Further hypothesis tests Mixed exercise

1 a confidence interval $=\left(\frac{(n-1)s^2}{\chi^2_{n-1}\left(\frac{\alpha}{2}\right)},\frac{(n-1)s^2}{\chi^2_{n-1}\left(1-\frac{\alpha}{2}\right)}\right)=\left(\frac{13\times 1.8}{24.736},\frac{13\times 1.8}{5.009}\right)$

$=(0.946, 4.672)$

b confidence interval $=\left(\frac{(n-1)s^2}{\chi^2_{n-1}\left(\frac{\alpha}{2}\right)},\frac{(n-1)s^2}{\chi^2_{n-1}\left(1-\frac{\alpha}{2}\right)}\right)=\left(\frac{13\times 1.8}{22.362},\frac{13\times 1.8}{5.892}\right)$

$=(1.046, 3.971)$

2 $\bar{x}=\frac{1428}{20}=71.4 \quad s^2=\frac{102\ 286-20\times 71.4^2}{19}=17.2$

a confidence interval $=\left(\frac{(n-1)s^2}{\chi^2_{n-1}\left(\frac{\alpha}{2}\right)},\frac{(n-1)s^2}{\chi^2_{n-1}\left(1-\frac{\alpha}{2}\right)}\right)=\left(\frac{19\times 17.2}{32.852},\frac{19\times 17.2}{8.907}\right)$

$=(9.948, 36.69)$

b $10=1.6449\times\sigma$ so $\sigma=\frac{10}{1.6449}=6.079$

c $\sqrt{36.69}<6.079$ so the supervisor should not be concerned.

3 a A confidence interval for a population parameter is a range of values defined so that there is a specific probability that the true value of the parameter lies within that range.

b The percentage points are:

$\chi^2_{19}(0.95)=10.117$ and $\chi^2_{19}(0.05)=30.144$.

We are given $s^2=3.75^2$ and so can calculate that the critical points are:

$\frac{(20-1)s^2}{\chi^2_{19}(0.95)}=\frac{19\times 3.75^2}{10.117}=26.4$ and $\frac{(20-1)s^2}{\chi^2_{19}(0.05)}=\frac{19\times 3.75^2}{30.144}=8.86.$

So the 90% confidence interval for **variance** is (8.86, 26.4).

The 90% confidence interval for the **standard deviation** has the square root of the limits of this interval as its limits. i.e. (2.98, 5.14).

4 Our hypotheses are $H_0 : \sigma = 2.7$ and $H_1 : \sigma \neq 2.7$
The significance level is 5% (2.5% at each tail) with $v = 6$ degrees of freedom.

From the table, we find critical values of $\chi_6^2(0.975) = 1.237$ and $\chi_6^2(0.025) = 14.449$.

The critical regions are $\frac{(n-1)s^2}{\sigma^2} \geqslant 14.449$ and $\frac{(n-1)s^2}{\sigma^2} \leqslant 1.237$.
We calculate an unbiased estimate of the variance to be

$$s^2 = \frac{1}{n-1}\left(\sum x^2 - \frac{\left(\sum x\right)^2}{n}\right)$$
$$= \frac{1}{7-1}\left(7338.07 - \frac{225.9^2}{7}\right)$$
$$= 7.99.$$

$s^2 = 7.99$ and $\sigma^2 = 2.7^2$.

So our test statistic is:

$$\frac{(n-1)s^2}{\sigma^2} = \frac{(7-1)\times 7.99}{2.7^2} = 6.58.$$

$1.237 < 6.58 < 14.449$

6.58 is not in the critical region so we do not have sufficient evidence to reject H_0.
We conclude that there has been no change in standard deviation.

5 **a** First we find s^2 using the equation

$$s^2 = \frac{1}{n-1}\left(\sum x^2 - \frac{\left(\sum x\right)^2}{n}\right)$$
$$= \frac{1}{10-1}\left(3127 - \frac{(171)^2}{10}\right)$$
$$= \frac{1}{9}\left(3127 - \frac{29241}{10}\right)$$
$$= \frac{2029}{90}$$

The percentage points are $\chi_9^2(0.975) = 2.700$ and $\chi_9^2(0.025) = 19.023$

The critical points ar $\frac{(10-1)s^2}{\chi_9^2(0.975)} = \frac{9\times\frac{2029}{90}}{2.700} = 75.148$ and $\frac{(10-1)s^2}{\chi_9^2(0.025)} = \frac{9\times\frac{2029}{90}}{19.023} = 10.667$.
The 95% confidence interval for the **variance** of the diameters is (10.667, 75.148).

Hence the 95% confidence interval for the **standard deviation** of the diameters is (3.266, 8.669).

5 b We have assumed that this sample is from a normal distribution.

c Since the probability of the population standard deviation being between the values of 3.266 and 75.148 is 0.95, that means there is at most a 5% chance that the standard deviation is outside of this range.
Due to Giovanna requiring a standard deviation less than 3.1 minutes (and this range lies entirely outside the 95% confidence interval), the dosage should be changed.

6 $\bar{x} = 45.1 \quad s = 6.838\ldots$

$H_0 : \sigma = 5 \quad H_1 : \sigma \neq 5$

Critical region > 19.023 and < 2.700

Test statistic $= \dfrac{(n-1)s^2}{\sigma^2} = \dfrac{9 \times 6.838\ldots^2}{5^2} = 16.836$

Since 16.836 is not in the critical region, there is insufficient evidence to reject H_0.

Therefore accept $\sigma = 5$ kg

7 $P(F_{5,10} \geqslant 3.33) = 0.05 \Rightarrow b = 3.33$

$P(F_{10,5} \geqslant 4.74) = 0.05 \Rightarrow P\left(F_{5,10} \leqslant \dfrac{1}{4.74}\right) = 0.05$

$\therefore a = 0.2110$ (4 s.f.)

8 a $H_0 : \sigma_1^2 = \sigma_2^2 \quad H_1 : \sigma_1^2 \neq \sigma_2^2$

$\dfrac{s_1^2}{s_2^2} = \dfrac{14^2}{8^2} = 3.0625 \quad \left(\text{or } \dfrac{s_2^2}{s_1^2} = \dfrac{8^2}{14^2} = 0.32653\ldots\right)$

Critical value $F_{12,7} = 3.57 \quad \left(\text{Critical value: } F_{7,12} = \dfrac{1}{3.57} = 0.28011\right)$

Since 3.0625 is not in the critical region there is insufficient evidence to reject H_0.

There is insufficient evidence of a difference in the variances of the lengths of the fence posts.

b The distribution of the population of lengths of fence posts is normally distributed.

9 $H_0 : \sigma_F^2 = \sigma_M^2 \quad H_1 : \sigma_F^2 \neq \sigma_M^2$

$s_F^2 = \dfrac{1}{6}(17\ 956.5 - 7 \times 50.6^2) = \dfrac{33.98}{6} = 5.66333....$

$s_M^2 = \dfrac{1}{9}(28\ 335.1 - 10 \times 53.2^2) = \dfrac{32.7}{9} = 3.63333....$

$\dfrac{s_F^2}{s_M^2} = 1.5587\ldots$(Reciprocal 0.6415)

$F_{6,9} = 3.37$ (or $F_{9,6} = 0.297$)

Not in critical region.

There is no reason to doubt the variances of the two distributions are the same.

Challenge

a We use the fact that $\text{Var}(a\times s^2)=a^2\,\text{Var}(s^2)$ as well as $v=n-1$ in order to find an expression for $\text{Var}(\chi_v^2)$ in terms of s, n and σ.

$$\begin{aligned}\text{Var}(\chi_v^2)&=\text{Var}\left(\frac{v\times s^2}{\sigma^2}\right)\\&=\text{Var}\left(\frac{(n-1)s^2}{\sigma^2}\right)\\&=\left(\frac{n-1}{\sigma^2}\right)^2\text{Var}(s^2)\\&=\frac{(n-1)^2}{\sigma^4}\text{Var}(s^2).\end{aligned}$$

From the question, we are also given $\text{Var}(\chi_v^2)=2v=2(n-1)$.

Now we equate these expressions in order to obtain an expression for $\text{Var}(s^2)$:

$$\frac{(n-1)^2}{\sigma^4}\text{Var}(s^2)=2(n-1)$$

$$\text{Var}(s^2)=\frac{2\sigma^4}{n-1}$$

b The variance of the estimator decreases as n increases.
This implies that it becomes more accurate and closes in on the population variance as the sample size grows large.