Combinations of random variables 4A

1 a $\text{E}(W) = \text{E}(X) + \text{E}(Y) = 80 + 50 = 130$
$\text{Var}(W) = \text{Var}(X) + \text{Var}(Y) = 9 + 4 = 13$
$W \sim \text{N}(130, 13)$

b $\text{E}(W) = \text{E}(X) - \text{E}(Y) = 80 - 50 = 30$
$\text{Var}(W) = \text{Var}(X) + \text{Var}(Y) = 9 + 4 = 13$
$W \sim \text{N}(30, 13)$

2 $\text{E}(R) = \text{E}(X) + \text{E}(Y) + \text{E}(W) = 45 + 54 + 49 = 148$
$\text{Var}(R) = \text{Var}(X) + \text{Var}(Y) + \text{Var}(W) = 6 + 4 + 8 = 18$
$R \sim \text{N}(148, 18)$

3 a $T = 3X$, so $T \sim \text{N}(3 \times 60, 3^2 \times 25)$
$T \sim \text{N}(180, 225)$

b $T = 7Y$, so $T \sim \text{N}(7 \times 50, 7^2 \times 16)$
$T \sim \text{N}(350, 784)$

c $T = 3X + 7Y$, so $T \sim \text{N}(180 + 350, 225 + 784)$
$T \sim \text{N}(530, 1009)$

d $T = X - 2Y$, so $T \sim \text{N}(60 - 2 \times 50, 25 + 2^2 \times 16)$
$T \sim \text{N}(-40, 89)$

4 a $A = X + Y + W$, so $A \sim \text{N}(8 + 12 + 15,\ 2 + 3 + 4)$
$A \sim \text{N}(35, 9)$

b $A = W - X$, so $A \sim \text{N}(15 - 8,\ 4 + 2)$
$A \sim \text{N}(7, 6)$

c $A = X - Y + 3W$, so $A \sim \text{N}(8 - 12 + 3 \times 15,\ 2 + 3 + 3^2 \times 4)$
$A \sim \text{N}(41, 41)$

d $A = 3X + 4W$, so $A \sim \text{N}(3 \times 8 + 4 \times 15,\ 3^2 \times 2 + 4^2 \times 4)$
$A \sim \text{N}(84, 82)$

e $A = 2X - Y + W$, so $A \sim \text{N}(2 \times 8 - 12 + 15,\ 2^2 \times 2 + 3 + 4)$
$A \sim \text{N}(19, 15)$

5 a Let $D = A + B$, then $D \sim \text{N}(50 + 60,\ 6 + 8)$, so $D \sim \text{N}(110, 14)$.
Then using the normal distribution function on a calculator gives:
$\text{P}(A + B < 115) = \text{P}(D < 115) = 0.9093$ (4 d.p.)

5 b Let $D = A + B + C$, then $D \sim \text{N}(50+60+80,\ 6+8+10)$, so $D \sim \text{N}(190,\ 24)$
$\text{P}(A+B+C > 198) = 1 - \text{P}(D < 198) = 1 - 0.9488 = 0.0512$ (4 d.p.)

c Let $D = B + C$, then $D \sim \text{N}(60+80,\ 8+10)$, so $D \sim \text{N}(140,\ 18)$
$\text{P}(B+C < 138) = \text{P}(D < 138) = 0.3187$ (4 d.p.)

d Let $D = 2A + B - C$, then $D \sim \text{N}(2\times 50+60-80,\ 4\times 6+8+10)$, so $D \sim \text{N}(80,\ 42)$
$\text{P}(2A+B-C < 70) = \text{P}(D < 70) = 0.0614$ (4 d.p.)

e Let $D = A + 3B - C$, then $D \sim \text{N}(50+3\times 60-80,\ 6+9\times 8+10)$, so $D \sim \text{N}(150,\ 88)$
$\text{P}(A+3B-C > 140) = 1 - \text{P}(D < 140) = 1 - 0.1432 = 0.8578$ (4 d.p.)

f Let $D = A + B$, then $D \sim \text{N}(50+60,\ 6+8)$, so $D \sim \text{N}(110,\ 14)$
$\text{P}(105 < A+B < 116) = \text{P}(D < 116) - \text{P}(D < 105) = 0.9456 - 0.0907 = 0.8549$ (4 d.p.)

6 a $\text{E}(X - Y) = \text{E}(X) - \text{E}(Y) = 20 - 10 = 10$

b $\text{Var}(X - Y) = \text{Var}(X) + \text{Var}(Y) = 5 + 4 = 9$

c Let $A = X - Y$, then $A \sim \text{N}(10,\ 9)$
$\text{P}(13 < X - Y < 16) = \text{P}(A < 16) - \text{P}(A < 13) = 0.9772 - 0.8413 = 0.1359$ (4 d.p.)

7 a Let $A = Y - X$, then $A \sim \text{N}(80-76,\ 10+15)$, i.e. $A \sim \text{N}(4,\ 25)$
$\text{P}(Y > X) = \text{P}(Y - X > 0) = \text{P}(A > 0) = 1 - \text{P}(A < 0) = 1 - 0.2119 = 0.7881$ (4 d.p.)

b $\text{P}(X > Y) = \text{P}(Y - X < 0) = \text{P}(A < 0) = 0.2119$ (4 d.p.)

c The probability that X and Y differ by less than 3 = $\text{P}(-3 < A < 3)$
$\text{P}(-3 < A < 3) = \text{P}(A < 3) - \text{P}(A < -3) = 0.42074 - 0.08076 = 0.3400$ (4 d.p.)

d The probability that X and Y differ by more than 7 = $\text{P}(A < -7) + \text{P}(A > 7)$
$\text{P}(A < -7) + \text{P}(A > 7) = \text{P}(A < -7) + 1 - \text{P}(A < 7) = 0.0139 + 1 - 0.7257 = 0.2882$ (4 d.p.)

8 a $\text{E}(R) = \text{E}(X) + 4\text{E}(Y) = 8 + (4\times 14) = 64$

b $\text{Var}(R) = \text{Var}(X) + 16\,\text{Var}(Y) = 2^2 + (16\times 3^2) = 148$

c $R \sim \text{N}(64,\ 148)$, $\text{P}(R < 41) = 0.0293$ (4 d.p.)

d $S = Y_1 + Y_2 + Y_3 - 0.5X$

$$\text{Var}(S) = 3\,\text{Var}(Y) + \left(\frac{1}{2}\right)^2 \text{Var}(X) = 3\times 9 + \frac{1}{4}\times 4 = 27 + 1 = 28$$

9 a Runner $A \sim \text{N}(13.2, 0.9^2)$, Runner $B \sim \text{N}(12.9, 1.3^2)$
Let $D = A - B$, then $D \sim \text{N}(13.2 - 12.9,\ 0.9^2 + 1.3^2)$, so $D \sim \text{N}(0.3,\ 2.5)$.
$\text{P}(A - B > 0.5) = \text{P}(D > 0.5) = 1 - \text{P}(D < 0.5) = 1 - 0.5503 = 0.4497$ (4 d.p.)

9 b $\text{P(photo finish)} = \text{P}(-0.1 < D < 0.1) = \text{P}(< 0.1) - \text{P}(< -0.1)$
$= 0.44967 - 0.40014 = 0.0495$ (4 d.p.)

10 Let R be the diameter of a steel rod and T be the internal diameter of a steel tube, then
$T \sim \text{N}(3.60,\ 0.02^2) R \sim \text{N}(3.55,\ 0.02^2)$
Let $A = T - R$, then $A \sim \text{N}(3.60 - 3.55,\ 0.02^2 + 0.02^2)$, so $A \sim \text{N}(0.05,\ 0.0008)$.
$\text{P}(T - R < 0) = \text{P}(A < 0) = 0.0385$ (4 d.p.)

11 Let T be the mass of a randomly selected jar of jam, B be the mass of a randomly selected box and then Y be the mass of a box of 6 jars, then
$T \sim \text{N}(1000,\ 12^2)$, $B \sim \text{N}(250,\ 10^2)$, $Y = T_1 + T_2 + T_3 + T_4 + T_5 + T_6 + B$
So $Y \sim \text{N}(6 \times 1000 + 250,\ 6 \times 12^2 + 10^2)$, hence $Y \sim \text{N}(6250,\ 964)$
Using a calculator gives $\text{P}(Y < 6200) = 0.0537$ (4 d.p.)

12 a i Let PB be the thickness of a randomly selected paperback and HB be the thickness of a randomly selected hardback, then $PB \sim \text{N}(2.1,\ 0.39)$ and $HB \sim \text{N}(4.0,\ 1.56)$
Let Y be the thickness of 15 randomly selected paperbacks, $Y = PB_1 + PB_2 + PB_3 + \ldots + PB_{15}$
$\text{E}(Y) = 15 \times 2.1 = 31.5 \qquad \text{Var}(Y) = 15 \times 0.39 = 5.85$
So $Y \sim \text{N}(31.5,\ 5.85)$
$\text{P}(Y < 30) = 0.2676$ (4 d.p.)

ii Let Z be the thickness of 5 randomly selected paperbacks and 5 randomly selected hardbacks, then $Z = PB_1 + PB_2 + PB_3 + PB_4 + PB_5 + HB_1 + HB_2 + HB_3 + HB_4 + HB_5$
$\text{E}(Z) = 5 \times 2.1 + 5 \times 4.0 = 30.5 \qquad \text{Var}(Z) = 5 \times 0.39 + 5 \times 1.56 = 9.75$
So $Z \sim \text{N}(30.5,\ 9.75)$
$\text{P}(Z < 30) = 0.4364$ (4 d.p.)

b Using $Y \sim \text{N}(31.5,\ 5.85)$ from part **ai**, find x such that $\text{P}(Y < x) = 0.99$
Using the inverse normal distribution function of the calculator, $x = 37.1$ cm (3 s.f.).

13 a Let A be the difference in mass of two randomly selected Yummies, so $A = Y_1 - Y_2$.
$\text{E}(A) = \text{E}(Y_1) - \text{E}(Y_2) = 0 \qquad \text{Var}(A) = \text{Var}(Y_1) + \text{Var}(Y_2) = 32$
So $A \sim \text{N}(0,\ 32)$
Required to find $\text{P}(A > 5) + \text{P}(A < -5) = 1 - \text{P}(A < 5) + \text{P}(A < -5) = 0.3768$ (4 d.p.)

b Let $B = Y - X$, then $B \sim \text{N}(32 - 30,\ 16 + 25)$, so $B \sim \text{N}(2,\ 41)$
Then $\text{P}(B > 0) = 1 - \text{P}(B < 0) = 1 - 0.3774 = 0.6226$ (4 d.p.)

c Let Z be the thickness of 6 randomly selected Xtras and 4 randomly selected Yummies.
$\text{E}(C) = 6 \times 30 + 4 \times 32 = 308 \qquad \text{Var}(A) = 6 \times 25 + 4 \times 16 = 214$
So $C \sim \text{N}(308,\ 214)$
$\text{P}(280 < C < 330) = \text{P}(C < 330) - \text{P}(C < 280) = 0.9337 - 0.0278 = 0.9059$ (4 d.p.)

14 Let B be the mass of a randomly selected biscuit, W be the mass of an individual wrapper, M be the mass of the packaging material and A be total mass of a packet of 6 biscuits, then:

$E(A) = 6\times75+6\times10+40 = 550 \qquad Var(A) = 6\times5^2+6\times2^2+3^2 = 183$

So $A \sim N(550, 183)$

$P(535 < A < 565) = P(A < 565) - P(A < 535) = 0.8662 - 0.1337 = 0.732$ (3 d.p.)

15 a i $E(Q) = 2E(X)+E(Y) = 2\times10+40 = 60$

ii $Var(Q) = 2^2\,Var(X)+Var(Y) = 2^2\times2^2+3^2 = 25$

b i $E(R) = 5E(X) = 5\times10 = 50$

$Var(R) = 5\times Var(X) = 5\times2^2 = 20$

So $R \sim N(50, 20)$

ii Let $S = Q - R$, so $S \sim N(60-50, 25+20)$, i.e. $S \sim N(10, 45)$ S

$P(Q > R) = P(Q - R > 0) = 1 - P(S < 0) = 1 - 0.0680 = 0.9320$ (4 d.p.)

16 a Let C be the usable capacity of a randomly selected games console, G be the storage required by a randomly selected game and A be storage required by 10 games, then:

$C \sim N(60, 2.5^2) \qquad G \sim N(5.5, 1.2^2) \qquad A \sim N(10\times5.5, 10\times1.2^2) \Rightarrow A \sim N(55, 14.4)$

Let $B = C - A$, so $B \sim N(60-55, 14.4+6.25) \Rightarrow B \sim N(5, 20.65)$

Required to find $P(B > 0) = 1 - P(B < 0) = 1 - 0.1356 = 0.8644$ (4 d.p.)

b Assuming that all random variables are independent, so the storage space required by each game and the usable capacity of the console are all independent.

17 $Y \sim N(3\times4, 3\times0.03)$, so $Y \sim N(12, 0.09)$

$Z \sim N(3\times4, 3^2\times0.03)$, so $Z \sim N(12, 0.27)$

Let $W = Z - Y$, so $W \sim N(12-12, 0.27+0.09) \Rightarrow W \sim N(0, 0.36)$

Required to find $P(-1 < W < 1) = P(W < 1) - P(W < -1) = 0.9522 - 0.0478 = 0.9044$ (4 d.p.)

18 a $L \sim N(75, 5^2)$, $S \sim N(40, 3^2)$

Let $D = S - 0.5L$, so $D \sim N(40 - 0.5\times75, 3^2+0.5^2\times5^2)$

So $D \sim N(2.5, 15.25)$

$P(D > 0) = 1 - P(D < 0) = 1 - 0.2610 = 0.7390$ (4 d.p.)

b $M \sim N(10\times40, 10\times3^2)$, so $M \sim N(400, 90)$

$P(|M - 400| < 5) = P(395 < M < 405) = P(M < 405) - P(M < 395)$

$= 0.7009 - 0.2991 = 0.4018$ (4 d.p.)

Challenge

$$\begin{aligned}
\text{Var}(X+Y) &= \text{E}\left((X+Y)^2\right) - \left(\text{E}(X+Y)\right)^2 \\
&= \text{E}(X^2+2XY+Y^2) - \left(\text{E}(X)+\text{E}(Y)\right)\left(\text{E}(X)+\text{E}(Y)\right) \\
&= \text{E}(X^2)+2\text{E}(XY)+\text{E}(Y^2) - \left(\text{E}(X)\text{E}(X)+2\text{E}(X)\text{E}(Y)+\text{E}(Y)\text{E}(Y)\right) \\
&= \text{E}(X^2)-\text{E}(X)\text{E}(X)+2\text{E}(XY)-2\text{E}(X)\text{E}(Y)+\text{E}(Y^2)-\text{E}(Y)\text{E}(Y) \\
&= \text{E}(X^2)-(\text{E}(X))^2+\text{E}(Y^2)-(\text{E}(Y))^2 \qquad \text{as } \text{E}(XY)=\text{E}(X)\text{E}(Y) \\
&= \text{Var}(X)+\text{Var}(Y)
\end{aligned}$$