Continuous distributions Mixed exercise 3

1 a $\text{E}(X)=\int x\,\text{f}(x)\text{d}x=\int_0^2 \frac{x}{3}\left(1+\frac{x}{2}\right)\text{d}x=\int_0^2 \frac{x}{3}+\frac{x^2}{6}\,\text{d}x$

$=\left[\frac{x^2}{6}+\frac{x^3}{18}\right]_0^2=\frac{2^2}{6}+\frac{2^3}{18}=\frac{2}{3}+\frac{4}{9}=\frac{10}{9}$

$\text{E}(3X+2)=3\,\text{E}(X)+2=3\times\frac{10}{9}+2=\frac{30+18}{9}=\frac{48}{9}=\frac{16}{3}$

b $\text{Var}(X)=\int x^2\,\text{f}(x)\text{d}x-(\text{E}(X))^2=\int_0^2 \frac{x^2}{3}\left(1+\frac{x}{2}\right)\text{d}x-\left(\frac{10}{9}\right)^2=\int_0^2 \frac{x^2}{3}+\frac{x^3}{6}\,\text{d}x-\frac{100}{81}$

$=\left[\frac{x^3}{9}+\frac{x^4}{24}\right]_0^2-\frac{100}{81}=\frac{2^3}{9}+\frac{2^4}{24}-\frac{100}{81}=\frac{8}{9}+\frac{2}{3}-\frac{100}{81}=\frac{72+54-100}{81}=\frac{26}{81}$

$\text{Var}(3X+2)=3^2\,\text{Var}(X)=9\times\frac{26}{81}=\frac{26}{9}$

c $\text{P}(X<1)=\int_0^1 \frac{1}{3}\left(1+\frac{x}{2}\right)\text{d}x=\int_0^1 \frac{1}{3}+\frac{x}{6}\text{d}x$

$=\left[\frac{x}{3}+\frac{x^2}{12}\right]_0^1=\frac{1}{3}+\frac{1}{12}=\frac{5}{12}$

d $\text{P}(X>\text{E}(X))=\text{P}\left(X>\frac{10}{9}\right)=1-\text{P}\left(X<\frac{10}{9}\right)$

$=1-\int_0^{\frac{10}{9}} \frac{1}{3}\left(1+\frac{x}{2}\right)\text{d}x=1-\int_0^{\frac{10}{9}} \frac{1}{3}+\frac{x}{6}\text{d}x=1-\left[\frac{x}{3}+\frac{x^2}{12}\right]_0^{\frac{10}{9}}$

$=1-\left(\frac{10}{27}+\frac{100}{972}\right)=1-\left(\frac{90}{243}+\frac{25}{243}\right)=1-\frac{115}{243}=\frac{128}{243}$

e $\text{P}(0.5<X<1.5)=\int_{0.5}^{1.5}\left(\frac{1}{3}+\frac{x}{6}\right)\text{d}x=\left[\frac{x}{3}+\frac{x^2}{12}\right]_{0.5}^{1.5}$

$=\left(\frac{1.5}{3}+\frac{1.5^2}{12}\right)-\left(\frac{0.5}{3}+\frac{0.5^2}{12}\right)=\frac{3}{6}+\frac{9}{48}-\frac{1}{6}-\frac{1}{48}$

$=\frac{2}{6}+\frac{8}{48}=\frac{2}{6}+\frac{1}{6}=\frac{3}{6}=0.5$

2 a $\text{E}(X)=\int x\,\text{f}(x)\text{d}x=\int_0^1 2x-2x^2\,\text{d}x=\left[x^2-\frac{2}{3}x^3\right]_0^1=1-\frac{2}{3}=\frac{1}{3}$

2 b $\text{Var}(X) = \int x^2 \, \text{f}(x)\text{d}x - (\text{E}(X))^2 = \int_0^1 2x^2 - 2x^3 \text{d}x - \left(\frac{1}{3}\right)^2$

$$= \left[\frac{2}{3}x^3 - \frac{1}{2}x^4\right]_0^1 - \frac{1}{9} = \frac{2}{3} - \frac{1}{2} - \frac{1}{9} = \frac{12-9-2}{18} = \frac{1}{18}$$

c $\text{E}(2X+1) = 2\text{E}(X) + 1 = 2 \times \frac{1}{3} + 1 = \frac{5}{3}$

$\text{Var}(2X+1) = 2^2 \text{Var}(X) = \frac{4}{18} = \frac{2}{9}$

d ***Method 1***

$\text{F}(x) = \int_0^x (2-2t) \text{ d}t = \left[2t - t^2\right]_0^x = 2x - x^2$

Method 2

$\text{F}(x) = \int 2 - 2x\text{d}x = 2x - x^2 + c$

$\text{F}(2) = 1$, so $2 - 1 + c = 1 \Rightarrow c = 0$

So the cumulative distribution function is:

$$\text{F}(x) = \begin{cases} 0 & x < 0 \\ 2x - x^2 & 0 \leqslant x \leqslant 1 \\ 1 & x > 1 \end{cases}$$

e $\text{F}(m) = 0.5$, so $2m - m^2 = 0.5 \Rightarrow 2m^2 - 4m + 1 = 0$

$m = \frac{4 \pm \sqrt{16-8}}{4} = 1 \pm \frac{\sqrt{2}}{2}$

As $0 \leqslant m \leqslant 1$, $m = 1 - \frac{\sqrt{2}}{2} = 0.293$ (3 s.f.)

3 a As $\text{F}(2) = 1$, $\text{F}(2) = k(4-2) = 1 \Rightarrow k = \frac{1}{2}$

b $\text{P}(Y < 1.5) = \text{F}(1.5) = \frac{1}{2} \times (1.5^2 - 1.5) = \frac{1}{2}\left(\frac{9}{4} - \frac{3}{2}\right) = \frac{3}{8} = 0.375$

c $\text{F}(m) = 0.5$, so $\frac{1}{2}(m^2 - m) = 0.5 \Rightarrow m^2 - m - 1 = 0$

$m = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1 \pm \sqrt{5}}{2}$

As $1 \leqslant m \leqslant 2$, $m = \frac{1+\sqrt{5}}{2} = 1.62$ (3 s.f.)

3 d Using $\frac{d}{dy}F(y) = f(y)$

$$\frac{d}{dy}\left(\frac{1}{2}(y^2 - y)\right) = y - \frac{1}{2}$$

So the probability density function is:

$$f(y) = \begin{cases} y - \frac{1}{2} & 1 \leqslant y \leqslant 2 \\ 0 & \text{otherwise} \end{cases}$$

4 a $P(X > 2.4) = 1 - P(X < 2.4) = 1 - F(2.4) = 1 - \frac{1}{5}(2.4^2 - 4) = 0.648$

Alternative method:

$$P(X > 2.4) = P(X < 3) - P(X < 2.4) = F(3) - F(2.4)$$
$$= \frac{1}{5}(3^2 - 4) - \frac{1}{5}(2.4^2 - 4) = 0.648$$

b $F(m) = 0.5$, so $\frac{1}{5}(m^2 - 4) = 0.5$

$\Rightarrow 2(m^2 - 4) = 5$ multiplying both sides by 10

$\Rightarrow 2m^2 = 13$

$\Rightarrow m = \sqrt{\frac{13}{2}} = 2.55$ (3 s.f.) as $-\sqrt{\frac{13}{2}}$ is not in the range $2 \leqslant m \leqslant 3$

c Using $\frac{d}{dx}F(x) = f(x)$

$$\frac{d}{dx}\left(\frac{1}{5}(x^2 - 4)\right) = \frac{2x}{5}$$

So the probability density function is:

$$f(x) = \begin{cases} \frac{2x}{5} & 2 \leqslant x \leqslant 3 \\ 0 & \text{otherwise} \end{cases}$$

d $E(X) = \int x\,f(x)\,dx = \int_2^3 \frac{2}{5}x^2\,dx = \left[\frac{2}{15}x^3\right]_2^3 = \frac{2}{15}(27 - 8) = \frac{38}{15}$

e The mode occurs at the maximum point of the probability density function graph. As f(x) is strictly increasing on the interval [2, 3] and 0 elsewhere, the mode must be 3.

5 a The area under the probability density function graph must be 1, so:

$$\int_0^2 kx^2 \, dx = 1 \Rightarrow \left[\frac{kx^3}{3}\right]_0^2 = 1$$

So $\frac{8k}{3} = 1 \Rightarrow k = \frac{3}{8}$

b $\mathrm{E}(X) = \int x \, \mathrm{f}(x) \mathrm{d}x = \int_0^2 \frac{3x^3}{8} \mathrm{d}x = \left[\frac{3x^4}{32}\right]_0^2 = \frac{3}{2} = 1.5$

c ***Method 1***

$$\mathrm{F}(x) = \int_0^x \frac{3}{8} t^2 \, \mathrm{d}t = \left[\frac{1}{8} t^3\right]_0^x = \frac{x^3}{8}$$

Method 2

$$\mathrm{F}(x) = \int \frac{3}{8} x^2 \mathrm{d}x = \frac{x^3}{8} + c$$

$$\mathrm{F}(2) = 1, \text{ so } \frac{2^3}{8} + c = 1 \Rightarrow c = 0$$

So the cumulative distribution function is:

$$\mathrm{F}(x) = \begin{cases} 0 & x < 0 \\ \frac{x^3}{8} & 0 \leqslant x \leqslant 2 \\ 1 & x > 2 \end{cases}$$

d $\mathrm{F}(m) = 0.5, \text{ so } \frac{m^3}{8} = 0.5$

$\Rightarrow m^3 = 4 \Rightarrow m = 1.59$ (3 s.f.)

e The mode occurs at the maximum point of the probability density function graph. As f(x) is strictly increasing on the interval [0, 2] and 0 elsewhere, the mode must be 2.

6 a The area under the probability density function graph must be 1, so:

$$\int_1^3 k(y^2 + 2y + 2) \, \mathrm{d}y = 1$$

$$\Rightarrow \left[k\left(\frac{y^3}{3} + y^2 + 2y\right)\right]_1^3 = 1$$

$$\Rightarrow k\left(\frac{3^3}{3} + 3^2 + 6\right) - k\left(\frac{1}{3} + 1 + 2\right) = 1$$

$$\Rightarrow k\left(24 - \frac{10}{3}\right) = 1 \Rightarrow \frac{62}{3} k = 1$$

So $k = \frac{3}{62}$

6 b $\text{F}(y)=\int_1^y \frac{3}{62}(t^2+2t+2)\,\text{d}t=\left[\frac{3}{62}\left(\frac{t^3}{3}+t^2+2t\right)\right]_1^y$

$$=\frac{3}{62}\left(\frac{y^3}{3}+y^2+2y\right)-\frac{3}{62}\left(\frac{1}{3}+1+2\right)=\frac{y^3}{62}+\frac{3y^2}{62}+\frac{3y}{31}-\frac{5}{31}$$

So the cumulative distribution function is:

$$\text{F}(y)=\begin{cases} 0 & y<1 \\ \frac{y^3}{62}+\frac{3y^2}{62}+\frac{3y}{31}-\frac{5}{31} & 1\leqslant y\leqslant 3 \\ 1 & y>3 \end{cases}$$

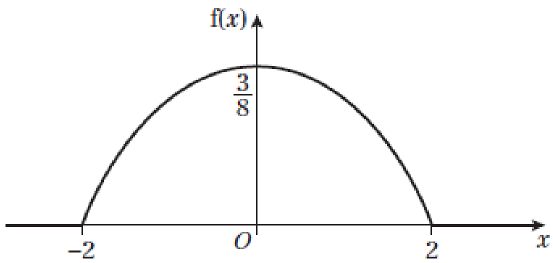
c $\text{P}(Y\leqslant 2)=\text{F}(2)=\frac{2^3}{62}+\frac{3\times 2^2}{62}+\frac{6}{31}-\frac{5}{31}=\frac{4+6+6-5}{31}=\frac{11}{31}$

Alternatively, the probability can be derived from the probability density function as follows:

$$\text{P}(Y\leq 2)=\int_1^2 \frac{3}{62}(y^2+2y+2)\,\text{d}y=\left[\frac{3}{62}\left(\frac{y^3}{3}+y^2+2y\right)\right]_1^2$$

$$=\frac{3}{62}\left(\frac{2^3}{3}+2^2+4\right)-\frac{3}{62}\left(\frac{1}{3}+1+2\right)=\frac{11}{31}$$

7 a The graph of the probability density function is a quadratic with a negative x^2 coefficient between (–2, 0) and (2, 0), with a maximum at (0, 0.375); otherwise it is 0. The sketch of the graph is:



b The mode occurs at the maximum point of the probability density function graph. From the graph, this occurs when $x = 0$. So the mode is 0.

7 c $\mathrm{F}(x)=\int_{-2}^{x}\frac{3}{32}(4-t^2)\,\mathrm{d}t=\left[\frac{12t}{32}-\frac{t^3}{32}\right]_{-2}^{x}$

$$=\left(\frac{12x}{32}-\frac{x^3}{32}\right)-\left(-\frac{24}{32}+\frac{8}{32}\right)=\frac{12x}{32}-\frac{x^3}{32}+\frac{1}{2}$$

So the cumulative distribution function is:

$$\mathrm{F}(x)=\begin{cases}0 & x<-2\\ \frac{12x}{32}-\frac{x^3}{32}+\frac{1}{2} & -2\leqslant x\leqslant 2\\ 1 & x>2\end{cases}$$

d $\mathrm{P}(0.5<X<1.5)=\mathrm{P}(X<1.5)-\mathrm{P}(X<0.5)=\mathrm{F}(1.5)-\mathrm{F}(0.5)$

$$=\left(\frac{18}{32}-\frac{1.5^3}{32}+\frac{1}{2}\right)-\left(\frac{6}{32}-\frac{0.5^3}{32}+\frac{1}{2}\right)=\frac{12}{32}-\frac{26}{256}=\frac{96-26}{256}=\frac{35}{128}=0.273\text{ (3 s.f.)}$$

8 a $\mathrm{E}(X)=\int x\,\mathrm{f}(x)\mathrm{d}x=\int_0^1\frac{x}{3}\mathrm{d}x+\int_1^2\frac{2x^3}{7}\,\mathrm{d}x$

$$=\left[\frac{x^2}{6}\right]_0^1+\left[\frac{2x^4}{28}\right]_1^2=\frac{1}{6}+\left(\frac{32}{28}-\frac{2}{28}\right)$$

$$=\frac{1}{6}+\frac{15}{14}=\frac{7}{42}+\frac{45}{42}=\frac{52}{42}=\frac{26}{21}=1.238\text{ (4 s.f.)}$$

b If $x\leqslant 0$, $\mathrm{F}(x)=0$ so $\mathrm{F}(0)=0$

If $0\leqslant x<1$

$$\mathrm{F}(x)=\mathrm{F}(0)+\int_0^x\frac{1}{3}\mathrm{d}t=\left[\frac{t}{3}\right]_0^x=\frac{x}{3}$$

So $\mathrm{F}(1)=\frac{1}{3}$

If $1\leqslant x\leqslant 2$

$$\mathrm{F}(x)=\mathrm{F}(1)+\int_1^x\frac{2}{7}t^2\,\mathrm{d}t=\frac{1}{3}+\left[\frac{2t^3}{21}\right]_1^x=\frac{1}{3}+\frac{2x^3}{21}-\frac{2}{21}=\frac{2x^3}{21}+\frac{5}{21}$$

So the cumulative distribution function is:

$$\mathrm{F}(x)=\begin{cases}0 & x<0\\ \frac{x}{3} & 0\leqslant x<1\\ \frac{2x^3}{21}+\frac{5}{21} & 1\leqslant x\leqslant 2\\ 1 & x>2\end{cases}$$

8 c i $F(1)=\frac{1}{3}$ therefore the median lies in the interval $1 \leqslant x \leqslant 2$, so

$$F(m)=0.5, \text{ so } \frac{2m^3}{21}+\frac{5}{21}=0.5$$
$$\Rightarrow 4m^3+10=21$$
$$\Rightarrow m^3=\frac{11}{4}=2.75$$
$$\Rightarrow m=1.401 \text{ (4 s.f.)}$$

ii $F(P_{15})=\frac{15}{100}$; so as $F(P_{15}) \leqslant \frac{1}{3}$, P_{15} is in the interval $0 \leqslant x < 1$

Therefore $\frac{1}{3}P_{15}=\frac{15}{100} \Rightarrow P_{15}=\frac{45}{100}=0.45 \Rightarrow x=0.45$

d Mean (1.238) < median (1.401) < mode (2), so negative skew.

9 $F(1)=0 \Rightarrow 0.05a-b=0 \Rightarrow b=0.05a$

$F(2)=1 \Rightarrow 0.05a^2-b=1$

$\Rightarrow 0.05(a^2-a)=1$ substituting for b

$\Rightarrow a^2-a-20=0$

$\Rightarrow (a+4)(a-5)=0$ factoring

Since a is positive, $a=5$

So $b=0.05a=\frac{1}{20}\times 5=\frac{1}{4}=0.25$

10 If F(x) is a cumulative distribution function, then the probability distribution function f(x) is found by:

$$\frac{d}{dx}F(x)=f(x)$$
$$\frac{d}{dx}\left(\frac{1}{5}(16x-x^2-55)\right)=\frac{1}{5}(16-2x)$$

So

$$f(x)=\begin{cases}\frac{2}{5}(8-x) & 5 \leqslant x \leqslant 10\\ 0 & \text{otherwise}\end{cases}$$

But this cannot be a probability distribution function as f(x) < 0 for $8 < x \leqslant 10$. So F(x) cannot be a cumulative distribution function.

11 a The area under the probability density function graph must be 1, so:

$$\int_1^3 kx - k\,\mathrm{d}x = 1$$

$$\Rightarrow k\left[\frac{x^2}{2} - x\right]_1^3 = 1$$

$$\Rightarrow k\left(\frac{9}{2} - 3 - \frac{1}{2} + 1\right) = 1$$

$$\Rightarrow 2k = 1 \Rightarrow k = \frac{1}{2}$$

b $\mathrm{E}(X) = \int x\,\mathrm{f}(x)\mathrm{d}x = \int_1^3 \frac{1}{2}(x^2 - x)\,\mathrm{d}x = \left[\frac{x^3}{6} - \frac{x^2}{4}\right]_1^3$

$$= \frac{27}{6} - \frac{9}{4} - \frac{1}{6} + \frac{1}{4} = \frac{13}{3} - 2 = \frac{7}{3} = 2.33 \text{ (3 s.f.)}$$

c $\mathrm{F}(x) = \int_1^x \frac{1}{2}(t - 1)\,\mathrm{d}t = \left[\frac{t^2}{4} - \frac{t}{2}\right]_1^x = \frac{x^2}{4} - \frac{x}{2} - \frac{1}{4}$

So the cumulative distribution function is:

$$\mathrm{F}(x) = \begin{cases} 0 & x < 1 \\ \dfrac{x^2}{4} - \dfrac{x}{2} + \dfrac{1}{4} & 1 \leqslant x \leqslant 3 \\ 1 & x > 3 \end{cases}$$

d $\mathrm{F}(2.4) = \frac{2.4^2}{4} - 1.2 + 0.25 = 0.49$

$\mathrm{F}(2.5) = \frac{2.5^2}{4} - 1.25 + 0.25 = 0.5625$

Since $\mathrm{F}(2.4) < 0.5 < \mathrm{F}(2.5)$, the median, m, when F(m) = 0.5 lies between 2.4 and 2.5.

e Mean (2.33) < median < mode (3), so negative skew.

11 f $\text{F}(P_{10}) = \frac{10}{100}$, so $\frac{P_{10}^2}{4} - \frac{P_{10}}{2} + \frac{1}{4} = \frac{1}{10}$

$\Rightarrow 5P_{10}^2 - 10P_{10} + 5 = 2$

$\Rightarrow 5P_{10}^2 - 10P_{10} + 3 = 0$

$\Rightarrow P_{10} = \frac{10 \pm \sqrt{100-60}}{10} = 1 + \frac{\sqrt{10}}{5} = 1.6325$

$\text{F}(P_{90}) = \frac{90}{100}$, so $\frac{P_{90}^2}{4} - \frac{P_{90}}{2} + \frac{1}{4} = \frac{9}{10}$

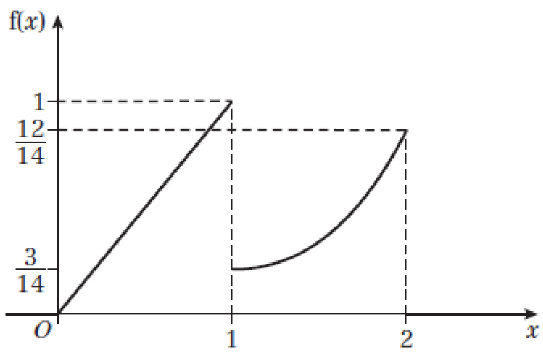
$\Rightarrow 5P_{90}^2 - 10P_{90} + 5 = 18$

$\Rightarrow 5P_{90}^2 - 10P_{90} - 13 = 0$

$\Rightarrow P_{90} = \frac{10 \pm \sqrt{100+260}}{10} = 1 + \frac{3\sqrt{10}}{5} = 2.8974$

So required percentile range $= P_{90} - P_{10} = 2.8974 - 1.6325 = 1.26$ (3 s.f.)

12 a The graph is a straight line from (0, 0) to (1, 1); part of a quadratic with a positive x^2 coefficient from $\left(1, \frac{3}{14}\right)$ to $\left(2, \frac{12}{14}\right)$; and otherwise 0. The sketch of the graph is:



b The mode occurs at the maximum point of the probability density function graph. From the graph, this occurs when $x = 1$. So the mode is 1.

c $\text{E}(X) = \int x\,\text{f}(x)\text{d}x = \int_0^1 x^2\,\text{d}x + \int_1^2 \frac{3x^3}{14}\text{d}x = \left[\frac{x^3}{3}\right]_0^1 + \left[\frac{3x^4}{56}\right]_1^2$

$= \frac{1}{3} + \frac{48}{56} - \frac{3}{56} = \frac{56}{168} + \frac{144}{168} - \frac{9}{168} = \frac{191}{168}$

$\text{E}(2X) = 2\text{E}(X) = 2 \times \frac{191}{168} = \frac{191}{84}$

12 d $\text{Var}(X)=\int x^2\text{f}(x)\text{d}x-(\text{E}(X))^2=\int_0^1 x^3\text{d}x+\int_1^2\frac{3x^4}{14}\text{d}x-\left(\frac{191}{168}\right)^2$

$=\left[\frac{x^4}{4}\right]_0^1+\left[\frac{3x^5}{70}\right]_1^2-\left(\frac{191}{168}\right)^2$

$=\frac{1}{4}+\frac{96}{70}-\frac{3}{70}-\left(\frac{191}{168}\right)^2=0.2860\text{ (4 s.f.)}$

$\text{Var}(2X+1)=2^2\,\text{Var}(X)=4\times0.28602=1.14\text{ (3 s.f.)}$

e If $x\leqslant0$, $\text{F}(x)=0$ so $\text{F}(0)=0$

If $0\leqslant x\leqslant1$

$\text{F}(x)=\text{F}(0)+\int_0^x t\,\text{d}t=\left[\frac{t^2}{2}\right]_0^x=\frac{x^2}{2}$

So $\text{F}(1)=\frac{1}{2}$

If $1<x\leqslant2$

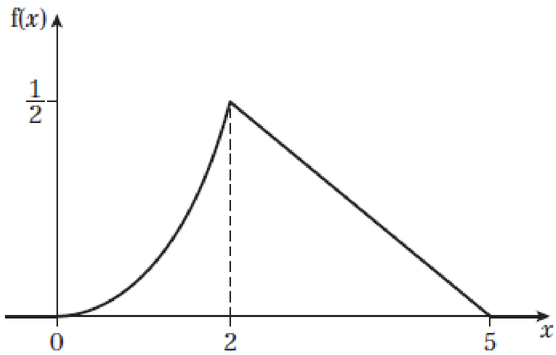
$\text{F}(x)=\text{F}(1)+\int_1^x\frac{3t^2}{14}\text{d}t=\frac{1}{2}+\left[\frac{3t^3}{42}\right]_1^x=\frac{1}{2}+\frac{3x^3}{42}-\frac{3}{42}=\frac{3x^3}{42}+\frac{18}{42}=\frac{3x^3}{42}+\frac{3}{7}$

So the cumulative distribution function is:

$$\text{F}(x)=\begin{cases}0 & x<0\\ \frac{x^2}{2} & 0\leqslant x\leqslant1\\ \frac{x^3}{14}+\frac{3}{7} & 1<x\leqslant2\\ 1 & x>2\end{cases}$$

f In deriving the cumulative distribution function in part **e**, it was shown that F(1) = 0.5. So the median is 1.

13 a The graph is part of a cubic with a positive x^3 coefficient from $(0,0)$ to $(2,0.5)$; a straight line from (2, 0.5) to (5, 0); and otherwise 0. The sketch of the graph is:



13 b The mode occurs at the maximum point of the probability density function graph. From the graph, this occurs when x = 2. So the mode is 2.

13 c Using the sketch, P(X > 2) = area of triangle with coordinates (2, 0), (2, 0.5) and (5, 0)

So $\text{P}(X > 2) = \frac{1}{2} \times 3 \times \frac{1}{2} = 0.75$

d If $x \leqslant 0$, $\text{F}(x) = 0$ so $\text{F}(0) = 0$

If $0 \leqslant x < 2$

$$\text{F}(x) = \text{F}(0) + \int_0^x \frac{t^3}{16}\,\text{d}t = \left[\frac{t^4}{64}\right]_0^x = \frac{x^4}{64}$$

So $\text{F}(2) = \frac{2^4}{64} = \frac{1}{4}$

If $2 \leqslant x \leqslant 5$

$$\text{F}(x) = \text{F}(2) + \int_2^x \frac{5-t}{6}\,\text{d}t = \frac{1}{4} + \left[\frac{5t}{6} - \frac{t^2}{12}\right]_2^x = \frac{1}{4} + \frac{5x}{6} - \frac{x^2}{12} - \frac{10}{6} + \frac{4}{12} = \frac{1}{12}(10x - x^2 - 13)$$

So the cumulative distribution function is:

$$\text{F}(x) = \begin{cases} 0 & x < 0 \\ \dfrac{x^4}{64} & 0 \leqslant x < 2 \\ \dfrac{10x - x^2 - 13}{12} & 2 \leqslant x \leqslant 5 \\ 1 & x > 5 \end{cases}$$

e As F(2) = 0.25 (from part **d**), the median, m, lies in the range $2 \leqslant m \leqslant 5$

$\text{F}(m) = 0.5 \Rightarrow \frac{10m - m^2 - 13}{12} = 0.5 \Rightarrow m^2 - 10m + 19 = 0$

So $m = \frac{10 \pm \sqrt{100 - 76}}{2} = 5 \pm \sqrt{6}$

As $5 + \sqrt{6}$ is outside the range, $m = 5 - \sqrt{6} = 2.55$ (3 s.f.)

14 a $\frac{\text{d}}{\text{d}x}\text{F}(x) = \text{f}(x)$, so

$$\text{f}(x) = \frac{\text{d}}{\text{d}x}\left(\frac{1}{81}(-2x^3 + 15x^2 - 44)\right) = \frac{1}{81}(-6x^2 + 30x) = -\frac{2}{27}x^2 + \frac{10}{27}x = \frac{2x}{27}(5 - x)$$

So $\text{f}(x) = \begin{cases} \dfrac{2x}{27}(5 - x) & 2 \leqslant x \leqslant 5 \\ 0 & \text{otherwise} \end{cases}$

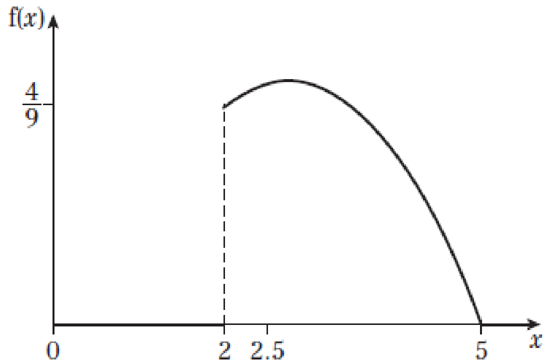
14 b To find the value of x when f(x) is a maximum, solve $\frac{d}{dx}f(x)=0$

$$\frac{d}{dx}\left(-\frac{2}{27}x^2+\frac{10}{27}x\right)=-\frac{4}{27}x+\frac{10}{27}$$

So the mode occurs when $-4x+10=0 \Rightarrow x=2.5$

Note that f(x) is clearly a maximum at this point as it is a negative quadratic.

c The graph is part of a quadratic with a negative x^2 coefficient from $\left(2,\frac{4}{9}\right)$ to $(5,0)$; and is otherwise 0. The quadratic has a maximum at $\left(2.5,\frac{25}{54}\right)$. The sketch of the graph is:



d $\mu=E(X)=\int x\,f(x)dx=\int_2^5 -\frac{2}{27}x^3+\frac{10}{27}x^2dx$

$$=\frac{2}{27}\left[-\frac{1}{4}x^4+\frac{5}{3}x^3\right]_2^5=\frac{2}{27}\left(-\frac{625}{4}+\frac{625}{3}+4-\frac{40}{3}\right)=\frac{2}{27}\left(\frac{-1875+2500+48-160}{12}\right)$$

$$=\frac{2}{27}\times\frac{513}{12}=\frac{2\times19}{12}=\frac{19}{6}$$

e $F(\mu)=F\left(\frac{19}{6}\right)=\frac{1}{81}\left(-2\left(\frac{19}{6}\right)^3+15\left(\frac{19}{6}\right)^2-44\right)=0.5297\text{ (4 s.f.)}$

f $F(2.5)=\frac{1}{81}\left(-2(2.5)^3+15(2.5)^2-44\right)=0.2284\text{ (4 s.f.)}$

If m is the median, F(m) =0.5, so, as 0.2284 < 0.5 < 0.5297, for this distribution, mode < median < mean, which means the distribution has a positive skew.

15 a As F(x) is a cumulative distribution function, $F(5)=1$

So $k(35\times5-2\times5^2)=1\Rightarrow125k=1\Rightarrow k=\frac{1}{125}$

15 b F(m) = 0.5, so $\frac{1}{125}(35m - 2m^2) = 0.5$

$\Rightarrow 4m^2 - 70m + 125 = 0$

$\Rightarrow m = \frac{70 \pm \sqrt{4900 - 2000}}{8} = \frac{70 \pm 53.8516\ldots}{8}$

As $\frac{70 + 53.8516\ldots}{8} > 5,$ this cannot be a solution as the median lies between 0 and 5

So the solution is $m = \frac{70 - 53.8516\ldots}{8} = 2.02$ (3 s.f.)

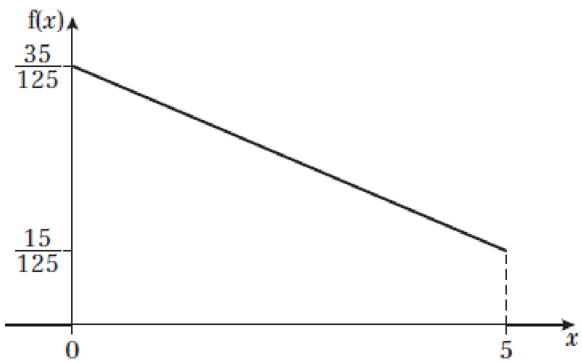
c $\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{F}(x) = \mathrm{f}(x)$, so

$\mathrm{f}(x) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{125}(35x - 2x^2)\right) = \frac{1}{125}(35 - 4x)$

So $\mathrm{f}(x) = \begin{cases} \frac{1}{125}(35 - 4x) & 0 \leqslant x \leqslant 5 \\ 0 & \text{otherwise} \end{cases}$

d The graph is a straight line from $\left(0, \frac{35}{125}\right)$ to $\left(5, \frac{15}{125}\right)$ and is otherwise 0.

The sketch of the graph is:



e The mode occurs at the maximum point of the probability density function graph. From the graph, this occurs when $x = 0$. So the mode is 0.

f $\mathrm{E}(X) = \int x\,\mathrm{f}(x)\mathrm{d}x = \int_0^5 \frac{1}{125}(35x - 4x^2)\mathrm{d}x = \frac{1}{125}\left[\frac{35}{2}x^2 - \frac{4}{3}x^3\right]_0^5$

$= \frac{1}{125}\left(\frac{35 \times 5^2}{2} - \frac{4 \times 5^3}{3}\right) = \frac{7}{2} - \frac{4}{3} = \frac{21}{6} - \frac{8}{6} = \frac{13}{6} = 2.17$ (3 s.f.)

g As mean (2.17) > median (2.02) > mode (0), the distribution has a positive skew.
The skewness can also be deduced directly from the graph (part **d**).

16 The area under the probability density function graph must be 1, so:

$$\int_0^2 ax+b\,\mathrm{d}x=\left[\frac{1}{2}ax^2+bx\right]_0^2=1\Rightarrow 2a+2b=1 \quad (1)$$

$$\mathrm{E}(X)=\int x\,\mathrm{f}(x)\mathrm{d}x=\int_0^2 ax^2+bx\,\mathrm{d}x=\left[\frac{1}{3}ax^3+\frac{1}{2}bx^2\right]_0^2=\frac{9}{8}\Rightarrow\frac{8}{3}a+2b=\frac{9}{8} \quad (2)$$

Subtracting equation (2) from equation (1) gives:

$$\frac{2}{3}a=\frac{1}{8}\Rightarrow a=\frac{3}{16}$$

Substituting for a in equation (1) gives:

$$\frac{6}{16}+2b=1\Rightarrow b=\frac{5}{16}$$

17 a The area under the probability density function graph must be 1, so:

$$\int_{-1}^0 k(x+1)^3\mathrm{d}x=\left[\frac{k(x+1)^4}{4}\right]_{-1}^0=1$$

$$\Rightarrow\frac{k}{4}=1\Rightarrow k=4$$

b $\mathrm{E}(X)=\int x\,\mathrm{f}(x)\mathrm{d}x=\int_{-1}^0 4x(x+1)^3\mathrm{d}x=\int_{-1}^0 4x^4+12x^3+12x^2+4x\mathrm{d}x$

$$=\left[\frac{4}{5}x^5+3x^4+4x^3+2x^2\right]_{-1}^0$$

$$=\frac{4}{5}-3+4-2=-0.2$$

c $\mathrm{F}(x)=\int 4(x+1)^3\mathrm{d}x=(x+1)^4+c$

$\mathrm{F}(-1)=0\Rightarrow c=0$

So the cumulative distribution function is:

$$\mathrm{F}(x)=\begin{cases}0 & x<-1\\(x+1)^4 & -1\leqslant x\leqslant 0\\1 & x>0\end{cases}$$

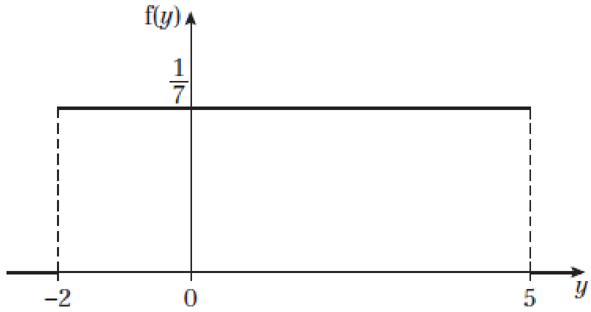
d $\mathrm{F}(m)=(m+1)^4=0.5$

$\Rightarrow m+1=0.8409$

$\Rightarrow m=-0.159$ (3 s.f.)

18 a $b-a=5-(-2)=7$, so the graph is a straight line from $\left(-2,\frac{1}{7}\right)$ to $\left(5,\frac{1}{7}\right)$ and 0 otherwise.

The sketch of the graph is:



b $\mathrm{E}(X)=\frac{a+b}{2}=\frac{-2+5}{2}=1.5$

c $\mathrm{Var}(X)=\frac{(b-a)^2}{12}=\frac{(5-(-2))^2}{12}=\frac{49}{12}$

d For $-2\leqslant x\leqslant 5$, $\mathrm{F}(x)=\int_{-2}^{x}\frac{1}{b-a}\,\mathrm{d}t=\int_{-2}^{x}\frac{1}{7}\,\mathrm{d}t=\left[\frac{t}{7}\right]_{-2}^{x}=\frac{x}{7}+\frac{2}{7}=\frac{x+2}{7}$

So:

$$\mathrm{F}(x)=\begin{cases}0 & x<-2\\ \frac{x+2}{7} & -2\leqslant x\leqslant 5\\ 1 & x>5\end{cases}$$

e $\mathrm{P}(3.5<X<5.5)=(5-3.5)\times\frac{1}{7}=\frac{3}{14}$

Note that as f(x) = 0 for x > 5, $\mathrm{P}(5<X<5.5)=0$

f As X is a continuous random variable P(X = n) = 0 for any discrete number n.
So P(X = 4) =0

g $\mathrm{P}(X>0\mid X<2)=\frac{\mathrm{P}(X>0\cap X<2)}{\mathrm{P}(X<2)}=\frac{\mathrm{P}(0<X<2)}{\mathrm{P}(X<2)}=\frac{\frac{2}{7}}{\frac{4}{7}}=\frac{1}{2}$

h $\mathrm{P}(X>3\mid X>0)=\frac{\mathrm{P}(X>3\cap X>0)}{\mathrm{P}(X>0)}=\frac{\mathrm{P}(X>3)}{\mathrm{P}(X>0)}=\frac{\frac{2}{7}}{\frac{5}{7}}=\frac{2}{5}$

19 a The area under the probability density function graph must be 1, so:
$0.2(k-(-4))=1\Rightarrow k=5-4=1$

b $\mathrm{P}(-2<X<-1)=(-1-(-2))\times\frac{1}{1-(-4)}=\frac{1}{5}=0.2$

19 c $\text{E}(X)=\frac{a+b}{2}=\frac{-4+1}{2}=-\frac{3}{2}=-1.5$

d $\text{Var}(X)=\frac{(b-a)^2}{12}=\frac{(1-(-4))^2}{12}=\frac{25}{12}$

e For $-4\leqslant x\leqslant 1$, $\text{F}(x)=\int_{-4}^{x}\frac{1}{b-a}\,\text{d}t=\int_{-4}^{x}\frac{1}{5}\,\text{d}t=\left[\frac{t}{5}\right]_{-4}^{x}=\frac{x}{5}+\frac{4}{5}=\frac{x+4}{5}$

So:

$$\text{F}(x)=\begin{cases}0 & x<-4\\ \frac{x+4}{5} & -4\leqslant x\leqslant 1\\ 1 & x>1\end{cases}$$

20 a $\text{E}(Y)=\frac{a+b}{2}=2\Rightarrow a+b=4\Rightarrow a=4-b$

$\text{Var}(Y)=\frac{(b-a)^2}{12}=3\Rightarrow(b-a)^2=36$

So $(b-4+b)^2=36$ substituting for a

$\Rightarrow(2b-4)^2=36$

$\Rightarrow 2b-4=\pm 6$

$\Rightarrow b=2\pm 3$

$b=-1\Rightarrow a=5$ reject as $b>a$

So solution is $b=5\Rightarrow a=-1$

b $\text{P}(Y>1.8)=\frac{5-1.8}{5-(-1)}=\frac{3.2}{6}=\frac{8}{15}=0.533$ (3 s.f.)

21 a As $10-5\times 0=10$ and $10-5\times 2=0$

So $Y\sim\text{U}[0,10]$

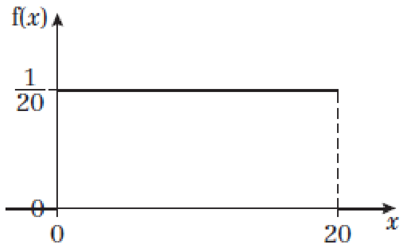
b $\text{P}(Y<3)=\frac{3-0}{10-0}=\frac{3}{10}=0.3$

c $\text{P}(Y>3\mid X>0.5)=\text{P}(Y>3\mid Y<7.5)=\frac{\text{P}(Y>3\cap Y<7.5)}{\text{P}(Y<7.5)}=\frac{\text{P}(3<Y<7.5)}{\text{P}(Y<7.5)}$

$=\frac{7.5-3}{7.5-0}=\frac{4.5}{7.5}=\frac{9}{15}=\frac{3}{5}=0.6$

22 a The variable X has a continuous uniform distribution, $X \sim \text{U}[0, 20]$.

The graph is a straight line from $\left(0, \frac{1}{20}\right)$ to $\left(20, \frac{1}{20}\right)$ and otherwise 0.



b $\text{E}(X) = \frac{a+b}{2} = \frac{0+20}{2} = 10$

$\text{Var}(X) = \frac{(b-a)^2}{12} = \frac{20^2}{12} = \frac{400}{12} = \frac{100}{3}$

c The string with the mark could be the shorter piece ($X < 10$) or the longer piece ($X > 10$)
Require the shorter length of string to be > 8
If the string with the mark on it is the shorter length of string, then require $8 < X < 10$
If the string with the mark on it is the longer length of string, then require $10 < X < 12$
So the required probability is:

$$\text{P}(8 < X < 10) \cup \text{P}(10 < X < 12) = \text{P}(8 < X < 12) = \frac{4}{20} = 0.2$$

23 a The temperature is between 28.5 °C and 29.5 °C, so X can be any value between –0.5 °C and 0.5 °C with equal probability. So a suitable model is $X \sim \text{U}[-0.5, 0.5]$

b $\text{P}(X < 0.2) = \frac{0.2-(-0.5)}{0.5-(-0.5)} = 0.7$

c $\text{Var}(X) = \frac{(b-a)^2}{12} = \frac{(0.5-(-0.5))^2}{12} = \frac{1}{12}$

24 a $X \sim \text{U}[-3, 10]$

The probability density function of X is:

$$\text{f}(x) = \begin{cases} \frac{1}{13} & -3 \leqslant x \leqslant 10 \\ 0 & \text{otherwise} \end{cases}$$

24 b Mean $= \mathrm{E}(X) = \frac{a+b}{2} = \frac{-3+10}{2} = 3.5$ minutes

c For $-3 \leqslant x \leqslant 10$, $\mathrm{F}(x) = \int_{-3}^{x} \frac{1}{b-a}\,\mathrm{d}t = \int_{-3}^{x} \frac{1}{13}\,\mathrm{d}t = \left[\frac{t}{13}\right]_{-3}^{x} = \frac{x}{13} + \frac{3}{13} = \frac{x+3}{13}$

So:

$$\mathrm{F}(x) = \begin{cases} 0 & x < -3 \\ \frac{x+3}{13} & -3 \leqslant x \leqslant 10 \\ 1 & x > 10 \end{cases}$$

d $\mathrm{P}(5 < X < 10) = (10-5) \times \frac{1}{13} = \frac{5}{13}$

25 a The difference between the true length and the measured length could be any value between –0.5 cm and 0.5 cm with equal probability. So a suitable model is $X \sim \mathrm{U}[-0.5, 0.5]$

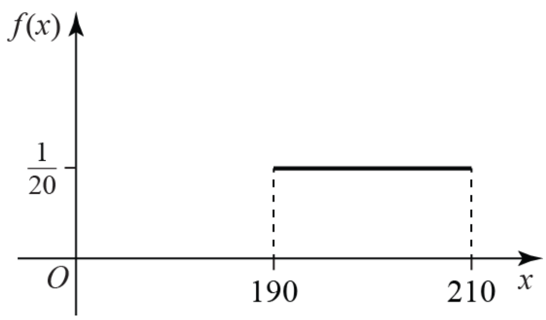
b $\mathrm{P}(-0.2 < X < 0.2) = \frac{0.2-(-0.2)}{0.5-(-0.5)} = 0.4$

c P(3 pipes between –0.2 and 0.2) $= 0.4^3 = 0.064$

26 a The volume can be any value between 190 ml and 210 ml with equal probability. So a suitable model is $X \sim \mathrm{U}[190, 210]$. The probability density function is:

$$\mathrm{f}(x) = \begin{cases} \frac{1}{20} & 190 \leqslant x \leqslant 210 \\ 0 & \text{otherwise} \end{cases}$$

The graph of f(x) is a straight line from $\left(190, \frac{1}{20}\right)$ to $\left(210, \frac{1}{20}\right)$ and otherwise 0.



b i $\mathrm{P}(X < 198) = \frac{198-190}{20} = \frac{8}{20} = \frac{2}{5} = 0.4$

ii As X is a continuous random variable P($X = n$) = 0 for any discrete number n.
So P(X = 198) =0

26 c The cumulative distribution function is:

$$F(x)=\begin{cases} 0 & x<190 \\ \dfrac{x-190}{20} & 190\leqslant x\leqslant 210 \\ 1 & x>210 \end{cases}$$

$F(Q_3)=0.75 \Rightarrow Q_3=205 \qquad F(Q_1)=0.25 \Rightarrow Q_1=195$

So $Q_3-Q_1=205-195=10$

d $P(X>200 \mid X>195)=\dfrac{P(X>195 \cap X>200)}{P(X>195)}=\dfrac{P(X>200)}{P(X>195)}$

$$=\frac{0.5}{0.75}=\frac{2}{3}$$

27 a Continuous uniform distribution. The difference between the true length and the measured length could be any value between –0.5 cm and 0.5 cm with equal probability. So a suitable model is $X \sim U[-0.5, 0.5]$

b Normal distribution.

28 a $F(t)=\int \frac{1}{72}(6-t)^2\,dt=-\frac{1}{216}(6-t)^3+c$

$F(0)=0 \Rightarrow c=1$

So the cumulative distribution function is:

$$F(t)=\begin{cases} 0 & t<0 \\ 1-\dfrac{(6-t)^3}{216} & 0\leqslant t\leqslant 6 \\ 1 & t>6 \end{cases}$$

b $F(m)=1-\dfrac{(6-m)^3}{216}=0.5$

$\Rightarrow (6-m)^3=108$

$\Rightarrow 6-m=4.7622\ldots$

$\Rightarrow m=1.24$ hours (3 s.f.)

c $E(T)=\int t\,f(t)\,dt=\int_0^6 \frac{t}{72}(6-t)^2\,dt=\int_0^6 \frac{t}{2}-\frac{t^2}{6}+\frac{t^3}{72}\,dt$

$$=\left[\frac{t^2}{4}-\frac{t^3}{18}+\frac{t^4}{288}\right]_0^6=\frac{36}{4}-\frac{216}{18}+\frac{1296}{288}=9-12+4.5=1.5 \text{ hours}$$

29 a As $5b - b = 4b$, he probability density function is

$$f(x) = \begin{cases} \frac{1}{4b} & b \leqslant x \leqslant 5b \\ 0 & \text{otherwise} \end{cases}$$

b $E(X) = \frac{b+5b}{2} = 3b$

c $\text{Var}(X) = \int x^2 f(x)dx - (E(X))^2 = \int_b^{5b} \frac{x^2}{4b} dx - (3b)^2$

$$= \left[\frac{x^3}{12b}\right]_b^{5b} - 9b^2 = \frac{125b^2 - b^2}{12} - \frac{108b^2}{12} = \frac{16b^2}{12} = \frac{4b^2}{3}$$

d $P(X > 10) = \frac{15-10}{12} = \frac{5}{12}$

e Let the random variable Y be the number of times that $X > 10$ in five observations, so Y has a binomial distribution, $Y \sim B\left(5, \frac{5}{12}\right)$

So $P(Y = 3) = \binom{5}{3}\left(\frac{5}{12}\right)^3\left(\frac{7}{12}\right)^2 = 0.246$ (3 s.f.)

30 $F(x) = \int_1^x \frac{2}{(2t-1)\ln 5} dt = \left[\frac{\ln(2t-1)}{\ln 5}\right]_1^x = \frac{\ln(2x-1)}{\ln 5}$

So the cumulative distribution function is:

$$F(x) = \begin{cases} 0 & x < 1 \\ \frac{\ln(2x-1)}{\ln 5} & 1 \leqslant x \leqslant 3 \\ 1 & x > 3 \end{cases}$$

31 a The area under the probability density function graph must be 1, so:

$\int_0^1 kx\sin(\pi x)dx = 1$

$\Rightarrow \left[-kx\frac{\cos(\pi x)}{\pi}\right]_0^1 - \int_0^1 -\frac{k\cos(\pi x)}{\pi} dx = 1$ using integration by parts

$\Rightarrow \left[-kx\frac{\cos(\pi x)}{\pi} + \frac{k\sin(\pi x)}{\pi^2}\right]_0^1 = 1$

$\Rightarrow \frac{k}{\pi} = 1$

$\Rightarrow k = \pi$

31 b $\mathrm{E}(X)=\int x\,\mathrm{f}(x)\mathrm{d}x=\int_0^1 \pi x^2\sin(\pi x)\mathrm{d}x$

$=\left[-x^2\cos(\pi x)\right]_0^1+\int_0^1 2x\cos(\pi x)\mathrm{d}x$ using integration by parts

$=\left[-x^2\cos(\pi x)+2x\frac{\sin(\pi x)}{\pi}\right]_0^1-\int_0^1 2\frac{\sin(\pi x)}{\pi}\mathrm{d}x$ using integration by parts again

$=\left[-x^2\cos(\pi x)+2x\frac{\sin(\pi x)}{\pi}+\frac{2\cos(\pi x)}{\pi^2}\right]_0^1$

$=1-\frac{2}{\pi^2}-\frac{2}{\pi^2}\simeq 0.5947=59.47\%$

32 a The area under the probability density function graph must be 1, so:

$\int_0^1 k\mathrm{d}x+\int_1^2\frac{k}{x^2}\mathrm{d}x=1$

$\Rightarrow [kx]_0^1+\left[-\frac{k}{x}\right]_1^2=1$

$\Rightarrow k-\frac{k}{2}+k=1$

$\Rightarrow \frac{3k}{2}=1$

$\Rightarrow k=\frac{2}{3}$

b $\mathrm{E}(X)=\int x\,\mathrm{f}(x)\mathrm{d}x=\int_0^1\frac{2x}{3}\mathrm{d}x+\int_1^2\frac{2}{3x}\mathrm{d}x$

$=\left[\frac{x}{3}\right]_0^1+\left[\frac{2\ln x}{3}\right]_1^2=\frac{1}{3}+\frac{2}{3}\ln 2=0.79543\ldots=0.795\text{ (3 s.f.)}$

c $\mathrm{E}(X^2)=\int x^2\,\mathrm{f}(x)\mathrm{d}x=\int_0^1\frac{2x^2}{3}\mathrm{d}x+\int_1^2\frac{2}{3}\mathrm{d}x$

$=\left[\frac{2x^3}{9}\right]_0^1+\left[\frac{2x}{3}\right]_1^2=\frac{2}{9}+\frac{2}{3}=\frac{8}{9}$

So $\mathrm{Var}(X)=\mathrm{E}(X^2)-(\mathrm{E}(X))^2=\frac{8}{9}-(0.7954)^2=0.256\text{ (3 s.f.)}$

Challenge

1 a As θ can take any value between 0 and 2π with equal probability, a continuous uniform distribution would be suitable, with $\theta \sim U[0, 2\pi]$

b $X = r|\sin\theta|$

So $E(X) = E(r|\sin\theta|) = \int_0^{2\pi} r|\sin\theta| \frac{1}{2\pi} d\theta$

$$= \frac{r}{2\pi}\left(\int_0^{\pi} \sin\theta d\theta + \int_{\pi}^{2\pi} (-\sin\theta) d\theta\right)$$

$$= \frac{r}{2\pi}\left([-\cos\theta]_0^{\pi} + [\cos\theta]_{\pi}^{2\pi}\right)$$

$$= \frac{r}{2\pi}(2+2)$$

$$= \frac{2r}{\pi} \approx 0.6366r$$

c Spin the spinner 100 times and measure X each time. Take the mean of these observations and divide $2r$ by this value.

2 a $E(X) = \int_0^{\infty} x f(x) dx = \int_0^{\infty} x\lambda e^{-\lambda x} dx$

$$= \left[x(\lambda e^{-\lambda x})\right]_0^{\infty} - \int_0^{\infty} -e^{-\lambda x} dx \qquad \text{using integration by parts}$$

$$= 0 + \int_0^{\infty} e^{-\lambda x} dx = \left[-\frac{e^{-\lambda x}}{\lambda}\right]_0^{\infty} = \frac{1}{\lambda}$$

$E(X^2) = \int_0^{\infty} x^2 f(x) dx = \int_0^{\infty} x^2(\lambda e^{-\lambda x}) dx$

$$= \left[x^2(\lambda e^{-\lambda x})\right]_0^{\infty} - \int_0^{\infty} -2xe^{-\lambda x} dx \qquad \text{using integration by parts}$$

$$= 0 + 2\int_0^{\infty} xe^{-\lambda x} dx = \frac{2}{\lambda} E(X) = \frac{2}{\lambda^2}$$

$$Var(X) = E(X^2) - (E(X))^2 = \frac{2}{\lambda^2} - \left(\frac{1}{\lambda}\right)^2 = \frac{1}{\lambda^2}$$

b $P(X > a) = 1 - P(X < a) = 1 - \int_0^{a} \lambda e^{-\lambda x} dx$

$$= 1 - \left[-e^{-\lambda x}\right]_0^{a} = 1 - (-e^{-\lambda a} + 1) = e^{-\lambda a}$$

Similarly, $P(X > b) = e^{-\lambda b}$ and $P(X > a + b) = e^{-\lambda(a+b)} = e^{-\lambda a} \times e^{-\lambda b}$

Hence, $P(X > a + b \mid X > a) = \frac{P(X > a + b)}{P(X > a)} = \frac{e^{-\lambda a} \times e^{-\lambda b}}{e^{-\lambda a}} = e^{-\lambda b} = P(X > b)$