Vectors 1B

1 a $\mathbf{a}\times\mathbf{b}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 1 & 1 & -4\\ 2 & -1 & -2\end{vmatrix}=-6\mathbf{i}-6\mathbf{j}-3\mathbf{k}$

$|\mathbf{a}\times\mathbf{b}|=\sqrt{(-6)^2+(-6)^2+(-3)^2}=\sqrt{81}=9$

Area of triangle $OAB=\frac{1}{2}|\mathbf{a}\times\mathbf{b}|=\frac{9}{2}=4.5$

b $\mathbf{a}\times\mathbf{b}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 3 & 4 & -5\\ 2 & 1 & -2\end{vmatrix}=-3\mathbf{i}-4\mathbf{j}-5\mathbf{k}$

$|\mathbf{a}\times\mathbf{b}|=\sqrt{(-3)^2+(-4)^2+(-5)^2}=\sqrt{50}=5\sqrt{2}$

Area of triangle $OAB=\frac{1}{2}|\mathbf{a}\times\mathbf{b}|=\frac{5\sqrt{2}}{2}$

c $\mathbf{a}\times\mathbf{b}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 2 & 3 & 0\\ 2 & 6 & -9\end{vmatrix}=-27\mathbf{i}+18\mathbf{j}+6\mathbf{k}=3(-9\mathbf{i}+6\mathbf{j}+2\mathbf{k})$

$|\mathbf{a}\times\mathbf{b}|=3\sqrt{(-9)^2+6^2+2^2}=3\sqrt{121}=33$

Area of triangle $OAB=\frac{1}{2}|\mathbf{a}\times\mathbf{b}|=\frac{33}{2}=16.5$

2 Find $\overrightarrow{AB}$ and $\overrightarrow{AC}$ and then use the formula: area of triangle $ABC=\frac{1}{2}|\overrightarrow{AB}\times\overrightarrow{AC}|$

a $\overrightarrow{AB}=\mathbf{b}-\mathbf{a}=3\mathbf{i}+2\mathbf{j}+2\mathbf{k}$ $\quad\overrightarrow{AC}=\mathbf{c}-\mathbf{a}=3\mathbf{i}-2\mathbf{j}+2\mathbf{k}$

$\overrightarrow{AB}\times\overrightarrow{AC}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 3 & 2 & 2\\ 3 & -2 & 2\end{vmatrix}=8\mathbf{i}+0\mathbf{j}-12\mathbf{k}$

Area of triangle $ABC=\frac{1}{2}|8\mathbf{i}-12\mathbf{k}|=\frac{1}{2}\sqrt{8^2+(-12)^2}=\frac{1}{2}\sqrt{208}=\sqrt{52}=2\sqrt{13}$

b $\overrightarrow{AB}=\mathbf{b}-\mathbf{a}=\begin{pmatrix}1\\-1\\0\end{pmatrix}$ $\quad\overrightarrow{AC}=\mathbf{c}-\mathbf{a}=\begin{pmatrix}2\\-1\\-12\end{pmatrix}$

$\overrightarrow{AB}\times\overrightarrow{AC}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 1 & -1 & 0\\ 2 & -1 & -12\end{vmatrix}=12\mathbf{i}+12\mathbf{j}+1\mathbf{k}$

Area of triangle $ABC=\frac{1}{2}|12\mathbf{i}+12\mathbf{j}+\mathbf{k}|=\frac{1}{2}\sqrt{12^2+12^2+1^2}=\frac{1}{2}\sqrt{289}=\frac{17}{2}=8.5$

3 $\overrightarrow{AB} = \begin{pmatrix} 2 \\ -2 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ -2 \end{pmatrix}$ $\quad \overrightarrow{AC} = \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}$

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -2 & -2 \\ 2 & -1 & -1 \end{vmatrix} = 0\mathbf{i} - 3\mathbf{j} + 3\mathbf{k}$$

Area of triangle $ABC = \frac{1}{2}|-3\mathbf{j} + 3\mathbf{k}| = \frac{1}{2}\sqrt{(-3)^2 + 3^2} = \frac{1}{2}\sqrt{18} = \frac{3\sqrt{2}}{2}$

4 $\overrightarrow{AB} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} - \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$ $\quad \overrightarrow{AC} = \begin{pmatrix} 0 \\ 3 \\ 4 \end{pmatrix} - \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 1 \\ 1 & 2 & 3 \end{vmatrix} = -5\mathbf{i} - 5\mathbf{j} + 5\mathbf{k}$$

Area of triangle $ABC = \frac{1}{2}|-5\mathbf{i} - 5\mathbf{j} + 5\mathbf{k}| = \frac{1}{2}\sqrt{(-5)^2 + 5^2 + 5^2} = \frac{1}{2}\sqrt{75} = \frac{5\sqrt{3}}{2}$

5 Find $\overrightarrow{AB}$ and $\overrightarrow{AD}$ and then use the formula: area of parallelogram $ABCD = |\overrightarrow{AB} \times \overrightarrow{AD}|$

$\overrightarrow{AB} = -3\mathbf{i} + 4\mathbf{j} + \mathbf{k} - (\mathbf{i} + \mathbf{j} + \mathbf{k}) = -4\mathbf{i} + 3\mathbf{j}$
$\overrightarrow{AD} = 2\mathbf{i} - \mathbf{j} - (\mathbf{i} + \mathbf{j} + \mathbf{k}) = \mathbf{i} - 2\mathbf{j} - \mathbf{k}$

$$\overrightarrow{AB} \times \overrightarrow{AD} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -4 & 3 & 0 \\ 1 & -2 & -1 \end{vmatrix} = -3\mathbf{i} - 4\mathbf{j} + 5\mathbf{k}$$

Area of parallelogram $ABCD = |-3\mathbf{i} - 4\mathbf{j} + 5\mathbf{k}| = \sqrt{(-3)^2 + (-4)^2 + 5^2} = \sqrt{50} = 5\sqrt{2}$

6 $\overrightarrow{AB} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} - \begin{pmatrix} 0 \\ 5 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ -4 \\ -4 \end{pmatrix}$ $\quad \overrightarrow{AD} = \begin{pmatrix} 1 \\ 6 \\ 6 \end{pmatrix} - \begin{pmatrix} 0 \\ 5 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}$

$$\overrightarrow{AB} \times \overrightarrow{AD} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -4 & -4 \\ 1 & 1 & 3 \end{vmatrix} = -8\mathbf{i} - 10\mathbf{j} + 6\mathbf{k}$$

Area of parallelogram $ABCD = |-8\mathbf{i} - 10\mathbf{j} + 6\mathbf{k}| = \sqrt{(-8)^2 + (-10)^2 + 6^2} = \sqrt{200} = 2\sqrt{50} = 10\sqrt{2}$

7 $\overrightarrow{AB} = \mathbf{i} + 4\mathbf{j} + \mathbf{k} - \mathbf{j} = \mathbf{i} + 3\mathbf{j} + \mathbf{k}$

$\overrightarrow{AD} = 2\mathbf{i} + 6\mathbf{j} + 3\mathbf{k} - \mathbf{j} = 2\mathbf{i} + 5\mathbf{j} + 3\mathbf{k}$

$$\overrightarrow{AB} \times \overrightarrow{AD} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 3 & 1 \\ 2 & 5 & 3 \end{vmatrix} = 4\mathbf{i} - \mathbf{j} - \mathbf{k}$$

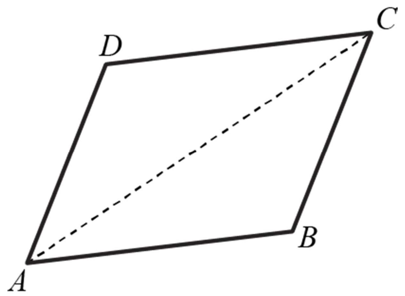
Area of parallelogram $ABCD = |4\mathbf{i} - \mathbf{j} - \mathbf{k}| = \sqrt{4^2 + (-1)^2 + (-1)^2} = \sqrt{18} = 3\sqrt{2}$

8 $$\mathbf{p} \times \mathbf{q} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a & a & 2a \\ 2a & a & 3a \end{vmatrix} = a^2\mathbf{i} + a^2\mathbf{j} - a^2\mathbf{k}$$

$|\mathbf{p} \times \mathbf{q}| = \sqrt{(a^2)^2 + (a^2)^2 + (-a^2)^2} = \sqrt{3a^4} = \sqrt{3}a^2$

Area of triangle $OPQ = \frac{1}{2}|\mathbf{p} \times \mathbf{q}| = \frac{\sqrt{3}}{2}a^2$

9 **a** The area of the parallelogram is twice the area of triangle ABC



$\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$ and $\overrightarrow{AC} = \mathbf{c} - \mathbf{a}$

Area of triangle $ABC = \frac{1}{2}|\overrightarrow{AB} \times \overrightarrow{AC}| = \frac{1}{2}|(\mathbf{b} - \mathbf{a}) \times (\mathbf{c} - \mathbf{a})|$

$\Rightarrow$ Area of parallelogram $ABCD = 2 \times \frac{1}{2}|(\mathbf{b} - \mathbf{a}) \times (\mathbf{c} - \mathbf{a})| = |(\mathbf{b} - \mathbf{a}) \times (\mathbf{c} - \mathbf{a})|$

b $(\mathbf{b} - \mathbf{a}) \times (\mathbf{c} - \mathbf{a}) = (\mathbf{b} - \mathbf{a}) \times (\mathbf{d} - \mathbf{a})$

$\Rightarrow (\mathbf{b} - \mathbf{a}) \times (\mathbf{c} - \mathbf{a}) - (\mathbf{b} - \mathbf{a}) \times (\mathbf{d} - \mathbf{a}) = 0$

$\Rightarrow (\mathbf{b} - \mathbf{a}) \times \left[(\mathbf{c} - \mathbf{a}) - (\mathbf{d} - \mathbf{a})\right] = 0$

$\Rightarrow (\mathbf{b} - \mathbf{a}) \times (\mathbf{c} - \mathbf{d}) = 0$

This means that $\overrightarrow{AB} \times \overrightarrow{DC} = 0$, i.e. $\overrightarrow{AB}$ is parallel to $\overrightarrow{DC}$.

10 a $\overrightarrow{AC} = (3\mathbf{i}+3\mathbf{j})-(2\mathbf{i}-\mathbf{j}-\mathbf{k}) = \mathbf{i}+4\mathbf{j}+\mathbf{k}$

$\overrightarrow{BC} = (3\mathbf{i}+3\mathbf{j})-(6\mathbf{i}-2\mathbf{k}) = -3\mathbf{i}+3\mathbf{j}+2\mathbf{k}$

$\overrightarrow{AC}\times\overrightarrow{BC} = (\mathbf{i}+4\mathbf{j}+\mathbf{k})\times(-3\mathbf{i}+3\mathbf{j}+2\mathbf{k})$

$= -3(\mathbf{i}\times\mathbf{i})+3(\mathbf{i}\times\mathbf{j})+2(\mathbf{i}\times\mathbf{k})-12(\mathbf{j}\times\mathbf{i})+12(\mathbf{j}\times\mathbf{j})+8(\mathbf{j}\times\mathbf{k})-3(\mathbf{k}\times\mathbf{i})+3(\mathbf{k}\times\mathbf{j})+2(\mathbf{k}\times\mathbf{k})$

$= 3\mathbf{k}-2\mathbf{j}+12\mathbf{k}+8\mathbf{i}-3\mathbf{j}-3\mathbf{i}$

$= 5\mathbf{i}-5\mathbf{j}+15\mathbf{k}$

b Area of triangle $ABC = \frac{1}{2}|\overrightarrow{AC}\times\overrightarrow{BC}| = \frac{1}{2}|5\mathbf{i}-5\mathbf{j}+15\mathbf{k}|$

$= \frac{1}{2}\sqrt{5^2+(-5)^2+15^2} = \frac{\sqrt{275}}{2} = \frac{5\sqrt{11}}{2}$

11 a $\overrightarrow{AB} = (-2\mathbf{i}-3\mathbf{j}+\mathbf{k})-(-3\mathbf{i}+2\mathbf{j}-4\mathbf{k}) = \mathbf{i}-5\mathbf{j}+5\mathbf{k}$

$\overrightarrow{AC} = (\mathbf{i}+2\mathbf{j}-\mathbf{k})-(-3\mathbf{i}+2\mathbf{j}-4\mathbf{k}) = 4\mathbf{i}+3\mathbf{k}$

$\overrightarrow{AB}\times\overrightarrow{AC} = (\mathbf{i}-5\mathbf{j}+5\mathbf{k})\times(4\mathbf{i}+3\mathbf{k})$

$= 4(\mathbf{i}\times\mathbf{i})+3(\mathbf{i}\times\mathbf{k})-20(\mathbf{j}\times\mathbf{i})-15(\mathbf{j}\times\mathbf{k})+20(\mathbf{k}\times\mathbf{i})+15(\mathbf{k}\times\mathbf{k})$

$= -3\mathbf{j}+20\mathbf{k}-15\mathbf{i}+20\mathbf{j}$

$= -15\mathbf{i}+17\mathbf{j}+20\mathbf{k}$

b Area of triangle $ABC = \frac{1}{2}|\overrightarrow{AB}\times\overrightarrow{AC}| = \frac{1}{2}|-15\mathbf{i}+17\mathbf{j}+20\mathbf{k}|$

$= \frac{1}{2}\sqrt{(-15)^2+17^2+20^2} = \frac{1}{2}\sqrt{225+289+400}$

$= \frac{\sqrt{914}}{2} = \frac{30.232}{2} = 15.12\,\text{m}^2$ (2 d.p.)

c The area of fabric needed will be larger as there will need to be excess fabric to attach to the masts and some slack in the sail to fill with air.

12 a $C = D + \overrightarrow{AB}$

$= -2\mathbf{i}+3\mathbf{k}+\left[(3\mathbf{i}-3\mathbf{j}-2\mathbf{k})-(-\mathbf{i}+2\mathbf{j})\right]$

$= -2\mathbf{i}+3\mathbf{k}+4\mathbf{i}-5\mathbf{j}-2\mathbf{k} = 2\mathbf{i}-5\mathbf{j}+\mathbf{k}$

So the coordinates of point C are (2, –5, 1)

12 b The area of the parallelogram $ABCD = |\overrightarrow{AB} \times \overrightarrow{AD}|$

$\overrightarrow{AB} = (3\mathbf{i} - 3\mathbf{j} - 2\mathbf{k}) - (-\mathbf{i} + 2\mathbf{j}) = 4\mathbf{i} - 5\mathbf{j} - 2\mathbf{k}$

$\overrightarrow{AD} = (-2\mathbf{i} + 3\mathbf{k}) - (-\mathbf{i} + 2\mathbf{j}) = -\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$

$$\begin{aligned}\overrightarrow{AB} \times \overrightarrow{AD} &= (4\mathbf{i} - 5\mathbf{j} - 2\mathbf{k}) \times (-\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}) \\ &= -4(\mathbf{i} \times \mathbf{i}) - 8(\mathbf{i} \times \mathbf{j}) + 12(\mathbf{i} \times \mathbf{k}) + 5(\mathbf{j} \times \mathbf{i}) + 10(\mathbf{j} \times \mathbf{j}) - 15(\mathbf{j} \times \mathbf{k}) \\ &\qquad + 2(\mathbf{k} \times \mathbf{i}) + 4(\mathbf{k} \times \mathbf{j}) - 6(\mathbf{k} \times \mathbf{k}) \\ &= -8\mathbf{k} - 12\mathbf{j} - 5\mathbf{k} - 15\mathbf{i} + 2\mathbf{j} - 4\mathbf{i} \\ &= -19\mathbf{i} - 10\mathbf{j} - 13\mathbf{k}\end{aligned}$$

Area of the parallelogram $ABCD = |-19\mathbf{i} - 10\mathbf{j} - 13\mathbf{k}| = \sqrt{361 + 100 + 169} = \sqrt{630} = 3\sqrt{70}$

The volume of one pendant $= 0.3 \times 3\sqrt{70} = 0.9\sqrt{70}\,\text{cm}^3$

So the cost to make one pendant $= 595 \times 0.9\sqrt{70} = £4480$ to the nearest pound.

Challenge

Area $ABFE = |\overrightarrow{AB} \times \overrightarrow{BF}| = |\mathbf{p} \times (\mathbf{q} + \mathbf{r})|$ As $\overrightarrow{BF} = \mathbf{q} + \mathbf{r}$

Area $ABCD = |\overrightarrow{AB} \times \overrightarrow{BC}| = |\mathbf{p} \times \mathbf{q}|$

Area $CDEF = |\overrightarrow{CD} \times \overrightarrow{CF}| = |\mathbf{p} \times \mathbf{r}|$ As $\overrightarrow{CD} = \mathbf{p}$ (because $ABCD$ is a parallelogram)

Area $ABFE$ = Area $ABCD$ + Area $CDEF$ + Area ADE − Area BCF

Using $\overrightarrow{AD} = \mathbf{q}$ and $\overrightarrow{DE} = \mathbf{r}$ (because $ABCD$ and $CDEF$ are parallelograms)

$\Rightarrow |\mathbf{p} \times (\mathbf{q} + \mathbf{r})| = |\mathbf{p} \times \mathbf{q}| + |\mathbf{p} \times \mathbf{r}| + \frac{1}{2}|\mathbf{q} \times \mathbf{r}| - \frac{1}{2}|\mathbf{q} \times \mathbf{r}|$

$\Rightarrow |\mathbf{p} \times (\mathbf{q} + \mathbf{r})| = |\mathbf{p} \times \mathbf{q}| + |\mathbf{p} \times \mathbf{r}|$