Confidence intervals and tests using the *t*-distribution 7E

1 a $s_p^2 = \frac{(n_{1st}-1)s_{1st}^2 + (n_{2nd}-1)s_{2nd}^2}{n_{1st}+n_{2nd}-2} = \frac{(19\times 12)+(10\times 12)}{20+11-2} = 12$

b H_0: $\mu_{1st} = \mu_{2nd}$ H_1: $\mu_{1st} \neq \mu_{2nd}$

Significance level 5%

$\upsilon = 29$

The critical value is $t_{29}(0.025) = 2.045$, so the critical regions are $t \leqslant -2.045$ and $t \geqslant 2.045$

Test statistic $t = \frac{(\bar{x}_{1st} - \bar{x}_{2nd}) - (\mu_{1st} - \mu_{2nd})}{s_p\sqrt{\frac{1}{n_{1st}} + \frac{1}{n_{2nd}}}} = \frac{(16-14)-0}{3.464\sqrt{\frac{1}{20}+\frac{1}{11}}} = 1.538$

$1.538 < 2.045$, so the result is not significant. Accept H_0.

There is evidence to suggest that the populations have the same mean.

2 H_0: $\mu_f = \mu_w$ H_1: $\mu_f > \mu_w$

Significance level 5%

$\upsilon = 6+4-2 = 8$

The critical value is $t_8(0.05) = 1.860$, so the critical region is $t \geqslant 1.86$

$$\bar{x}_w = \frac{\sum x_w}{n_w} = \frac{42.8+40+38.2+37.5+37.0+36.5}{6} = \frac{232}{6} = 38.667$$

$$\sum x_w^2 = 8998.58 \text{ so } s_w^2 = \frac{1}{n_w - 1}\left(\sum x_w^2 - n_w \bar{x}_w^2\right) = \frac{1}{5}(8998.58 - 6\times 38.667^2) = 5.5827$$

$$\bar{x}_c = \frac{\sum x_c}{n_c} = \frac{42.0+43.0+41.5+40.0}{4} = \frac{166.5}{4} = 41.625$$

$$\sum x_c^2 = 6935.25 \text{ so } s_c^2 = \frac{1}{n_c - 1}\left(\sum x_c^2 - n_c \bar{x}_c^2\right) = \frac{1}{3}(6935.25 - 4\times 41.625^2) = 1.5625$$

$$s_p^2 = \frac{(n_w-1)s_w^2 + (n_c-1)s_c^2}{n_w+n_c-2} = \frac{(5\times 5.5827)+(3\times 1.5625)}{6+4-2} = 4.075 \quad \text{so } s_p = 2.0187$$

Test statistic $t = \frac{(\bar{x}_c - \bar{x}_w) - (\mu_c - \mu_w)}{s_p\sqrt{\frac{1}{n_c}+\frac{1}{n_w}}} = \frac{41.625-38.667}{2.0187\sqrt{\frac{1}{6}+\frac{1}{4}}} = 2.270$

$2.27 > 1.86$, so the result is significant. Reject H_0.

There is evidence to suggest that the salmon are not wild but have been obtained from a fish farm.

3 a $H_0: \mu_t = \mu_e$ $H_1: \mu_t \neq \mu_e$

Significance level 5% (2.5% in each tail)

$\upsilon = 6+6-2=10$

The critical value is $t_{10}(0.025)=1.812$, so the critical regions are $t \leqslant -2.228$ and $t \geqslant 2.228$

$$\bar{x}_t = \frac{\sum x_t}{n_t} = \frac{0.711}{6} = 0.1185$$

$$\sum x_t^2 = 0.086867 \text{ so } s_t^2 = \frac{1}{n_t - 1}\left(\sum x_t^2 - n_t \bar{x}_t^2\right) = \frac{1}{5}(0.086867 - 6\times 0.1185^2) = 0.0005227$$

$$\bar{x}_e = \frac{\sum x_e}{n_e} = \frac{0.855}{6} = 0.1425$$

$$\sum x_e^2 = 0.127497 \text{ so } s_e^2 = \frac{1}{n_e - 1}\left(\sum x_e^2 - n_e \bar{x}_e^2\right) = \frac{1}{5}(0.127497 - 6\times 0.1425^2) = 0.0011319$$

$$s_p^2 = \frac{(n_t-1)s_t^2 + (n_e-1)s_e^2}{n_t+n_e-2} = \frac{(5\times 0.0005227)+(5\times 0.0011319)}{6+6-2} = 0.0008273 \quad \text{so } s_p = 0.02876$$

$$\text{Test statistic } t = \frac{(\bar{x}_e - \bar{x}_t) - (\mu_e - \mu_t)}{s_p\sqrt{\frac{1}{n_e}+\frac{1}{n_t}}} = \frac{0.1425 - 0.1185}{0.02876\sqrt{\frac{1}{6}+\frac{1}{6}}} = 1.445$$

$1.445 < 2.228$ so the result is not significant. Accept H_0.

There is evidence to suggest that tetracycline and erythromycin are equally as effective.

b $H_0: \mu_s = \mu_{te}$ $H_1: \mu_s > \mu_{te}$

Significance level 5%

$\upsilon = 6+12-2=16$

The critical value is $t_{16}(0.05)=1.746$, so the critical region is $t \geqslant 1.746$

$$\bar{x}_s = \frac{\sum x_s}{n_s} = \frac{1.432}{6} = 0.2387$$

$$\sum x_s^2 = 0.34425 \text{ so } s_s^2 = \frac{1}{n_s - 1}\left(\sum x_s^2 - n_s \bar{x}_s^2\right) = \frac{1}{5}(0.34425 - 6\times 0.2387^2) = 0.00049587$$

$$\bar{x}_{te} = \frac{\sum x_{te}}{n_{te}} = \frac{1.566}{12} = 0.1305$$

$$\sum x_{te}^2 = 0.214364 \text{ so } s_{te}^2 = \frac{1}{n_{te} - 1}\left(\sum x_{te}^2 - n_{te} \bar{x}_{te}^2\right) = \frac{1}{11}(0.214364 - 6\times 0.1305^2) = 0.00090918$$

$$s_p^2 = \frac{(n_s-1)s_s^2 + (n_{te}-1)s_{te}^2}{n_s+n_{te}-2} = \frac{(5\times 0.00049587)+(11\times 0.00090918)}{6+12-2} = 0.000780 \quad \text{so } s_p = 0.0279$$

$$\text{Test statistic } t = \frac{(\bar{x}_s - \bar{x}_{te}) - (\mu_s - \mu_{te})}{s_p\sqrt{\frac{1}{n_s}+\frac{1}{n_{te}}}} = \frac{0.2387 - 0.1305}{0.0279\sqrt{\frac{1}{6}+\frac{1}{12}}} = 7.756$$

$7.756 > 1.746$, so the result is significant. Reject H_0.

There is evidence to suggest that streptomycin is more effective than the other two drugs.

4 a $H_0: \mu_{old} = \mu_{new}$ $H_1: \mu_{old} > \mu_{new}$

b $H_0: \mu_{old} = \mu_{new}$ $H_1: \mu_{old} > \mu_{new}$

Significance level 5%

$\upsilon = 7 + 9 - 2 = 14$

The critical value is $t_{14}(0.05) = 1.761$, so the critical region is $t \geqslant 1.761$

$$\bar{x}_{new} = \frac{\sum x_{new}}{n_{new}} = \frac{4.9 + 6.3 + 9.6 + 5.2 + 4.1 + 7.2 + 4.0}{7} = \frac{41.3}{7} = 5.9$$

$$\sum x_{new}^2 = 267.55 \text{ so } s_{new}^2 = \frac{1}{n_{new} - 1}\left(\sum x_{new}^2 - n_{new}\bar{x}_{new}^2\right) = \frac{1}{6}(267.55 - 7 \times 5.9^2) = 3.98$$

$$\bar{x}_{old} = \frac{\sum x_{old}}{n_{old}} = \frac{71.2}{9} = 7.911$$

$$s_{old}^2 = \frac{1}{n_{old} - 1}\left(\sum x_{old}^2 - n_{old}\bar{x}_{old}^2\right) = \frac{1}{8}(604.92 - 9 \times 7.911^2) = 5.2061$$

$$s_p^2 = \frac{(n_{new} - 1)s_{new}^2 + (n_{old} - 1)s_{old}^2}{n_{new} + n_{old} - 2} = \frac{(6 \times 3.98) + (8 \times 5.2061)}{7 + 9 - 2} = 4.6806 \text{ so } s_p = 2.1635$$

$$\text{Test statistic } t = \frac{(\bar{x}_{old} - \bar{x}_{new}) - (\mu_{old} - \mu_{new})}{s_p\sqrt{\frac{1}{n_{old}} + \frac{1}{n_{new}}}} = \frac{7.911 - 5.9}{2.1635\sqrt{\frac{1}{7} + \frac{1}{9}}} = 1.844$$

$1.844 > 1.761$, so the result is significant. Reject H_0.

There is evidence to suggest that the new language does allow faster task completion.

c Once a task has been completed once, a programmer should be quicker next time with either language.

5 a $\sum x = \sum y - \sum v = 27 \times \bar{y} - 384 = 27 \times 34 - 384 = 918 - 384 = 534$

$$s^2 = \frac{1}{n-1}\left(\sum y^2 - n\bar{y}^2\right) = \frac{1}{26}\left(\sum y^2 - 27 \times 34^2\right)$$

$$\Rightarrow \sum y^2 = 26 \times s^2 + 27 \times 34^2 = 26 \times 23 + 31\,212 = 31\,810$$

$$\text{So } \sum x^2 = \sum y^2 - \sum v^2 = 31\,810 - 12\,480 = 19\,330$$

b $\bar{x}_v = \frac{\sum v}{n_v} = \frac{384}{12} = 32$ $\quad s_v^2 = \frac{1}{n_v - 1}\left(\sum v^2 - n_v\bar{v}^2\right) = \frac{1}{11}(12\,480 - 12 \times 32^2) = 17.455$

$\bar{x}_s = \frac{\sum x}{n_s} = \frac{534}{15} = 35.6$ $\quad s_s^2 = \frac{1}{n_s - 1}\left(\sum x^2 - n_s\bar{x}_s^2\right) = \frac{1}{14}(19\,330 - 15 \times 35.6^2) = 22.829$

$$s_p^2 = \frac{(n_v - 1)s_v^2 + (n_s - 1)s_s^2}{n_v + n_s - 2} = \frac{(11 \times 17.455) + (14 \times 22.829)}{12 + 15 - 2} = 20.464$$

5 c $H_0: \mu_v = \mu_s \quad H_1: \mu_v \neq \mu_s$

Significance level 5% (2.5% in each tail)

$\upsilon = 12 + 15 - 2 = 25$

The critical value is $t_{25}(0.025) = 2.060$, so the critical regions are $t \leqslant -2.06$ and $t \geqslant 2.06$

$s_p^2 = 20.464$ so $s_p = 4.5238$

$$\text{Test statistic } t = \frac{(\bar{x}_s - \bar{x}_v) - (\mu_s - \mu_v)}{s_p\sqrt{\frac{1}{n_s} + \frac{1}{n_v}}} = \frac{35.6 - 32}{4.5238\sqrt{\frac{1}{15} + \frac{1}{12}}} = 2.0547$$

$2.0547 < 2.06$, so the result is not significant. Accept H_0.

There is insufficient evidence to suggest that the mean consumption of the two models is different.

d The test assumes that the populations (consumption of each model) are normally distributed.

e To get a fair comparison, the fuel consumption of the cars in the test should be measured over the same types of driving, roads and weather.

6 a $H_0: \sigma_A^2 = \sigma_B^2 \qquad H_1: \sigma_A^2 \neq \sigma_B^2$

$\upsilon_l = 13 - 1 = 12 \qquad \upsilon_s = 11 - 1 = 10$

$s_l^2 = 3.3$ and $s_s^2 = 2.1$

The critical value is $F_{12,10}(0.05) = 2.91$

$$\text{The test statistic is } \frac{s_l^2}{s_s^2} = \frac{3.3}{2.1} = 1.57$$

1.57 < 2.91, so accept H_0.

There is no evidence that there is a difference in the variability of the weights.

The test assumes that the samples are taken from populations that are normally distributed.

b The result in part **a** supports the assumption that the variances of the population are equal, so the use of a *t*-distribution to test the hypothesis is justified providing that two other requirements are met: that the populations that are normally distributed and the two samples are independent.

c $H_0: \mu_A = \mu_B \quad H_1: \mu_A \neq \mu_B$

Significance level 5% (2.5% in each tail)

$\upsilon = 11 + 13 - 2 = 22$

The critical value is $t_{22}(0.025) = 2.074$, so the critical regions are $t \leqslant -2.074$ and $t \geqslant 2.074$

$$s_p^2 = \frac{(n_A - 1)s_A^2 + (n_B - 1)s_B^2}{n_A + n_B - 2} = \frac{(10 \times 2.1) + (12 \times 3.3)}{11 + 13 - 2} = 2.755 \quad \text{so } s_p = 1.6597$$

$$\text{Test statistic } t = \frac{(\bar{x}_B - \bar{x}_A) - (\mu_B - \mu_A)}{s_p\sqrt{\frac{1}{n_B} + \frac{1}{n_A}}} = \frac{46.3 - 42.1}{1.6597\sqrt{\frac{1}{13} + \frac{1}{11}}} = 6.177$$

$6.178 > 2.074$, so the result is significant. Reject H_0.

There is evidence that there is a difference in the average weight of the potatoes.

Challenge

$$\mathrm{E}\left((n_x - 1)s_x^2 + (n_y - 1)s_y^2\right) = \mathrm{E}\left((n_x - 1)s_x^2\right) + \mathrm{E}\left((n_y - 1)s_y^2\right)$$
$$= (n_x - 1)s^2 + (n_y - 1)s^2$$
$$= \left((n_x - 1) + (n_y - 1)\right)s^2$$

So

$$\mathrm{E}\left(S_p^2\right) = \mathrm{E}\left(\frac{(n_x - 1)s_x^2 + (n_y - 1)s_y^2}{(n_x - 1) + (n_y - 1)}\right)$$
$$= \frac{\mathrm{E}\left((n_x - 1)s_x^2 + (n_y - 1)s_y^2\right)}{(n_x - 1) + (n_y - 1)}$$
$$= \sigma$$

So S_p^2 is an unbiased estimator.