{10}0.003t^4\,\text{d}t-7.5^2=\left[\frac{0.003t^5}{5}\right]_0^{10}-7.5^2=60-56.25=3.75$

d $\text{P}(7<T<9)=\int_7^9 0.003t^2\,\text{d}t=\left[\frac{0.003t^3}{3}\right]_7^9=0.729-0.343=0.386$

e Between $(0,0)$ and $(10,0.3)$, the curve is a positive quadratic with a minimum at $(0,0)$. Otherwise f(x) is 0.



11 a $\text{E}(X)=\int_0^2 \frac{3}{4}x-\frac{3}{16}x^3\,\text{d}x=\left[\frac{3}{8}x^2-\frac{3}{64}x^4\right]_0^2=\frac{12}{8}-\frac{48}{64}=\frac{6}{4}-\frac{3}{4}=\frac{3}{4}$

b $\text{E}(X^2)=\int_0^2 \frac{3}{4}x^2-\frac{3}{16}x^4\,\text{d}x=\left[\frac{1}{4}x^3-\frac{3}{80}x^5\right]_0^2=\frac{8}{4}-\frac{96}{80}=2-\frac{6}{5}=\frac{4}{5}$

c $\text{Var}(X)=\text{E}(X^2)-(\text{E}(X))^2=\frac{4}{5}-\left(\frac{3}{4}\right)^2=\frac{4}{5}-\frac{9}{16}=\frac{64-45}{80}=\frac{19}{80}$

12 a $\text{f}(x)=\frac{\text{d}}{\text{d}x}\text{F}(x)=\frac{x}{50}$ for $0\leqslant x\leqslant 10$; and $\text{f}(x)=0$ otherwise

$\text{E}(X)=\int_0^{10}\frac{x^2}{50}\text{d}x=\left[\frac{x^3}{150}\right]_0^{10}=\frac{1000}{150}=\frac{20}{3}$

$\text{E}(X^2)=\int_0^{10}\frac{x^3}{50}\text{d}x=\left[\frac{x^4}{2000}\right]_0^{10}=\frac{10000}{200}==50$

$\text{Var}(X)=\text{E}(X^2)-(\text{E}(X))^2=50-\frac{400}{9}=\frac{50}{9}$

b $\text{E}(X^3)=\int x^3\,\text{f}(x)\text{d}x=\int_0^{10}\frac{x^4}{50}\text{d}x=\left[\frac{x^5}{250}\right]_0^{10}=\frac{100000}{250}=400$

13 a The area under the curve must equal 1, so:

$$\int_1^3 \frac{k}{x}\,dx = 1$$

$$\left[k\ln x\right]_1^3 = 1$$

$$k\ln 3 = 1$$

$$k = \frac{1}{\ln 3}$$

b $E(X) = \int_1^3 \frac{1}{\ln 3}\,dx = \frac{1}{\ln 3}\left[x\right]_1^3 = \frac{3-1}{\ln 3} = \frac{2}{\ln 3}$

c $E(X^2) = \int_1^3 \frac{x}{\ln 3}\,dx = \frac{1}{\ln 3}\left[\frac{x^2}{2}\right]_1^3 = \frac{9-1}{2\ln 3} = \frac{4}{\ln 3}$

$\text{Var}(X) = E(X^2) - (E(X))^2 = \frac{4}{\ln 3} - \left(\frac{2}{\ln 3}\right)^2 = 3.64096 - 3.3141 = 0.3268 \text{ (4 d.p.)}$

14 a The area under the curve must equal 1, so:

$$\int_1^2 \frac{c}{x(3-x)}\,dx = \frac{c}{3}\int_1^2\left(\frac{1}{x} + \frac{1}{3-x}\right)dx = 1$$

$$\frac{c}{3}\left[\ln x - \ln(3-x)\right]_1^2 = 1$$

$$\frac{c}{3}(\ln 2 - (-\ln 2)) = 1$$

$$\frac{c}{3}(2\ln 2) = 1$$

$$\frac{c}{3}\ln 4 = 1$$

$$c = \frac{3}{\ln 4}$$

b $\mu = E(X) = \frac{3}{\ln 4}\int_1^2 \frac{1}{3-x}\,dx = \frac{3}{\ln 4}\left[-\ln(3-x)\right]_1^2 = \frac{3\ln 2}{\ln 4} = \frac{3\ln 2}{2\ln 2} = 1.5$

$E(X^2) = \frac{3}{\ln 4}\int_1^2 \frac{x}{3-x}\,dx = \frac{3}{\ln 4}\int_1^2 -1 + \frac{3}{3-x}\,dx = \frac{3}{\ln 4}\left[-x - 3\ln(3-x)\right]_1^2 = \frac{3}{\ln 4}(3\ln 2 - 1)$

$\sigma^2 = \text{Var}(X) = \frac{3}{\ln 4}(3\ln 2 - 1) - 1.5^2 = 2.33596 - 2.25 = 0.0860 \text{ (4 d.p.)}$

15 Using integration by parts

$E(\ln X) = \int_0^1 2x\ln x\,dx = \left[x^2\ln x\right]_0^1 - \int_0^1 x\,dx = \left[x^2\ln x\right]_0^1 - \left[0.5x^2\right]_0^1$

Since $\lim_{x\to 0}(x^2\ln x) = 0, \left[x^2\ln x\right]_0^1 = 0$

So $E(\ln X) = -\left[0.5x^2\right]_0^1 = -0.5$

Challenge

$$\begin{aligned}
\text{Var}(X) &= \int_{-\infty}^{\infty} (x-\mu)^2 \, \text{f}(x)\text{d}x \\
&= \int_{-\infty}^{\infty} (x^2 - 2\mu x + \mu^2)\,\text{f}(x)\text{d}x \\
&= \int_{-\infty}^{\infty} x^2 \,\text{f}(x)\text{d}x - 2\mu\int_{-\infty}^{\infty} x\,\text{f}(x)\text{d}x + \mu^2\int_{-\infty}^{\infty} \text{f}(x)\text{d}x \\
&= \int_{-\infty}^{\infty} x^2 \,\text{f}(x)\text{d}x - 2\mu(\mu) + \mu^2(1) \\
&= \int_{-\infty}^{\infty} x^2 \,\text{f}(x)\ \text{d}x - \mu^2 \\
&= \text{E}(X^2) - (\text{E}(X))^2
\end{aligned}$$