Confidence intervals and tests using the *t*-distribution Mixed exercise 7

1 $H_0: \mu = 28 \quad H_1: \mu \neq 28$

Significance level 5%, probability in each tail = 0.025

$\upsilon = 13$

The critical value is $t_{13}(0.025) = 2.160$, so the critical regions are $t \leqslant -2.16$ and $t \geqslant 2.16$

$s^2 = 36 \Rightarrow s = 6$

Test statistic $t = \dfrac{\bar{x} - \mu}{\frac{s}{\sqrt{n}}} = \dfrac{30.4 - 28}{\frac{6}{\sqrt{14}}} = 1.497$

$1.497 < 2.16$, so the result is not significant. Accept H_0.

There is not enough evidence to suggest that μ is not 28.

2 $H_0: \mu = 10 \quad H_1: \mu > 10$

Significance level 5%

$\upsilon = 7$

The critical value is $t_7(0.05) = 1.895$, so the critical region is $t \geqslant 1.895$

$\bar{x} = \dfrac{\sum x}{n} = \dfrac{85}{8} = 10.625$

$s^2 = \dfrac{1}{n-1}\left(\sum x^2 - n\bar{x}^2\right) = \dfrac{1}{7}(970.25 - 8 \times 10.625^2) = 9.5892$

$s = 3.0967$

Test statistic $t = \dfrac{\bar{x} - \mu}{\frac{s}{\sqrt{n}}} = \dfrac{10.625 - 10.0}{\frac{3.0967}{\sqrt{8}}} = 0.571$

$0.571 < 1.895$, so the result is not significant. Accept H_0.

There is not enough evidence to suggest that μ is > 10.

3 **a** Using a calculator gives $\bar{x} = 52.833$ and $s^2 = 2.9667$, so

$s = \sqrt{2.9667} = 1.7224$

The 95% confidence limits for the mean are

$$\bar{x} \pm t_{(n-1)}\left(\frac{\alpha}{2}\right) \times \frac{s}{\sqrt{n}} = 52.833 \pm t_5(0.025) \times \frac{1.7224}{\sqrt{6}}$$
$$= 52.833 \pm 2.571 \times \frac{1.7224}{\sqrt{6}}$$
$$= 52.833 \pm 1.808$$

So the 95% confidence interval is (51.025, 54.641)

b $\dfrac{(n-1)s^2}{\chi^2_{n-1}(0.025)} = \dfrac{5s^2}{\chi^2_5(0.025)} = \dfrac{5 \times 2.9667}{12.832} = 1.1560$

$\dfrac{(n-1)s^2}{\chi^2_{n-1}(0.975)} = \dfrac{5s^2}{\chi^2_5(0.975)} = \dfrac{5 \times 2.9667}{0.831} = 17.850$

So the 95% confidence interval for the population variance is (1.156, 17.850)

3 c Parts **a** and **b** assume that the weights of the eggs are normally distributed.

4 a $s^2 = 0.49 \Rightarrow s = 0.7$

The 95% confidence limits for the mean are

$$\bar{x} \pm t_{(n-1)}\left(\frac{\alpha}{2}\right) \times \frac{s}{\sqrt{n}} = 9.8 \pm t_{17}(0.025) \times \frac{0.7}{\sqrt{18}}$$
$$= 9.8 \pm 2.110 \times \frac{0.7}{\sqrt{18}}$$
$$= 9.8 \pm 0.348$$

So the 95% confidence interval is (9.452, 10.148)

b $$\frac{(n-1)s^2}{\chi^2_{n-1}(0.025)} = \frac{17s^2}{\chi^2_{17}(0.025)} = \frac{17 \times 0.49}{30.191} = 0.276$$
$$\frac{(n-1)s^2}{\chi^2_{n-1}(0.975)} = \frac{17s^2}{\chi^2_{17}(0.975)} = \frac{17 \times 0.49}{7.564} = 1.101$$

So the 95% confidence interval for the population variance is (0.276, 1.101)

5 $H_0: \mu = 21.5 \quad H_1: \mu < 21.5$

Significance level 5%

$\upsilon = 7$

The critical value is $t_7(0.05) = 1.895$, so the critical region is $t \leqslant -1.895$

$$\bar{x} = \frac{\sum x}{n} = \frac{167.6}{8} = 20.95 \qquad \sum x^2 = 3561.28$$
$$s^2 = \frac{1}{n-1}\left(\sum x^2 - n\bar{x}^2\right) = \frac{1}{7}(3561.28 - 8 \times 20.95^2) = 7.1514$$
$$s = 2.6742$$

Test statistic $t = \dfrac{\bar{x} - \mu}{\frac{s}{\sqrt{n}}} = \dfrac{20.95 - 21.5}{\frac{2.6742}{\sqrt{8}}} = -0.5817$

$-0.582 > -1.895$, so the result is not significant. Accept H_0.

There is not enough evidence to suggest that the batteries have a shorter mean lifetime than that claimed by the manufacturer.

6 a Using a calculator gives $\bar{x} = 6.1917$ and $s^2 = 0.56992$, so

$s = \sqrt{0.56992} = 0.75493$

The 95% confidence limits for the mean are

$$\bar{x} \pm t_{(n-1)}\left(\frac{\alpha}{2}\right) \times \frac{s}{\sqrt{n}} = 6.1917 \pm t_{11}(0.025) \times \frac{0.75493}{\sqrt{12}}$$
$$= 6.1917 \pm 2.201 \times \frac{0.75493}{\sqrt{12}}$$
$$= 6.1917 \pm 0.4797$$

So the 95% confidence interval is (5.712, 6.671)

6 b $\frac{(n-1)s^2}{\chi^2_{n-1}(0.025)} = \frac{11s^2}{\chi^2_{11}(0.025)} = \frac{11 \times 0.5699}{21.920} = 0.286$

$\frac{(n-1)s^2}{\chi^2_{n-1}(0.975)} = \frac{11s^2}{\chi^2_{11}(0.975)} = \frac{11 \times 0.5699}{3.816} = 1.643$

So the 95% confidence interval for the population variance is (0.286, 1.643)
Therefore, the 95% confidence interval for the population standard deviation is (0.535, 1.282)

c To get a better assessment of his blood glucose levels, the patient should measure his blood glucose at the same time(s) each day.

7 a Using a calculator gives $\bar{x} = 11.5$ and $s^2 = 4.3$, so

$s = \sqrt{4.3} = 2.0736$

The 95% confidence limits for the mean are

$$\bar{x} \pm t_{(n-1)}\left(\frac{\alpha}{2}\right) \times \frac{s}{\sqrt{n}} = 11.5 \pm t_5(0.025) \times \frac{2.0736}{\sqrt{6}}$$
$$= 11.5 \pm 2.571 \times \frac{2.0736}{\sqrt{6}}$$
$$= 11.5 \pm 2.176$$

So the 95% confidence interval is (9.324, 13.676)

b $\frac{(n-1)s^2}{\chi^2_{n-1}(0.025)} = \frac{5s^2}{\chi^2_5(0.025)} = \frac{5 \times 4.3}{12.832} = 1.675$

$\frac{(n-1)s^2}{\chi^2_{n-1}(0.975)} = \frac{5s^2}{\chi^2_5(0.975)} = \frac{5 \times 4.3}{0.831} = 25.872$

So the 95% confidence interval for the population variance is (1.675, 25.872)

8 a $H_0: \sigma = 4 \qquad H_1: \sigma > 4$

Significance level 5%

$\upsilon = 9$

$\chi^2_9(0.05) = 16.919$

The critical region is $\frac{(n-1)s^2}{\sigma^2} \geqslant 16.919$

Test statistic $\frac{(n-1)s^2}{\sigma^2} = \frac{9 \times 5.2^2}{4^2} = 15.21$

$15.21 < 16.919$ so there is insufficient evidence for rejecting H_0. There is no evidence that the standard deviation is different from 4.

8 b $H_0: \mu = 24 \qquad H_1: \mu > 24$

Significance level 5%

$\upsilon = 9$

The critical value is $t_9(0.05) = 1.833$, so the critical region is $t \geqslant 1.833$

Test statistic $t = \dfrac{\bar{x} - \mu}{\frac{s}{\sqrt{n}}} = \dfrac{27.2 - 24}{\frac{5.2}{\sqrt{10}}} = 1.946$

$1.946 > 1.833$, so the result is significant and H_0 is rejected.

There is evidence to support the garage's claim that the mean lifetime of the car batteries is greater than 24 months.

c The tests assume that the lifetimes of the batteries are normally distributed.

9 a $\bar{x} = \dfrac{\sum fx}{\sum f} = \dfrac{3 \times 705 + 5 \times 715 + 9 \times 725 + 2 \times 735 + 1 \times 745}{3 + 5 + 9 + 2 + 1} = \dfrac{14430}{20} = 721.5$

$s^2 = \dfrac{\sum f(x - \bar{x})^2}{\left(\sum f\right) - 1} = \dfrac{3 \times 16.5^2 + 5 \times 6.5^2 + 9 \times 3.5^2 + 2 \times 13.5^2 + 1 \times 23.5^2}{19} = \dfrac{2055}{19} = 108.1579$

$s = \sqrt{108.1579} = 10.3999$

The 95% confidence limits for the mean are

$$\bar{x} \pm t_{(n-1)}\left(\frac{\alpha}{2}\right) \times \frac{s}{\sqrt{n}} = 721.5 \pm t_{19}(0.025) \times \frac{10.3999}{\sqrt{20}}$$

$$= 721.5 \pm 2.093 \times \frac{10.3999}{\sqrt{20}}$$

$$= 721.5 \pm 4.867$$

So the 95% confidence interval is (716.6, 726.3)

b $\dfrac{(n-1)s^2}{\chi^2_{n-1}(0.025)} = \dfrac{19s^2}{\chi^2_{19}(0.025)} = \dfrac{19 \times 108.1579}{32.852} = 62.553$

$\dfrac{(n-1)s^2}{\chi^2_{n-1}(0.975)} = \dfrac{19s^2}{\chi^2_{19}(0.975)} = \dfrac{19 \times 108.1579}{8.907} = 230.72$

So the 95% confidence interval for the population variance is (62.553, 230.72)

Therefore, the 95% confidence interval for the population standard deviation is (7.909, 15.189)

c As 725 is within confidence interval, there is no evidence to reject this hypothesis.

10 a $\bar{x} = \dfrac{34.2}{10} = 3.42 \qquad s^2 = \dfrac{\sum x^2 - n\bar{x}^2}{n-1} = \dfrac{121.6 - 10 \times 3.42^2}{9} = 0.5151$ (4 s.f.)

10 b i The 95% confidence limits for the mean are

$$\bar{x} \pm t_{(n-1)}\left(\frac{\alpha}{2}\right) \times \frac{s}{\sqrt{n}} = 3.42 \pm t_9(0.025) \times \frac{\sqrt{0.5151}}{\sqrt{10}}$$

$$= 3.42 \pm 2.262 \times \frac{\sqrt{0.5151}}{\sqrt{10}}$$

$$= 3.42 \pm 0.513$$

So the 95% confidence interval is (2.907, 3.933)

ii $$\frac{(n-1)s^2}{\chi^2_{n-1}(0.025)} = \frac{9s^2}{\chi^2_9(0.025)} = \frac{9 \times 0.5151}{19.023} = 0.244$$

$$\frac{(n-1)s^2}{\chi^2_{n-1}(0.975)} = \frac{9s^2}{\chi^2_9(0.975)} = \frac{9 \times 0.5151}{2.700} = 1.717$$

So the 95% confidence interval for the population variance is (0.244, 1.717)
Therefore, the 95% confidence interval for the population standard deviation is (0.494, 1.310)

c As 3.5 hours is inside the confidence interval on the mean, so there is no evidence of a change in the meantime, and 0.5 hours is inside the confidence interval on the standard deviation so there is no evidence of a change in the variability of the time, there is insufficient evidence to support changing the repair method.

d Use a 'matched pairs' experiment, getting each engineer to carry out a similar repair using the old method and the new method and use a paired *t*-test.

11 a As the sample size is relatively large, use a normal approximation.

So $\bar{X}$ is approximately $\sim \text{N}\left(\mu, \frac{\sigma^2}{\sqrt{n}}\right)$ and $Z = \frac{\bar{X} - \mu}{\frac{\sigma}{\sqrt{n}}} \sim \text{N}(0, 1^2)$

Using tables, $\text{P}(Z > 1.6449) = 0.05$, so $\text{P}(-1.6449 < Z < 1.6449) = 0.90$
So the 90% confidence interval for the mean is

$$\left(\bar{x} - 1.6449 \times \frac{\sigma}{\sqrt{n}}, \bar{x} + 1.6449 \times \frac{\sigma}{\sqrt{n}}\right), \text{ i.e. } \left(24 - 1.6449 \times \frac{\sqrt{2.1}}{\sqrt{60}}, 24 + 1.6449 \times \frac{\sqrt{2.1}}{\sqrt{60}}\right)$$

So the 90% confidence interval for the mean is (23.69, 24.31)

b The sample size is far too small to use the normal approximation.

c $\text{H}_0: \mu = 25 \qquad \text{H}_1: \mu > 25$
Significance level 5%
$\upsilon = 5$
The critical value is $t_5(0.05) = 2.015$, so the critical region is $t \geqslant 2.015$

Test statistic $t = \frac{\bar{x} - \mu}{\frac{s}{\sqrt{n}}} = \frac{27 - 25}{\frac{\sqrt{2.7}}{\sqrt{6}}} = 2.981$

$2.981 > 2.015$, so the result is significant and H_0 is rejected.
There is evidence that the average length of males raccoons is greater than 25 cm.

12 $H_0: \mu_D = 0 \qquad H_1: \mu_D > 0$

Significance level 10%

$\upsilon = 7$

The critical value is $t_7(0.1) = 1.415$, so the critical region is $t \geqslant 1.415$

$\sum d = 5 + 13 - 8 + 2 - 3 + 4 + 11 - 1 = 23$, so $\bar{d} = \frac{23}{8} = 2.875$

$\sum d^2 = 25 + 169 + 64 + 4 + 9 + 16 + 121 + 1 = 409$

$s^2 = \frac{1}{n-1}\left(\sum d^2 - n\bar{d}^2\right) = \frac{1}{7}(409 - 8 \times 2.875^2) = 48.982$

$s = 6.9987$

Test statistic $t = \frac{\bar{d} - \mu_D}{\frac{s}{\sqrt{n}}} = \frac{2.875 - 0}{\frac{6.9987}{\sqrt{8}}} = 1.162$

$1.162 < 1.415$, so the result is not significant: accept H_0.

There is insufficient evidence to support the chemist's claim.

13 a The data were not collected in pairs.

b Use data from twin lambs and compare the weight gain in each pair of twins.

c Farmers might also consider the age, weight and gender of the lambs.

d $H_0: \mu_D = 0 \qquad H_1: \mu_D \neq 0$

Significance level 5% – probability in each tail = 0.025

$\upsilon = 9$

The critical value is $t_9(0.025) = 2.262$, so the critical region is $|t| \geqslant 2.262$

$\sum d = 2 + 1.2 + 1 + 1.8 - 1 + 2.2 + 2 - 1.2 + 1.1 + 2.8 = 11.9$, so $\bar{d} = \frac{11.9}{10} = 1.19$

$\sum d^2 = 30.01$

$s^2 = \frac{1}{n-1}\left(\sum d^2 - n\bar{d}^2\right) = \frac{1}{9}(30.01 - 10 \times 1.19^2) = 1.761$

$s = 1.327$

Test statistic $t = \frac{\bar{d} - \mu_D}{\frac{s}{\sqrt{n}}} = \frac{1.19}{\frac{1.327}{\sqrt{10}}} = 2.836$

$2.836 > 2.262$, so the result is significant: reject H_0.

There is evidence of a difference in weight gain by lambs using diet *A* compared with those using diet *B*.

e Recommend diet *B* as this has the higher mean.

14 a $H_0: \mu_D = 0 \qquad H_1: \mu_D \neq 0$

Significance level 10% – probability in each tail = 0.05

$\upsilon = 9$

The critical value is $t_9(0.05) = 1.833$, so the critical region is $|t| \geqslant 1.833$

$\sum d = 14 + 2 + 18 + 25 + 0 - 8 + 4 + 4 + 12 + 20 = 91$, so $\bar{d} = \frac{91}{10} = 9.1$

$\sum d^2 = 1789$

$s^2 = \frac{1}{n-1}\left(\sum d^2 - n\bar{d}^2\right) = \frac{1}{9}(1789 - 10 \times 9.1^2) = 106.767$

$s = 10.3328$

Test statistic $t = \frac{\bar{d} - \mu_D}{\frac{s}{\sqrt{n}}} = \frac{9.1}{\frac{10.3328}{\sqrt{10}}} = 2.785$

$2.785 > 1.833$, so the result is significant: reject H_0.

There is evidence that the two methods give different results.

b The difference in measurements of blood pressure is normally distributed.

15 $H_0: \mu_D = 0 \qquad H_1: \mu_D > 0$

Significance level 1%

$\upsilon = 9$

The critical value is $t_9(0.01) = 2.821$, so the critical region is $t \geqslant 2.821$

$\sum d = 2.1 - 0.7 + 2.6 - 1.7 + 3.3 + 1.6 + 1.7 + 1.2 + 1.6 + 2.4 = 14.1$, so $\bar{d} = \frac{14.1}{10} = 1.41$

$\sum d^2 = 40.65$

$s^2 = \frac{1}{n-1}\left(\sum d^2 - n\bar{d}^2\right) = \frac{1}{9}(40.65 - 10 \times 1.41^2) = 2.3077$

$s = 1.519$

Test statistic $t = \frac{\bar{d} - \mu_D}{\frac{s}{\sqrt{n}}} = \frac{1.41}{\frac{1.519}{\sqrt{10}}} = 2.935$

$2.935 > 2.821$, so the result is significant: reject H_0.

There is evidence to support the finding that the diet causes an increase in the mean weight of the mice.

16 a $\bar{x}_{old} = \frac{225}{10} = 22.5 \qquad \bar{x}_{new} = \frac{234}{9} = 26$

$s_{old}^2 = \frac{1}{n_{old} - 1}\left(\sum x_{old}^2 - n_{old}\bar{x}_{old}^2\right) = \frac{1}{9}(5136.3 - 10 \times 22.5^2) = 8.2$

$s_{new}^2 = \frac{1}{n_{new} - 1}\left(\sum x_{new}^2 - n_{new}\bar{x}_{new}^2\right) = \frac{1}{8}(6200 - 9 \times 26^2) = 14.5$

16 b $\text{H}_0: \sigma^2_{old} = \sigma^2_{new}$ $\quad$ $\text{H}_1: \sigma^2_{old} < \sigma^2_{new}$

$\upsilon_l = 9 - 1 = 8 \quad \upsilon_s = 10 - 1 = 9$

$s_l^2 = 14.5$ and $s_s^2 = 8.2$

The critical value is $F_{8,9}(0.05) = 3.23$

The test statistic is $\frac{s_l^2}{s_s^2} = \frac{14.5}{8.2} = 1.77$

$1.77 < 3.23$, so accept H_0.

There is no evidence that there is a difference in the variance of the times of using the two sets of equipment.

c $\text{H}_0: \mu_{old} = \mu_{new}$ $\quad$ $\text{H}_1: \mu_{old} \neq \mu_{new}$

Significance level 2% – so 1% in each tail

$\upsilon = 10 + 9 - 2 = 17$

The critical value is $t_{17}(0.01) = 2.567$, so the critical regions are $t \geqslant 2.567$ and $t \leqslant -2.567$

$$s_p^2 = \frac{(n_{new}-1)s^2_{new} + (n_{old}-1)s^2_{old}}{n_{new}+n_{old}-2} = \frac{(8\times 14.5)+(9\times 8.2)}{9+10-2} = 11.1647 \quad \text{so } s_p = 3.3414$$

$$\text{Test statistic } t = \frac{(\bar{x}_{old} - \bar{x}_{new}) - (\mu_{old} - \mu_{new})}{s_p\sqrt{\frac{1}{n_{old}} + \frac{1}{n_{new}}}} = \frac{22.5 - 26}{3.3414\sqrt{\frac{1}{10} + \frac{1}{9}}} = -2.280$$

$-2.28 > -2.567$, so the result is not significant. Accept H_0.

There is evidence to suggest that there is no difference in mean treatment times between the old and new equipment.

d $t_{17}(0.025) = 2.110$, so the confidence limits are:

$$(\bar{x}_{new} - \bar{x}_{old}) \pm t_c s_p\sqrt{\frac{1}{n_{new}} + \frac{1}{n_{old}}} = (26 - 22.5) \pm 2.110 \times 3.3414\sqrt{\frac{1}{9} + \frac{1}{10}}$$

$$= 3.5 \pm 3.239 = 0.261 \text{ and } 6.739$$

The 95% confidence interval is $(0.261, 6.74)$ to 3 s.f.

e Staff might need to training or practice to learn how to use new equipment efficiently

f Gather data on the new equipment only after staff have mastered the equipment.

17 a $\text{H}_0: \sigma^2_A = \sigma^2_B$ $\quad$ $\text{H}_1: \sigma^2_A \neq \sigma^2_B$

$\upsilon_l = 25 - 1 = 24 \quad \upsilon_s = 19 - 1 = 18$

$s_l^2 = 2.6$ and $s_s^2 = 1.7$

The critical value is $F_{24,18}(0.05) = 2.15$

The test statistic is $\frac{s_l^2}{s_s^2} = \frac{2.6}{1.7} = 1.529$

$1.529 < 2.15$, so accept H_0.

There is no evidence that there is a difference in the variability of the blood counts.

The test assumes that the samples are taken from populations that are normally distributed.

17 b As the test in part **a** supports the assumption that the variances of the population are equal, the use of a *t*-distribution to find the confidence interval is justified providing that two other requirements are met: that the populations that are normally distributed and the two samples are independent..

c $s_p^2 = \frac{(n_a-1)s_a^2+(n_b-1)s_b^2}{n_a+n_b-2} = \frac{(24\times 2.6)+(18\times 1.7)}{25+19-2} = 2.21429$

$s_p = 1.48804$

$t_{42}(0.025) = 2.018$

The confidence limits are:

$$(\bar{x}_a-\bar{x}_b)\pm t_c s_p\sqrt{\frac{1}{n_a}+\frac{1}{n_b}} = (5.9-4.8)\pm 2.018\times 1.48804\sqrt{\frac{1}{25}+\frac{1}{19}}$$

$$= 1.1 \pm 0.914 = 0.186 \text{ and } 2.014$$

The 99% confidence interval is $(0.186, 2.01)$ to 3 s.f.

Challenge

a $S_p^2 = \frac{(n_x-1)S_x^2+(n_y-1)S_y^2+(n_z-1)S_z^2}{(n_x-1)+(n_y-1)+(n_z-1)} = \frac{(n_x-1)S_x^2+(n_y-1)S_y^2+(n_z-1)S_z^2}{n_x+n_y+n_z-3}$

b $\mathrm{E}(S_p^2) = \mathrm{E}\left(\frac{(n_x-1)S_x^2+(n_y-1)S_y^2+(n_z-1)S_z^2}{n_x+n_y+n_z-3}\right)$

$$= \frac{\mathrm{E}\left((n_x-1)S_x^2+(n_y-1)S_y^2+(n_z-1)S_z^2\right)}{n_x+n_y+n_z-3}$$

$$= \frac{\left((n_x-1)+(n_y-1)+(n_z-1)\right)\sigma^2}{n_x+n_y+n_z-3}$$

$$= \sigma^2$$

So S_p^2 is an unbiased estimator for σ^2