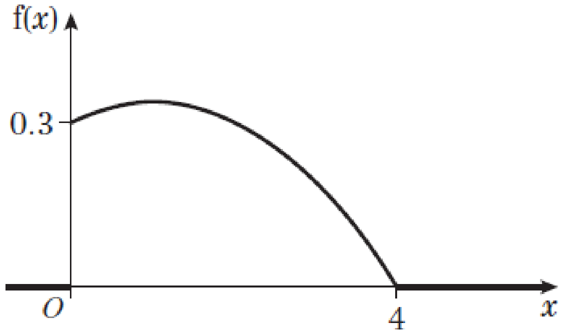
Continuous distributions 3D

1 a Between $(0,0.3)$ and $(4,0)$ the curve is a negative quadratic. There is a maximum between $x = 0$ and $x = 4$ (as when $x = 1$, for example, $f(x) > 0.3$). Otherwise $f(x)$ is 0.



b To find the mode solve, $\frac{d}{dx}f(x) = 0$

$$\frac{d}{dx}\frac{3}{80}(8+2x-x^2)=0$$
$$\frac{3}{80}(2-2x)=0$$
$$2-2x=0$$
$$x=1$$

The mode is 1.

(To check this is a maximum, either use the sketch or differentiate again and see if $f''(1) < 0$.)

2 a If $x < 0$, $F(x) = 0$ so $F(0) = 0$

If $0 \leqslant x \leqslant 4$

$$F(x) = F(0) + \int_0^x \frac{1}{8}t \, dt = \left[\frac{1}{16}t^2\right]_0^x = \frac{1}{16}x^2$$

So the full solution is:

$$F(x) = \begin{cases} 0 & x < 0 \\ \frac{1}{16}x^2 & 0 \leqslant x \leqslant 4 \\ 1 & x > 4 \end{cases}$$

b i $F(m) = \frac{1}{16}m^2 = 0.5 \Rightarrow m^2 = 8 \Rightarrow m = \sqrt{8}$ (Note $-\sqrt{8}$ is note in the range $0 \leqslant x \leqslant 4$)

So median = 2.83 (2 d.p.)

ii $F(P_{10}) = \frac{1}{16}P_{10}^2 = 0.1 \Rightarrow P_{10}^2 = 1.6 \Rightarrow P_{10} = 1.26$ (2 d.p.)

iii $F(P_{80}) = \frac{1}{16}P_{80}^2 = 0.8 \Rightarrow P_{80}^2 = 12.8 \Rightarrow P_{80} = 3.58$ (2 d.p.)

3 a As F(2) = $\frac{2}{3}$ and F(m) = 0.5, the median must lie in the range $0 \leqslant x \leqslant 2$

So F(m) = $\frac{m^2}{6} = 0.5 \Rightarrow m^2 = 3 \Rightarrow m = +\sqrt{3} = 1.732$ (3 d.p.) (as $-\sqrt{3}$ is not in the range)

b Lower quartile is less then the median so it lies in the range $0 \leqslant x \leqslant 2$

$$\frac{Q_1^2}{6} = 0.25 \Rightarrow Q_1^2 = 1.5 \Rightarrow Q_1 = \sqrt{1.5} = 1.2247... = 1.225 \text{ (3 d.p.)}$$

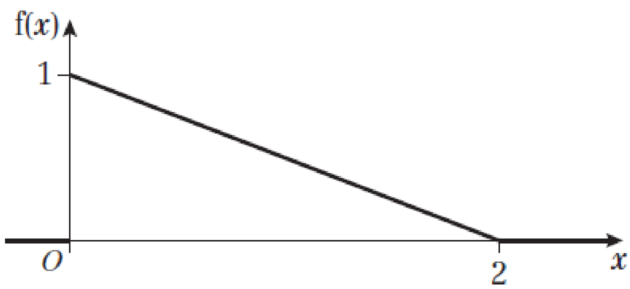
As F(2) = $\frac{2}{3}$, upper quartile lies in the range $2 \leqslant x \leqslant 3$

$$-\frac{Q_3^2}{3} + 2Q_3 - 2 = 0.75$$
$$-Q_3^2 + 6Q_3 - 6 = 2.25$$
$$-Q_3^2 + 6Q_3 - 8.25 = 0$$
$$Q_3 = \frac{-6 \pm \sqrt{36 - 33}}{-2}$$
$$Q_3 = 2.134 \text{ (3 d.p.) or } 3.866 \text{ (3 d.p.)}$$

$Q_3 = 2.134$ (3 d.p.) as 3.866 does not lie in the range

Interquartile range $= 2.1340 - 1.2247 = 0.909$ (3 d.p.)

4 a The graph is a straight line between $(0,1)$ and $(2,0)$. Otherwise f(x) is 0.



b 0 (the mode occurs at the maximum point of the probability density function graph)

4 c Using Method 1:

For $0 \leqslant x \leqslant 2$, $\mathrm{F}(x) = \int_0^x \left(1 - \frac{1}{2}t\right) \, \mathrm{d}t = \left[t - \frac{1}{4}t^2\right]_0^x = x - \frac{1}{4}x^2$

Using Method 2:

For $0 \leqslant x \leqslant 2$, $\mathrm{F}(x) = \int 1 - \frac{1}{2}x \, \mathrm{d}x = x - \frac{1}{4}x^2 + c$

As F(2) = 1, $2 - 1 + c = 1 \Rightarrow c = 0$

So the full solution is:

$$\mathrm{F}(x) = \begin{cases} 0 & x < 0 \\ x - \frac{1}{4}x^2 & 0 \leqslant x \leqslant 2 \\ 1 & x > 2 \end{cases}$$

d $m - \frac{1}{4}m^2 = 0.5$

$m^2 - 4m + 2 = 0$

$m = \frac{4 \pm \sqrt{16 - 8}}{2} = 2 \pm \sqrt{2}$

As $2 + \sqrt{2}$ is not in range, median = $2 - \sqrt{2} = 0.586$ (3 s.f.)

e $Q_3 - \frac{1}{4}Q_3^2 = 0.75$

$Q_3^2 - 4Q_3 + 3 = 0$

$(Q_3 - 1)(Q_3 - 3) = 0$

So $Q_3 = 1$ (other solution is not in the range $0 \leqslant x \leqslant 2$)

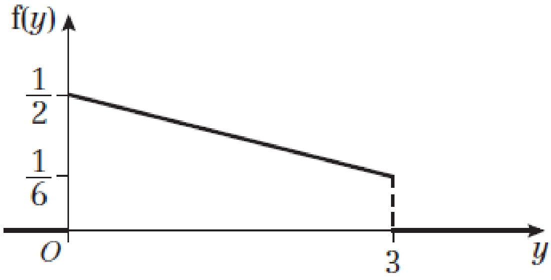
f $P_5 - \frac{1}{4}P_5^2 = 0.05$

$5P_5^2 - 20P_5 + 1 = 0$

$P_5 = \frac{20 \pm \sqrt{400 - 20}}{10} = 2 \pm \sqrt{3.8}$

So $P_5 = 2 - \sqrt{3.8} = 0.0506$ (3 s.f.) (other solution is not in the range $0 \leqslant x \leqslant 2$)

5 a The graph is a straight line between $\left(0, \frac{1}{2}\right)$ and $\left(3, \frac{1}{6}\right)$. Otherwise f($x$) is 0.



b The distribution is positively skewed, the mass of the distribution is concentrated at the left-hand end.

c 0 (the mode occurs at the maximum point of the probability density function graph)

d Using Method 1:

For $0 \leqslant y \leqslant 3$, $F(y) = \int_0^y \frac{1}{2} - \frac{1}{9}t \, dt = \left[\frac{t}{2} - \frac{1}{18}t^2\right]_0^y = \frac{y}{2} - \frac{1}{18}y^2$

Using Method 2:

For $0 \leqslant y \leqslant 3$, $F(y) = \int \frac{y}{2} - \frac{1}{9}y \, dy = \frac{y}{2} - \frac{1}{18}y^2 + x$

As F(3) = 1, $\frac{3}{2} - \frac{9}{18} + c = 1 \Rightarrow c = 0$

So the full solution is:

$$F(y) = \begin{cases} 0 & y < 0 \\ \frac{y}{2} - \frac{1}{18}y^2 & 0 \leqslant y \leqslant 3 \\ 1 & y > 3 \end{cases}$$

e $\frac{m}{2} - \frac{1}{18}m^2 = 0.5$

$m^2 - 9m + 9 = 0$

$m = \frac{9 \pm \sqrt{81 - 36}}{2} = \frac{9 \pm \sqrt{45}}{2} = \frac{9 \pm 3\sqrt{5}}{2}$

As $\frac{9 + 3\sqrt{5}}{2} = 7.85$ lies outside the range, median $= \frac{9 - 3\sqrt{5}}{2} = 1.15$ (3 s.f.)

5 f $\frac{1}{2}P_{90} - \frac{1}{18}P_{90}^2 = 0.9 \Rightarrow P_{90}^2 - 9P_{90} + 16.2 = 0$

$P_{90} = \frac{9 \pm \sqrt{81-64.8}}{2} = \frac{9 \pm \sqrt{16.2}}{2} = \frac{9 \pm 4.0249}{2}$

$P_{90} = \frac{9-4.0249}{2} = 2.4875\ldots$ (other solution is not in the range)

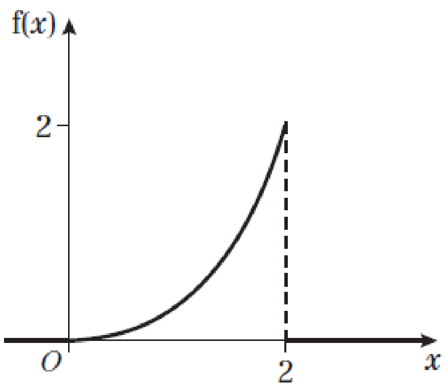
$\frac{1}{2}P_{10} - \frac{1}{18}P_{10}^2 = 0.1 \Rightarrow P_{10}^2 - 9P_{10} + 1.8 = 0$

$P_{10} = \frac{9 \pm \sqrt{81-7.2}}{2} = \frac{9 \pm \sqrt{73.8}}{2} = \frac{9 \pm 8.5907}{2}$

$P_{10} = \frac{9-8.5907}{2} = 0.2047\ldots$ (other solution is not in the range)

$P_{90} - P_{10} = 2.4875 - 0.2047 = 2.28$ (3 s.f.)

6 a The graph is a positive cubic between $(0,0)$ and $(2,2)$. Otherwise f(x) is 0.



b 2 (the mode occurs at the maximum point of the probability density function graph)

c For $0 \leqslant x \leqslant 2$, $\mathrm{F}(x) = \int_0^x \frac{1}{4}t^3 \, \mathrm{d}t = \left[\frac{1}{16}t^4\right]_0^x = \frac{1}{16}x^4$

So the full solution is:

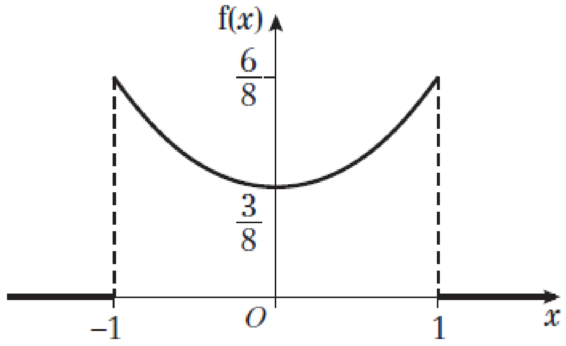
$$\mathrm{F}(x) = \begin{cases} 0 & x < 0 \\ \frac{1}{16}x^4 & 0 \leqslant x \leqslant 2 \\ 1 & x > 2 \end{cases}$$

d $\frac{1}{16}m^4 = 0.5 \Rightarrow m^4 = 8 \Rightarrow m = \sqrt[4]{8}$ (median must be the positive root to be in the range)

So median = 1.68 (3 s.f.)

7 a The graph is a positive quadratic between $\left(-1, \frac{3}{4}\right)$ and $\left(1, \frac{3}{4}\right)$, with a minimum at $\left(0, \frac{3}{8}\right)$.

Otherwise f(x) is 0.



b The distribution is bimodal; it has two modes. They are at –1 and 1.

c The distribution is symmetrical. Median = 0

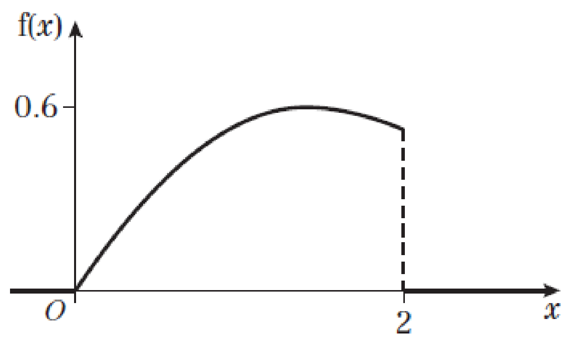
d For $-1 \leqslant x \leqslant 1$, $\mathrm{F}(x) = \int_{-1}^{x} \frac{3}{8}x^2 + \frac{3}{8}\,\mathrm{d}x = \left[\frac{1}{8}x^3 + \frac{3}{8}x\right]_{-1}^{x} = \left[\frac{1}{8}x^3 + \frac{3}{8}x\right] - \left[-\frac{1}{8} - \frac{3}{8}\right] = \frac{1}{8}x^3 + \frac{3}{8}x + \frac{1}{2}$

So the full solution is:

$$\mathrm{F}(x) = \begin{cases} 0 & x < -1 \\ \frac{1}{8}x^3 + \frac{3}{8}x + \frac{1}{2} & -1 \leqslant x \leqslant 1 \\ 1 & x > 1 \end{cases}$$

8 a The graph is a negative quadratic between $(0,0)$ and $\left(2, \frac{6}{10}\right)$, with a maximum when $x = 1.5$.

Otherwise f(x) is 0.



b The distribution is negatively skewed, the mass of the distribution is concentrated at the right-hand end.

8 c Find the mode by solving

$$\frac{d}{dx}\left(\frac{9}{10}x-\frac{3}{10}x^2\right)=0$$

$$\Rightarrow \frac{9}{10}-\frac{6}{10}x=0$$

$$\Rightarrow \text{mode}=\frac{3}{2}=1.5$$

d For $0\leqslant x\leqslant 2$, $F(x)=\int_0^x\left(\frac{9}{10}t-\frac{3}{10}t^2\right)dt=\left[\frac{9}{20}t^2-\frac{1}{10}t^3\right]_0^x=\frac{9}{20}x^2-\frac{1}{10}x^3$

So the full solution is:

$$F(x)=\begin{cases}0 & x<0\\ \frac{9}{20}x^2-\frac{1}{10}x^3 & 0\leqslant x\leqslant 2\\ 1 & x>2\end{cases}$$

e $F(1.23)=\frac{9}{20}\times 1.23^2-\frac{1}{10}\times 1.23^3=0.495$

$F(1.24)=\frac{9}{20}\times 1.24^2-\frac{1}{10}\times 1.24^3=0.501$

Since $F(m)=0.5$, $F(1.23)<F(m)<F(1.24)$ and as $F(x)$ is a cumulative distribution function this shows that the median, m, lies between 1.23 and 1.24.

9 a $f(x)=\frac{d}{dx}F(x)$

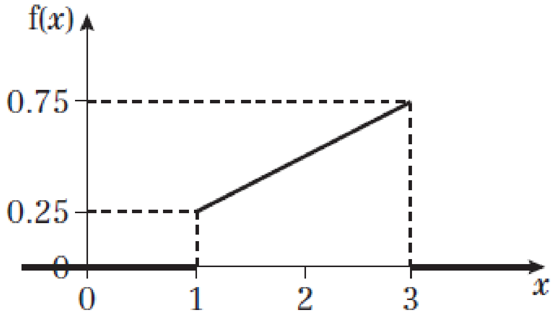
So where $F(x)$ is constant, $f(x)=0$

For $1\leqslant x\leqslant 3$, $f(x)=\frac{d}{dx}\left(\frac{1}{8}x^2-\frac{1}{8}\right)=\frac{1}{4}x$

So the probability density function is:

$$f(x)=\begin{cases}\frac{1}{4}x & 1\leqslant x\leqslant 3\\ 0 & \text{othrewise}\end{cases}$$

9 b The graph of f(x) is a straight line between $(1, 0.25)$ and $(3, 0.75)$. Otherwise f(x) is 0.



The mode = 3 (the mode occurs at the maximum point of the probability density function graph)

c $\frac{1}{8}m^2 - \frac{1}{8} = 0.5$

$\Rightarrow \frac{1}{8}m^2 = \frac{5}{8} \Rightarrow m = \sqrt{5}$

Median $= \sqrt{5} = 2.24$ (3 s.f.)

d The distribution is negatively skewed, the mode > median and the mass of the distribution is concentrated at the right-hand end.

e $\mathrm{P}(k < X < k+1) = \mathrm{P}(X < k+1) - \mathrm{P}(X < k) = \mathrm{F}(k+1) - \mathrm{F}(k)$

So $\frac{1}{8}\left((k+1)^2 - 1\right) - \frac{1}{8}(k^2 - 1) = 0.6$

$\Rightarrow k^2 + 2k + 1 - 1 - k^2 + 1 = 4.8$

$\Rightarrow 2k = 3.8$

$\Rightarrow k = 1.9$

10 a $\mathrm{f}(x) = \frac{\mathrm{d}}{\mathrm{d}x}\mathrm{F}(x)$

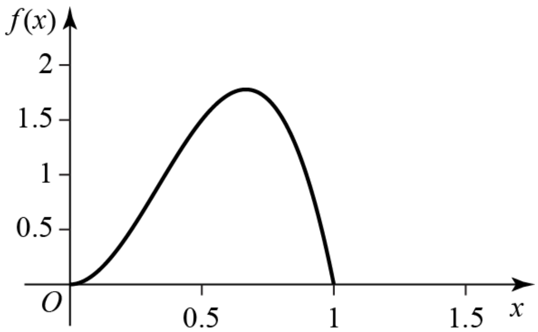
So where F(x) is constant, f(x) =0

For $0 \leqslant x \leqslant 1$, $\mathrm{f}(x) = \frac{\mathrm{d}}{\mathrm{d}x}(4x^3 - 3x^4) = 12x^2 - 12x^3$

So the probability density function is:

$$\mathrm{f}(x) = \begin{cases} 12x^2(1-x) & 0 \leqslant x \leqslant 1 \\ 0 & \text{otherwise} \end{cases}$$

10 b The graph of f(x) is negative cubic between $(0,0)$ and $(1,0)$ with a maximum between $x = 0$ and $x = 1$. Otherwise f(x) is 0



To find the mode solve $\frac{d}{dx}f(x)=0$

$$\frac{d}{dx}(12x^2-12x^3)=24x-36x^2=0$$
$$\Rightarrow 12x(2-3x)=0$$
$$\Rightarrow x=0 \text{ or } \frac{2}{3}$$

Checking whether f(x) is a maximum or minimum at these points:

$$f''(x)=\frac{d}{dx}(24x-36x^2)=24-72x$$

At $x=0$, $f''(x)=24$. As $f''(x)>0$, there is a minimum at this point

At $x=\frac{2}{3}$, $f''(x)=-24$. As $f''(x)<0$, there is a maximum at this point

So the mode $=\frac{2}{3}$

c $P(0.2<X<0.5)=F(0.5)-F(0.2)$
$$=(4\times 0.5^3-3\times 0.5^4)-(4\times 0.2^3-3\times 0.2^4)$$
$$=0.5-0.1875-0.032+0.0048=0.2853$$

11 a For $0\leqslant w\leqslant 5$, $F(w)=\int_0^w \frac{20}{5^5}t^3(5-t)\,dt=\left[\frac{100}{4\times 5^5}t^4-\frac{20}{5\times 5^5}t^5\right]_0^w=\frac{25}{5^5}w^4-\frac{4}{5^5}w^5=\frac{w^4}{5^5}(25-4w)$

So the full solution is:

$$F(w)=\begin{cases}0 & w<0\\ \frac{w^4}{5^5}(25-4w) & 0\leqslant w\leqslant 5\\ 1 & w>5\end{cases}$$

11 b $F(3.4) = \frac{3.4^4(25-13.6)}{5^5} = 0.4875$ (4 d.p.)

$F(3.5) = \frac{3.5^4(25-14)}{5^5} = 0.5282$ (4 d.p.)

So $F(3.4) < 0.5 < F(3.5)$, hence the median lies between 3.4 kg and 3.5 kg

c To find the mode, solve $\frac{d}{dx}f(w) = 0$

$\frac{d}{dx}\left(\frac{20}{5^5}w^3(5-w)\right) = \frac{60}{5^4}w^2 - \frac{80}{5^5}w^3$

$\Rightarrow \frac{20}{5^5}w^2(15-4w) = 0$

$\Rightarrow w = 0$ or $\frac{15}{4}$

$f'(w) > 0$ when $w < \frac{15}{4}$ and < 0 when $w > \frac{15}{4}$, so $w = \frac{15}{4}$ is a maximum

Hence mode = $\frac{15}{4}$

(Alternatively justify the maximum by sketching f(w) or showing that $f''(3.75) < 0$)

d Median , mode so negative skew.

12 a $E(X) = \int_0^1 \frac{x}{4}dx + \int_1^2 \frac{x^4}{5}dx = \left[\frac{x^2}{8}\right]_0^1 + \left[\frac{x^5}{25}\right]_1^2$

$= \frac{1}{8} + \frac{32}{25} - \frac{1}{25} = \frac{25+248}{200} = \frac{273}{200} = 1.365$

12 b If $x < 0$, $F(x) = 0$ so $F(0) = 0$

If $0 \leqslant x < 1$

$$F(x) = \int \frac{1}{4}\,dx = \frac{x}{4} + c$$

As $F(0) = 0 \Rightarrow c = 0$

If $1 \leqslant x \leqslant 2$

$$F(x) = \int \frac{x^3}{5}\,dx = \frac{x^4}{20} + d$$

As $F(2) = 1 \Rightarrow \frac{16}{20} + d = 1 \Rightarrow d = \frac{4}{20} = \frac{1}{5}$

So the full solution is:

$$F(x) = \begin{cases} 0 & x < 0 \\ \frac{x}{4} & 0 \leqslant x < 1 \\ \frac{x^4}{20} + \frac{1}{5} & 1 \leqslant x \leqslant 2 \\ 1 & x > 2 \end{cases}$$

c $F(m) = \frac{m^4}{20} + \frac{1}{5} = 0.5$

$m^4 + 4 = 10$

$m^4 = 6$

$m = 1.565$ (3 d.p.)

So median = 1.565 (3 d.p.)

Lower quartile

$\frac{Q_1^4}{20} + \frac{1}{5} = 0.25$

$Q_1^4 + 4 = 5$

$Q_1 = 1$

Upper quartile

$\frac{Q_3^4}{20} + \frac{1}{5} = 0.75$

$Q_3^4 + 4 = 15$

$Q_3^4 = 11$

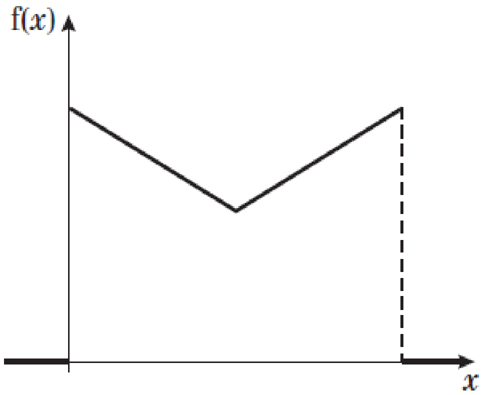
$Q_3 = 1.8212$ (4 d.p.) (Note -1.8212 is not in range)

Interquartile range = 1.8212 – 1 = 0.821 (3 d.p.)

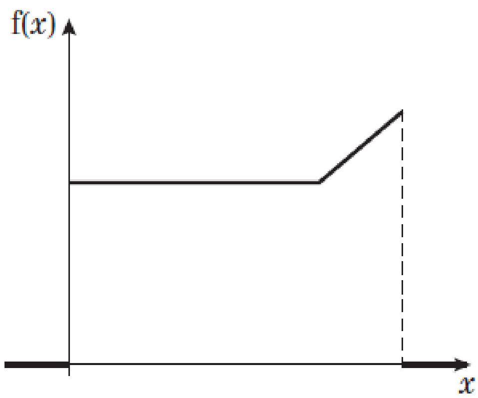
d The mean (1.365) < median (1.565) so the distribution is negatively skewed.

13 There are many possible answers. The sketches below show one set of graphs that satisfy the respective conditions:

a The mode $\neq$ median because there is no maximum.

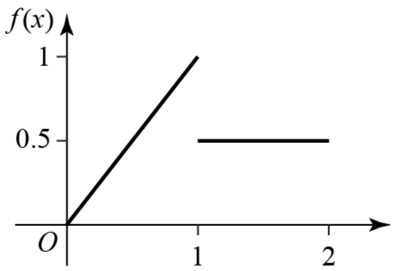


b The mode lies outside the interquartile range because the maximum is at an end point.



14 There are many possible answers. Consider this function:

$$f(x)=\begin{cases} x & 0\leqslant x\leqslant 1\\ \frac{1}{2} & 1<x\leqslant 2\\ 0 & \text{otherwise}\end{cases}$$



It is a probability distribution function as:

$$\int_0^1 x\,dx+\int_1^2 \frac{1}{2}dx=\left[\frac{x^2}{2}\right]_0^1+\left[\frac{x}{2}\right]_1^2=\frac{1}{2}+1-\frac{1}{2}=1$$

The cumulative distribution function is:

$$F(x)=\begin{cases} 0 & x<0\\ \frac{x^2}{2} & 0\leqslant x<1\\ \frac{x}{2} & 1<x\leqslant 2\\ 1 & x>2\end{cases}$$

From the sketch, the mode = 1. From the cumulative distribution function F(1) = 0.5, so the median is 1 and the median and the mode are therefore equal.

15 a f(x) is a continuously decreasing function in the range $2\leqslant x\leqslant 10$, as x increases $\frac{1}{x\ln 5}$ decreases

So the mode occurs at $x = 2$

b For $2\leqslant x\leqslant 10$, $F(x)=\int_2^x \frac{1}{t\ln 5}\,dt=\left[\frac{\ln t}{\ln 5}\right]_2^x=\frac{\ln x}{\ln 5}-\frac{\ln 2}{\ln 5}=\frac{\ln x-\ln 2}{\ln 5}=\frac{\ln(0.5x)}{\ln 5}$

So the full solution is

$$F(x)=\begin{cases} 0 & x<2\\ \frac{\ln(0.5x)}{\ln 5} & 2\leqslant x\leqslant 10\\ 1 & x>10\end{cases}$$

15 c $\mathrm{F}(m) = \dfrac{\ln 0.5m}{\ln 5} = \dfrac{\ln m - \ln 2}{\ln 5} = 0.5$

$\Rightarrow \ln m = 0.5\ln 5 + \ln 2 = \ln(2\sqrt{5})$

$\Rightarrow m = 2\sqrt{5}$

d Lower quartile

$\dfrac{\ln Q_1 - \ln 2}{\ln 5} = 0.25$

$\Rightarrow \ln Q_1 = 0.25\ln 5 + \ln 2 = 1.09551 \text{ (5 d.p.)}$

$\Rightarrow Q_1 = \mathrm{e}^{1.09551} = 2.9907 \text{ (4 d.p.)}$

Upper quartile

$\dfrac{\ln Q_3 - \ln 2}{\ln 5} = 0.75$

$\Rightarrow \ln Q_3 = 0.75\ln 5 + \ln 2 = 1.90023 \text{ (5 d.p.)}$

$\Rightarrow Q_1 = e^{1.90023} = 6.6874 \text{ (4 d.p.)}$

Interquartile range = 6.6874 – 2.9907 = 3.697 (3 d.p.)

16 a For $x \geqslant 0$, $\mathrm{F}(x) = \int_0^x 2.5\mathrm{e}^{-2.5t}\,\mathrm{d}t = \left[-\mathrm{e}^{-2.5t}\right]_0^x = 1 - \mathrm{e}^{-2.5x}$

So the cumulative distribution function is

$$\begin{cases} 0 & x < 0 \\ 1 - \mathrm{e}^{-2.5x} & x \geqslant 0 \end{cases}$$

$\mathrm{F}(m) = 1 - \mathrm{e}^{-2.5m} = 0.5$

$\Rightarrow \mathrm{e}^{-2.5m} = 0.5$

$\Rightarrow -2.5m = \ln 0.5 = -0.6931$

$\Rightarrow m = 0.277 \text{ (3 d.p.)}$

So median is 277 hours (to the nearest hour)

16 b Lower quartile

$1-e^{-2.5Q_1}=0.25$

$\Rightarrow e^{-2.5Q_1}=0.75$

$\Rightarrow -2.5Q_1=\ln 0.75=-0.28768$

$\Rightarrow Q_1=0.1151$ (4 d.p.)

Upper quartile

$1-e^{-2.5Q_3}=0.75$

$\Rightarrow e^{-2.5Q_3}=0.25$

$\Rightarrow -2.5Q_3=\ln 0.25=-1.38629$

$\Rightarrow Q_3=0.5545$ (4 d.p.)

Interquartile range = 0.5545 – 0.1151 = 0.439 (3 d.p.)

So to the nearest hour, the lower quartile is 115 hours, the upper quartile is 555 hours and the interquartile range is 439 hours.

17 a The area under the curve must equal 1, so:

$$\int_0^{0.25} k\sec^2(\pi x)\,dx=1$$

$$\Rightarrow \left[\frac{k}{\pi}\tan(\pi x)\right]_0^{0.25}=1$$

$$\Rightarrow \frac{k}{\pi}\left(\tan(0.25\pi)-\tan 0\right)=\frac{k}{\pi}=1$$

$$\Rightarrow k=\pi$$

b For $0\leqslant x\leqslant 2.5$, $\mathrm{F}(x)=\int_0^x \pi\sec^2(\pi t)\,dt=\left[\tan(\pi t)\right]_0^x=\tan(\pi x)$

So the cumulative distribution function is

$$\mathrm{F}(x)=\begin{cases}0 & x<0\\ \tan(\pi x) & 0\leqslant x\leqslant 0.25\\ 1 & x>0.25\end{cases}$$

c $\mathrm{F}(m)=\tan(\pi m)=0.5$

$\Rightarrow \pi m=0.4636\Rightarrow m=0.1476$ (4 d.p.)

18 a The area under the curve must equal 1, so:

$$\int_2^4 \frac{k}{x(5-x)}\,dx = 1$$

$$\Rightarrow k\int_2^4 \frac{1}{5x} + \frac{1}{5(5-x)}\,dx = 1$$

$$\Rightarrow \frac{k}{5}\left[\ln x - \ln(5-x)\right]_2^4 = 1$$

$$\Rightarrow k(\ln 4 - \ln 2 + \ln 3) = 5$$

$$\Rightarrow k\ln\left(\frac{4\times 3}{2}\right) = k\ln 6 = 5$$

$$\Rightarrow k = \frac{5}{\ln 6}$$

b $$\mathrm{E}(X) = \frac{5}{\ln 6}\int_2^4 \frac{x}{x(5-x)}\,dx = \frac{5}{\ln 6}\int_2^4 \frac{1}{(5-x)}\,dx$$

$$= \frac{5}{\ln 6}\left[-\ln(5-x)\right]_2^4 = \frac{5\ln 3}{\ln 6} = 3.0657\ldots = 3.066 \text{ (3 d.p.)}$$

c $$\mathrm{E}(X^2) = \frac{5}{\ln 6}\int_2^4 \frac{x^2}{x(5-x)}\,dx = \frac{5}{\ln 6}\int_2^4 \frac{x}{(5-x)}\,dx = \frac{5}{\ln 6}\int_2^4 -1 + \frac{5}{(5-x)}\,dx$$

$$= \frac{5}{\ln 6}\left[-x - 5\ln(5-x)\right]_2^4 = \frac{5\ln 3}{\ln 6} = \frac{5}{\ln 6}(-4+2+5\ln 3) = \frac{5(5\ln 3 - 2)}{\ln 6} = 9.7476 \text{ (4 d.p.)}$$

$$\mathrm{Var}(X) = \mathrm{E}(X^2) - (\mathrm{E}(X))^2 = 9.7476 - (3.0657)^2 = 0.349 \text{ (3 d.p.)}$$

d For $2 \leqslant x \leqslant 4$, $\mathrm{F}(x) = \frac{5}{\ln 6}\int_2^x \frac{1}{t(5-t)}\,dt = \frac{5}{\ln 6}\int_2^x \frac{1}{5t} + \frac{1}{5(5-t)}\,dt$

$$= \frac{1}{\ln 6}\left[\ln t - \ln(5-t)\right]_2^x = \frac{1}{\ln 6}\left(\ln x - \ln(5-x) - \ln 2 + \ln 3\right) = \frac{1}{\ln 6}\left(\ln\left(\frac{3x}{2(5-x)}\right)\right)$$

So the cumulative distribution function is

$$\mathrm{F}(x) = \begin{cases} 0 & x < 2 \\ \frac{1}{\ln 6}\left(\ln\left(\frac{3x}{10-2x}\right)\right) & 2 \leqslant x \leqslant 4 \\ 1 & x > 4 \end{cases}$$

18 e $\mathrm{F}(m) = \frac{1}{\ln 6}\left(\ln \frac{3m}{10-2m}\right) = 0.5$

$\Rightarrow \ln\left(\frac{3m}{10-2m}\right) = 0.5\ln 6 = \ln\sqrt{6}$

$\Rightarrow \frac{3m}{10-2m} = \sqrt{6}$

$\Rightarrow m = \frac{10\sqrt{6}}{3+2\sqrt{6}} = 3.101$ (3 d.p.)

f 4 because the pdf curve is U-shaped and the maximum value of the pdf is at the end-point 4.

g As the mean < median < mode, the distribution is negatively skewed.