Confidence intervals and tests using the *t*-distribution 7A

1 a $t_{12}(0.025) = 2.179$, so $P(X > t) = 0.025$ when $t = 2.179$
So, by symmetry, $P(X < t) = 0.025$ when $t = -2.179$

b $t_{12}(0.05) = 1.782$, so $P(X > t) = 0.05$ when $t = 1.782$

c $P(|X| > t) = P(X < -t) + P(X > t)$
From part **a**, $P(X > 2.179) = 0.025$ and $P(X < -2.179) = 0.025$
Hence $P(|X| > 2.179) = P(X < -2.179) + P(X > 2.179) = 0.025 + 0.025 = 0.05$
Solution: $|t| = 2.179$

2 a $t_{26}(0.01) = 2.479$ (from tables)

b $t_{26}(0.05) = 1.706$ (from tables)

3 a $t_{10}(0.05) = 1.812$, so $P(X > t) = 0.05$ when $t = 1.812$
Hence $P(Y < t) = 0.95$ when $t = 1.812$

b $t_{32}(0.005) = 2.738$, so $P(X > t) = 0.005$ when $t = 2.738$
Hence $P(Y < t) = 0.005$ when $t = -2.738$

c $t_5(0.025) = 2.571$, so $P(X > t) = 0.025$ when $t = 2.571$
Hence $P(Y < t) = 0.025$ when $t = -2.571$

d $t_{16}(0.01) = 2.583$, so $P(Y > t) = 0.01$ when $t = 2.583$ and $P(Y < -t) = 0.01$ when $t = -2.583$
So $P(|Y| > t) = P(Y < -t) + P(Y > t) = 0.02$ when $|t| = 2.583$
Hence $P(|Y| < t) = 0.98$ when $|t| = 2.583$

e $t_{18}(0.05) = 1.734$, so $P(Y > t) = 0.05$ when $t = 1.734$ and $P(Y < -t) = 0.05$ when $t = -1.734$
Hence $P(|Y| > t) = P(Y < -t) + P(Y > t) = 0.1$ when $|t| = 1.734$

4 Using a calculator gives $\bar{x} = 20.95$ and $s^2 = 12.0542857$, so
$s = \sqrt{12.0542857} = 3.4719\ldots$
The 90% confidence limits for the mean are

$$\bar{x} \pm t_{(n-1)}\left(\frac{\alpha}{2}\right) \times \frac{s}{\sqrt{n}} = 20.95 \pm t_7(0.05) \times \frac{3.4719\ldots}{\sqrt{8}}$$
$$= 20.95 \pm 1.895 \times \frac{3.4719\ldots}{\sqrt{8}}$$
$$= 20.95 \pm 2.3261$$

So the 90% confidence interval is (18.624, 23.276)

5 As $\bar{x} = 12.4$ and $s^2 = 21.0$, $s = \sqrt{21} = 4.58257\ldots$

The 95% confidence limits for the mean are

$$\bar{x} \pm t_{(n-1)}\left(\frac{\alpha}{2}\right) \times \frac{s}{\sqrt{n}} = 12.4 \pm t_{15}(0.025) \times \frac{\sqrt{21}}{\sqrt{16}}$$
$$= 12.4 \pm 2.131 \times \frac{\sqrt{21}}{\sqrt{16}}$$
$$= 12.4 \pm 2.4413$$

So the 95% confidence interval is (9.959, 14.841)

6 a Using a calculator gives $\bar{x} = 179.333\ldots$ and $s^2 = 30.2666\ldots$, so
$s = \sqrt{30.2666\ldots} = 5.5015\ldots$

The 90% confidence limits for the mean are

$$\bar{x} \pm t_{(n-1)}\left(\frac{\alpha}{2}\right) \times \frac{s}{\sqrt{n}} = 179.3333 \pm t_5(0.05) \times \frac{5.5015}{\sqrt{6}}$$
$$= 179.3333 \pm 2.015 \times \frac{5.5015}{\sqrt{6}}$$
$$= 179.3333 \pm 4.5256$$

So the 90% confidence interval is (174.808, 183.859)

b The 95% confidence limits for the mean are

$$\bar{x} \pm t_{(n-1)}\left(\frac{\alpha}{2}\right) \times \frac{s}{\sqrt{n}} = 179.3333 \pm t_5(0.025) \times \frac{5.5015}{\sqrt{6}}$$
$$= 179.3333 \pm 2.571 \times \frac{5.5015}{\sqrt{6}}$$
$$= 179.3333 \pm 5.7744$$

So the 95% confidence interval is (173.559, 185.108)

7 a Using a calculator gives $\bar{x} = 10.36$ and $s^2 = 0.538222\ldots$, so
$s = \sqrt{0.538222\ldots} = 0.73363\ldots$

The 98% confidence limits for the mean are

$$\bar{x} \pm t_{(n-1)}\left(\frac{\alpha}{2}\right) \times \frac{s}{\sqrt{n}} = 10.36 \pm t_9(0.01) \times \frac{0.73363}{\sqrt{10}}$$
$$= 10.36 \pm 2.821 \times \frac{0.73363}{\sqrt{10}}$$
$$= 10.36 \pm 0.654$$

So the 98% confidence interval is (9.706, 11.014)

b The masses of the nails are normally distributed.

8 $\bar{x} = \frac{\sum x}{n} = \frac{224.1}{8} = 28.0125$ and $s^2 = \frac{1}{n-1}\left(\sum x^2 - n\bar{x}^2\right) = \frac{1}{7}\left(6337.39 - \frac{224.1^2}{8}\right) = 8.54125$, so

$s = \sqrt{8.54125} = 2.92254\ldots$

The 99% confidence limits for the mean are

$$\bar{x} \pm t_{(n-1)}\left(\frac{\alpha}{2}\right) \times \frac{s}{\sqrt{n}} = 28.0125 \pm t_7(0.005) \times \frac{2.99254}{\sqrt{8}}$$
$$= 28.0125 \pm 3.499 \times \frac{2.99254}{\sqrt{8}}$$
$$= 28.0125 \pm 3.6154$$

So the 99% confidence interval is (24.397, 31.628)

9 As $\bar{x} = 122$ and $s^2 = 225$, $s = 15$

The 95% confidence limits for the mean are

$$\bar{x} \pm t_{(n-1)}\left(\frac{\alpha}{2}\right) \times \frac{s}{\sqrt{n}} = 122 \pm t_{25}(0.025) \times \frac{15}{\sqrt{26}}$$
$$= 122 \pm 2.060 \times \frac{15}{\sqrt{26}}$$
$$= 122 \pm 6.0599$$

So the 95% confidence interval is (115.94, 128.06)

10 The completed table is:

	Normal	χ^2	***t***
For the population mean, using a sample of size 50 from a population of unknown variance			✓
For the population mean, using a sample of size 6 from a population of known variance	✓		
For the population variance, using a sample of size 20		✓	

Use the *t*-distribution to find a confidence interval for the mean of a normal distribution when the population variance is unknown (covered in Chapter 7 of the textbook).
Use the standardised normal distribution to find a confidence interval for the mean of a normal distribution when the population variance is known (covered in Chapter 5 of the textbook).
Use the chi-squared distribution to find a confidence interval for the population variance (covered in Chapter 6 of the textbook).

11 a The *t*-distribution must be used because the population variance is unknown and the sample size is small. (If the sample size is large, the sample variance can be used as an approximation of the normal variance.)

11 b $\bar{x} = \frac{\sum x}{n} = \frac{7338}{15} = 489.2$ and $s^2 = \frac{1}{n-1}\left(\sum x^2 - n\bar{x}^2\right) = \frac{1}{14}\left\{3618260 - \frac{7338^2}{15}\right\} = 2036.4571$, so

$s = \sqrt{2036.4571} = 45.1271\ldots$

The 90% confidence limits for the mean are

$$\bar{x} \pm t_{(n-1)}\left(\frac{\alpha}{2}\right) \times \frac{s}{\sqrt{n}} = 489.2 \pm t_{14}(0.05) \times \frac{45.1271}{\sqrt{15}}$$
$$= 489.2 \pm 1.761 \times \frac{45.1271}{\sqrt{15}}$$
$$= 489.2 \pm 20.519$$

So the 90% confidence interval is (468.7, 509.7)

c The lifespan of the light bulbs is normally distributed

d $$\frac{(n-1)s^2}{\chi^2_{n-1}(0.025)} = \frac{14 \times 2036.46}{26.119} = 1091.56$$
$$\frac{(n-1)s^2}{\chi^2_{n-1}(0.975)} = \frac{14 \times 2036.46}{5.629} = 5064.92$$

So the 95% confidence interval for the population variance is (1091.56, 5064.92)