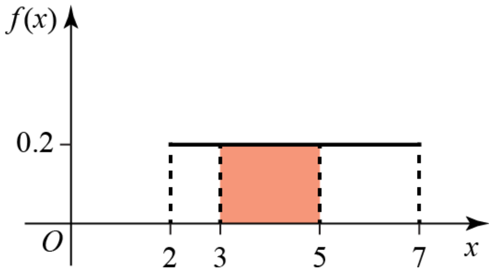
Continuous distributions 3E

1 a $\frac{1}{b-a}=\frac{1}{7-2}=0.2$

So a sketch of the probability distribution function (with the required probability shaded) is:



$P(3<X<5)=(5-3)\times 0.2=0.4$

b $P(X>4)=(7-4)\times 0.2=0.6$

2 a Area under the probability distribution function curve is 1, so:

$(k-2.6)\times 0.1=1$

$(k-2.6)=10$

$k=12.6$

b $P(4<X<7.9)=(7.9-4)\times 0.1=0.39$

3 a Area under the probability distribution function curve is 1, so:

$k\times(6-(-2))=1$

$8k=1$

$k=\frac{1}{8}$

b $P(-1.3<X<4.2)=\frac{1}{8}\times(4.2-(-1.3))=\frac{1}{8}\times 5.5=0.6875$

c $3X<X+p\Rightarrow X<\frac{p}{2}$

$P\left(X<\frac{p}{2}\right)=0.5$

$\Rightarrow\frac{p}{2}=2\Rightarrow p=4$

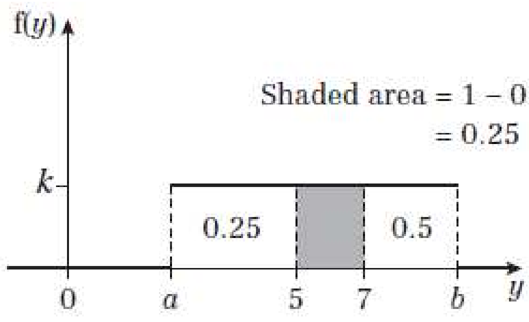
d Using $P(A|B)=\frac{P(A\cap B)}{P(B)}$

$P(X>5|X>0)=\frac{P(X>5\cap X>0)}{P(X>0)}=\frac{P(X>5)}{P(X>0)}=\frac{\frac{1}{8}}{\frac{6}{8}}=\frac{1}{6}$

3 e $P(X>0|X<3)=\dfrac{P(X>0\cap X<3)}{P(X<3)}=\dfrac{P(0<X<3)}{P(X<3)}=\dfrac{\frac{3}{8}}{\frac{5}{8}}=\dfrac{3}{5}$

f $P(X<1|0<X<2)=\dfrac{P(X<1\cap 0<X<2)}{P(0<X<2)}=\dfrac{P(0<X<1)}{P(0<X<2)}=\dfrac{\frac{1}{8}}{\frac{2}{8}}=\dfrac{1}{2}$

4 Sketching the probability distribution function:



$P(5<Y<7)=1-P(Y<5)-P(Y>7)=1-0.25-0.5=0.25$

But from the sketch, $P(5<Y<7)=(7-5)\times k=2k$

So $2k=0.25\Rightarrow k=0.125=\dfrac{1}{8}$

$P(X>7)=(b-7)k=\dfrac{(b-7)}{8}=\dfrac{1}{2}\Rightarrow b=4+7=11$

$P(X<5)=(5-a)k=\dfrac{(5-a)}{8}=\dfrac{1}{4}\Rightarrow a=5-2=3$

5 a As $2\times2+5=9$ and $2\times8+5=21$

Hence $Y\sim U[9,21]$

b For Y, $\dfrac{1}{b-a}=\dfrac{1}{21-9}=\dfrac{1}{12}$

$P(12<Y<20)=(20-12)\times\dfrac{1}{12}=\dfrac{2}{3}$

6 a Continuous uniform distribution

b As $20-2\times2=16$ and $20-2\times12=-4$

Hence $Y\sim U[-4,16]$

By symmetry, $E(Y)=\dfrac{16+(-4)}{2}=\dfrac{12}{2}=6$

c For Y, $\dfrac{1}{b-a}=\dfrac{1}{16-(-4)}=\dfrac{1}{20}$

$P(Y<4)=(4-(-4))\times\dfrac{1}{20}=\dfrac{8}{20}=\dfrac{2}{5}$

6 d If $X < 10$, then $Y > 10$, so

$$P(Y>4 \mid X<10) == P(Y>4 \mid Y>0) \frac{P(Y>4 \cap Y>0)}{P(Y>0)} = \frac{P(Y>4)}{P(Y>0)} = \frac{\frac{12}{20}}{\frac{16}{20}} = \frac{12}{16} = \frac{3}{4}$$

7 a $E(X) = \frac{5+(-3)}{2} = 1$

b $\text{Var}(X) = \frac{(b-a)^2}{12} = \frac{(5-(-3))^2}{12} = \frac{64}{12} = \frac{16}{3}$

c $\text{Var}(X) = E(X^2) - (E(X))^2$, so $E(X^2) = \frac{16}{3} + 1^2 = \frac{19}{3}$

d For $-3 \leqslant x \leqslant 5$, $F(x) = \int_{-3}^{x} \frac{1}{b-a} dt = \int_{-3}^{x} \frac{1}{8} dt = \left[\frac{t}{8}\right]_{-3}^{x} = \frac{x}{8} + \frac{3}{8} = \frac{x+3}{8}$

So the cumulative distribution function is

$$F(x) = \begin{cases} 0 & x < -3 \\ \frac{x+3}{8} & -3 \leqslant x \leqslant 5 \\ 1 & x > 5 \end{cases}$$

8 a $E(X) = \frac{a+b}{2} = \frac{5+1}{2} = 3$ $\quad \text{Var}(X) = \frac{(b-a)^2}{12} = \frac{(5-1)^2}{12} = \frac{16}{12} = \frac{4}{3}$

b $E(X) = \frac{a+b}{2} = \frac{6+(-2)}{2} = 2$ $\quad \text{Var}(X) = \frac{(b-a)^2}{12} = \frac{(6-(-2))^2}{12} = \frac{64}{12} = \frac{16}{3}$

9 a $E(X) = \frac{5.5+3.5}{2} = 4.5$

b $\text{Var}(X) = \frac{(5.5-3.5)^2}{12} = \frac{4}{12} = \frac{1}{3}$

c $\text{Var}(X) = E(X^2) - (E(X))^2$

So $\frac{1}{3} = E(X^2) - 4.5^2$

$\Rightarrow E(X^2) = 20.25 + \frac{1}{3} = \frac{247}{12} = 20.6$ (3 s.f.)

d For $3.5 \leqslant x \leqslant 5.5$, $F(x) = \int_{3.5}^{x} \frac{1}{b-a} dt = \int_{3.5}^{x} \frac{1}{2} dt = \left[\frac{t}{2}\right]_{3.5}^{x} = \frac{x}{2} - \frac{7}{4} = \frac{2x-7}{4}$

So the cumulative distribution function is

$$F(x) = \begin{cases} 0 & x < 3.5 \\ \frac{2x-7}{4} & 3.5 \leqslant x \leqslant 5.5 \\ 1 & x > 5.5 \end{cases} = \begin{cases} 0 & x < 3.5 \\ 0.5x - 1.75 & 3.5 \leqslant x \leqslant 5.5 \\ 1 & x > 5.5 \end{cases}$$

10 $\text{E}(Y)=\frac{a+b}{2}=1 \Rightarrow b=2-a$ **(1)**

$\text{Var}(Y)=\frac{(b-a)^2}{12}=\frac{4}{3} \Rightarrow (b-a)^2=16$ **(2)**

Substituting equation (**1**) in equation (**2**) gives:

$(2-a-a)^2=16$

$\Rightarrow 2-2a=\pm 4$

$\Rightarrow a=3 \text{ or } -1$

If $a=3, b=2-3=-1$

If $a=-1, b=2-(-1)=3$

As $b>a,$ the solution is $a=-1$ and $b=3$

11 $\text{E}(X)=\frac{5+(-1)}{2}=2$

$\text{Var}(X)=\frac{(5-(-1))^2}{12}=3$

$\text{E}(Y)=\text{E}(4X-6)=4\text{E}(X)-6=8-6=2$

$\text{Var}(Y)=\text{Var}(4X-6)=4^2\,\text{Var}(X)=16\times 3=48$

12 a $\text{P}\left(X<\frac{3}{7}\alpha+\frac{4}{7}\beta\right)=\frac{\frac{3}{7}\alpha+\frac{4}{7}\beta-\alpha}{\beta-\alpha}=\frac{\frac{4}{7}(\beta-\alpha)}{\beta-\alpha}=\frac{4}{7}$

b $\text{P}\left(X>\frac{2}{5}\alpha+\frac{3}{5}\beta\right)=\frac{\beta-\frac{2}{5}\alpha-\frac{3}{5}\beta}{\beta-\alpha}=\frac{\frac{2}{5}(\beta-\alpha)}{\beta-\alpha}=\frac{2}{5}$

13 a $\text{E}(R)=\frac{\alpha+\beta}{2}=5 \Rightarrow \beta=10-\alpha$

$\text{Var}(R)=\frac{(\beta-\alpha)^2}{12}=\frac{4}{3} \Rightarrow (10-2\alpha)^2=16 \Rightarrow \alpha=5\pm 2 \Rightarrow \alpha=3 \text{ or } 7$

If $\alpha=7, \beta=3$; if $\alpha=3, \beta=7$

But $\alpha<\beta$ so the solution is $\alpha=3, \beta=7$

b $\text{P}(R<5.2)=\frac{5.2-\alpha}{\beta-\alpha}=\frac{2.2}{4}=0.55 \quad \frac{2.2}{4}=0.55$

14 a The probability density function of X is

$$\text{f}(x)=\begin{cases}\frac{1}{\beta-\alpha} & \alpha\leqslant x\leqslant\beta\\ 0 & \text{otherwise}\end{cases}$$

14 b $\text{E}(X)=\frac{\alpha+\beta}{2}=2.5\Rightarrow\beta=5-\alpha$

$\text{P}(X<1)=\frac{1-\alpha}{\beta-\alpha}=\frac{4}{11}\Rightarrow\frac{1-\alpha}{5-2\alpha}=\frac{4}{11}$ substituting for β

$\Rightarrow 11-11\alpha=20-8\alpha$

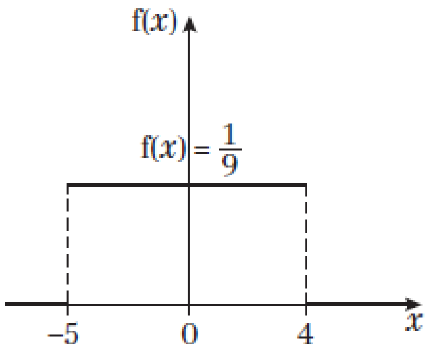
$\Rightarrow 3\alpha=-9$

$\Rightarrow \alpha=-3, \beta=5-\alpha=8$

15 a The probability density function of X is

$$\text{f}(x)=\begin{cases}\frac{1}{9} & -5\leqslant x\leqslant 4\\ 0 & \text{otherwise}\end{cases}$$

b



c $\text{E}(X)=\frac{a+b}{2}=\frac{4+(-5)}{2}=-\frac{1}{2}$ $\text{Var}(X)=\frac{(b-a)^2}{12}=\frac{(4-(-5))^2}{12}=\frac{81}{12}=\frac{27}{4}$

$\text{Var}(X)=\text{E}(X^2)-(\text{E}(X))^2$

So $\frac{27}{4}=\text{E}(X^2)-\frac{1}{2^2}\Rightarrow\text{E}(X^2)=\frac{28}{4}=7$

d $\text{P}(-0.2<X<0.6)=\frac{0.6-(-0.2)}{4-(-5)}=\frac{0.8}{9}=\frac{4}{45}$

16 a $\text{P}(X<0)=\text{F}(0)=\frac{3}{7}$

b $\text{f}(x)=\frac{\text{d}}{\text{d}x}\text{F}(x)$

So the probability density function is:

$$\text{f}(x)=\begin{cases}\frac{1}{7} & -3\leqslant x\leqslant 4\\ 0 & \text{otherwise}\end{cases}$$

c $X\sim\text{U}[-3,4]$, it has a continuous uniform distribution

16 d $\text{E}(X)=\dfrac{a+b}{2}=\dfrac{4+(-3)}{2}=\dfrac{1}{2}$

$\text{Var}(X)=\dfrac{(b-a)^2}{12}=\dfrac{(4-(-3))^2}{12}=\dfrac{49}{12}$

17 a $\text{E}(X)=\dfrac{4+(-1)}{2}=\dfrac{3}{2}=1.5\ \dfrac{-1+4}{2}=1.5$

b $\text{Var}(X)=\dfrac{(4-(-1))^2}{12}=\dfrac{25}{12}$

c $\text{Var}(X)=\text{E}(X^2)-(\text{E}(X))^2$

So $\dfrac{25}{12}=\text{E}(X^2)-\dfrac{9}{4}\Rightarrow \text{E}(X^2)=\dfrac{25+27}{12}=\dfrac{52}{12}=\dfrac{13}{3}$

d $\text{P}(X<1.4)=\dfrac{1.4-(-1)}{4-(-1)}=\dfrac{2.4}{5}=0.48$

e Let the random variable Y represent the number of times that X is observed to be less than 1.4 in 6 observations, then using the result from part **d**, $Y\sim \text{B}(6,0.48)$

$\text{P}(Y=4)=\binom{6}{4}0.48^4(1-0.48)^2=\dfrac{6\times 5}{2}\times 0.48^4\times 0.52^2=0.2153\text{ (4 d.p.)}$

18 a $\text{E}(X)=\dfrac{\alpha+\beta}{2}=7.5\Rightarrow \beta=15-\alpha$

$P(X\leqslant 10.5)=1-P(X>10.5)=1-0.25=0.75$

$P(X\leqslant 10.5)=\dfrac{10.5-\alpha}{\beta-\alpha}=0.75\Rightarrow \dfrac{10.5-\alpha}{15-2\alpha}=0.75$ substituting for β

$\Rightarrow 10.5-\alpha=11.25-1.5\alpha$

$\Rightarrow \alpha=1.5,\ \beta=15-\alpha=13.5$

b i $\text{P}(x<c)=\dfrac{c-\alpha}{\beta-\alpha}=\dfrac{c-1.5}{12}=\dfrac{1}{3}\Rightarrow c=5.5$

ii $\text{P}(c<X<9)=\text{P}(5.5<X<9)=\dfrac{3.5}{12}=\dfrac{7}{24}$