Vectors 1E

1 a Find two directions in the plane and take their vector product to give a normal to the plane.

Two directions are $\begin{pmatrix}1\\1\\2\end{pmatrix}-\begin{pmatrix}0\\4\\2\end{pmatrix}=\begin{pmatrix}1\\-3\\0\end{pmatrix}$ and $\begin{pmatrix}-1\\5\\0\end{pmatrix}-\begin{pmatrix}0\\4\\2\end{pmatrix}=\begin{pmatrix}-1\\1\\-2\end{pmatrix}$

A normal to the plane $\mathbf{n}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\1&-3&0\\-1&1&-2\end{vmatrix}=6\mathbf{i}+2\mathbf{j}-2\mathbf{k}$

Dividing by 2, this gives $3\mathbf{i}+\mathbf{j}-\mathbf{k}$, which is also normal to the plane.
Using $\mathbf{r.n}=\mathbf{a.n}$, with $\mathbf{n}=3\mathbf{i}+\mathbf{j}-\mathbf{k}$ and $\mathbf{a}=4\mathbf{j}+2\mathbf{k}$ (note $\mathbf{a}$ can be the position vector of any point on the plane), this gives the equation of the plane as:
$\mathbf{r}.(3\mathbf{i}+\mathbf{j}-\mathbf{k})=(4\mathbf{j}+2\mathbf{k}).(3\mathbf{i}+\mathbf{j}-\mathbf{k})=2$
In Cartesian form this may be written as $3x+y-z=2$

b Two directions in the plane are $\begin{pmatrix}2\\3\\-3\end{pmatrix}-\begin{pmatrix}1\\1\\0\end{pmatrix}=\begin{pmatrix}1\\2\\-3\end{pmatrix}$ and $\begin{pmatrix}3\\7\\-2\end{pmatrix}-\begin{pmatrix}1\\1\\0\end{pmatrix}=\begin{pmatrix}2\\6\\-2\end{pmatrix}$

A normal to the plane $\mathbf{n}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\1&2&-3\\2&6&-2\end{vmatrix}=14\mathbf{i}-4\mathbf{j}+2\mathbf{k}$

Dividing by 2, this gives $7\mathbf{i}-2\mathbf{j}+\mathbf{k}$, which is also normal to the plane.
Using $\mathbf{a}=\mathbf{i}+\mathbf{j}$, this gives the equation of the plane as:
$\mathbf{r}.(7\mathbf{i}-2\mathbf{j}+\mathbf{k})=(\mathbf{i}+\mathbf{j}).(7\mathbf{i}-2\mathbf{j}+\mathbf{k})=5$
In Cartesian form this is $7x-2y+z=5$

c Two directions in the plane are $\begin{pmatrix}2\\0\\-1\end{pmatrix}-\begin{pmatrix}3\\0\\0\end{pmatrix}=\begin{pmatrix}-1\\0\\-1\end{pmatrix}$ and $\begin{pmatrix}4\\1\\3\end{pmatrix}-\begin{pmatrix}3\\0\\0\end{pmatrix}=\begin{pmatrix}1\\1\\3\end{pmatrix}$

A normal to the plane $\mathbf{n}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\-1&0&-1\\1&1&3\end{vmatrix}=\mathbf{i}+2\mathbf{j}-\mathbf{k}$

Using $\mathbf{a}=3\mathbf{i}$, this gives the equation of the plane as:
$\mathbf{r}.(\mathbf{i}+2\mathbf{j}-\mathbf{k})=3\mathbf{i}.(\mathbf{i}+2\mathbf{j}-\mathbf{k})=3$
In Cartesian form this is $x+2y-z=3$

1 d Two directions in the plane are $\begin{pmatrix}3\\1\\-2\end{pmatrix}-\begin{pmatrix}1\\-1\\6\end{pmatrix}=\begin{pmatrix}2\\2\\-8\end{pmatrix}$ and $\begin{pmatrix}4\\1\\0\end{pmatrix}-\begin{pmatrix}1\\-1\\6\end{pmatrix}=\begin{pmatrix}3\\2\\-6\end{pmatrix}$

The normal to the plane is $\mathbf{n}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\2 & 2 & -8\\3 & 2 & -6\end{vmatrix}=4\mathbf{i}-12\mathbf{j}-2\mathbf{k}$

Dividing by 2, this gives $2\mathbf{i}-6\mathbf{j}-\mathbf{k}$, which is also normal to the plane.
Using $\mathbf{a}=\mathbf{i}-\mathbf{j}+6\mathbf{k}$, this gives the equation of the plane as:
$\mathbf{r}.(2\mathbf{i}-6\mathbf{j}-\mathbf{k})=(\mathbf{i}-\mathbf{j}+6\mathbf{k}).(2\mathbf{i}-6\mathbf{j}-\mathbf{k})=2$
In Cartesian form this is $2x-6y-z=2$

2 In these problems, the equation of the line includes the position vector of another point on the plane (for example, take $\lambda=0$) and includes a direction vector in the plane.

a The line has direction $2\mathbf{i}-\mathbf{k}$, and this is a direction in the plane.

The vector $4\mathbf{i}+3\mathbf{j}+\mathbf{k}-(\mathbf{i}+\mathbf{j}-2\mathbf{k})=3\mathbf{i}+2\mathbf{j}+3\mathbf{k}$ also lies in the plane.

A normal to the plane $\mathbf{n}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\2 & 0 & -1\\3 & 2 & 3\end{vmatrix}=2\mathbf{i}-9\mathbf{j}+4\mathbf{k}$

So the equation of the plane is
$\mathbf{r}.(2\mathbf{i}-9\mathbf{j}+4\mathbf{k})=(4\mathbf{i}+3\mathbf{j}+\mathbf{k}).(2\mathbf{i}-9\mathbf{j}+4\mathbf{k})$
$\Rightarrow \mathbf{r}.(2\mathbf{i}-9\mathbf{j}+4\mathbf{k})=8-27+4$
$\Rightarrow \mathbf{r}.(2\mathbf{i}-9\mathbf{j}+4\mathbf{k})=-15$

b The line has direction $2\mathbf{i}+\mathbf{j}-3\mathbf{k}$

Another vector in the plane is $3\mathbf{i}+5\mathbf{j}+\mathbf{k}-(\mathbf{i}+2\mathbf{j}+2\mathbf{k})=2\mathbf{i}+3\mathbf{j}-\mathbf{k}$

A normal to the plane $\mathbf{n}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\2 & 1 & -3\\2 & 3 & -1\end{vmatrix}=8\mathbf{i}-4\mathbf{j}+4\mathbf{k}$

So the equation of the plane is
$\mathbf{r}.(8\mathbf{i}-4\mathbf{j}+4\mathbf{k})=(3\mathbf{i}+5\mathbf{j}+\mathbf{k}).(8\mathbf{i}-4\mathbf{j}+4\mathbf{k})$
$\Rightarrow \mathbf{r}.(8\mathbf{i}-4\mathbf{j}+4\mathbf{k})=24-20+4$
$\Rightarrow \mathbf{r}.(8\mathbf{i}-4\mathbf{j}+4\mathbf{k})=8$
$\Rightarrow \mathbf{r}.(2\mathbf{i}-\mathbf{j}+\mathbf{k})=2$

2 c The line has direction $\mathbf{i}+2\mathbf{j}+2\mathbf{k}$

Two points in the plane have position vectors $7\mathbf{i}+8\mathbf{j}+6\mathbf{k}$ and $2\mathbf{i}-\mathbf{j}+\mathbf{k}$, the vector joining these points is $5\mathbf{i}+9\mathbf{j}+5\mathbf{k}$, which lies in the plane.

A normal to the plane $\mathbf{n}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 1 & 2 & 2\\ 5 & 9 & 5\end{vmatrix}=-8\mathbf{i}+5\mathbf{j}-\mathbf{k}$

So the equation of the plane is

$\mathbf{r}.(-8\mathbf{i}+5\mathbf{j}-\mathbf{k})=(7\mathbf{i}+8\mathbf{j}+6\mathbf{k}).(-8\mathbf{i}+5\mathbf{j}-\mathbf{k})$

$\Rightarrow \mathbf{r}.(-8\mathbf{i}+5\mathbf{j}-\mathbf{k})=-56+40-6$

$\Rightarrow \mathbf{r}.(-8\mathbf{i}+5\mathbf{j}-\mathbf{k})=-22$

$\Rightarrow \mathbf{r}.(8\mathbf{i}-5\mathbf{j}+\mathbf{k})=22$

3 a $3\mathbf{i}-2\mathbf{j}-\mathbf{k}$ is normal to Π_1 and $4\mathbf{i}-\mathbf{j}-2\mathbf{k}$ is normal to Π_2

As the line is perpendicular to both normal vectors, the direction vector of the line is given by

$$\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 3 & -2 & -1\\ 4 & -1 & -2\end{vmatrix}=3\mathbf{i}+2\mathbf{j}+5\mathbf{k}$$

The Cartesian equation of both planes are:

$\Pi_1:\ 3x-2y-z=5$ **(1)**

$\Pi_2:\ 4x-y-2z=5$ **(2)**

Setting $y=0$ and then multiplying equation **(1)** by 2 and subtracting equation **(2)** gives

$2x=5\Rightarrow x=\frac{5}{2}$

Substituting for x in equation **(2)** gives

$4\times\frac{5}{2}-2z=5\Rightarrow z=\frac{5}{2}$

So $\frac{5}{2}\mathbf{i}+\frac{5}{2}\mathbf{z}$ is a point on the line

An equation passing though a point with position vector **a** and parallel to vector **b** has the vector equation $\mathbf{r}=\mathbf{a}+\lambda\mathbf{b}$, so the equation of the line of intersection of the two planes is

$\mathbf{r}=\frac{5}{2}\mathbf{i}+\frac{5}{2}\mathbf{k}+\lambda(3\mathbf{i}+2\mathbf{j}+5\mathbf{k})$

3 b $5\mathbf{i}-\mathbf{j}-2\mathbf{k}$ is normal to Π_1 and $16\mathbf{i}-5\mathbf{j}-4\mathbf{k}$ is normal to Π_2

As the line is perpendicular to both normal vectors, the direction vector of the line is given by

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 5 & -1 & -2 \\ 16 & -5 & -4 \end{vmatrix} = -6\mathbf{i}-12\mathbf{j}-9\mathbf{k}$$

Dividing by the scalar –3 to get this vector in a simple form, gives the direction of the line as $2\mathbf{i}+4\mathbf{j}+3\mathbf{k}$

The Cartesian equation of both planes are:

Π_1: $5x-y-2z=16$ **(1)**

Π_2: $16x-5y-4z=53$ **(2)**

Setting $z = 0$ and then multiplying equation **(1)** by 5 and subtracting equation **(2)** gives

$(25-16)x=80-53 \Rightarrow 9x=27 \Rightarrow x=3$

Substituting for x in equation **(1)** gives

$5\times3-y=16 \Rightarrow y=-1$

So $3\mathbf{i}-\mathbf{j}$ is a point on the line

The equation of the line of intersection of the two planes is

$\mathbf{r}=3\mathbf{i}-\mathbf{j}+\lambda(2\mathbf{i}+4\mathbf{j}+3\mathbf{k})$

c $\mathbf{i}-3\mathbf{j}+\mathbf{k}$ is normal to Π_1 and $4\mathbf{i}-3\mathbf{j}-2\mathbf{k}$ is normal to Π_2

As the line is perpendicular to both normal vectors, the direction vector of the line is given by

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -3 & 1 \\ 4 & -3 & -2 \end{vmatrix} = 9\mathbf{i}+6\mathbf{j}+9\mathbf{k}$$

Dividing by the scalar 3 to get this vector in a simple form, gives the direction of the line as $3\mathbf{i}+2\mathbf{j}+3\mathbf{k}$

The Cartesian equation of both planes are:

Π_1: $x-3y+z=10$ **(1)**

Π_2: $4x-3y-2z=1$ **(2)**

Setting $z = 0$ and then subtracting equation **(2)** from equation **(1)** gives

$3x=-9 \Rightarrow x=-3$

Substituting for x in equation **(1)** gives

$-3-3y=10 \Rightarrow y=-\frac{13}{3}$

So $-3\mathbf{i}-\frac{13}{3}\mathbf{j}$ is a point on the line

The equation of the line of intersection of the two planes is

$\mathbf{r}=-3\mathbf{i}-\frac{13}{3}\mathbf{j}+\lambda(3\mathbf{i}+2\mathbf{j}+3\mathbf{k})$

4 First find a normal **n** to the plane

$$\mathbf{n}=(4\mathbf{i}-\mathbf{j}-\mathbf{k})\times(4\mathbf{i}-5\mathbf{j}-3\mathbf{k})=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 4 & -1 & -1\\ 4 & -5 & 3\end{vmatrix}=-8\mathbf{i}-16\mathbf{j}-16\mathbf{k}$$

Dividing by 8, this simplifies to $-\mathbf{i}-2\mathbf{j}-2\mathbf{k}$

Let θ be the angle between the line and the normal to the plane.

Using the definition of the scalar product $\mathbf{a}.\mathbf{b}=|\mathbf{a}||\mathbf{b}|\cos\theta$ gives

$$\cos\theta=\frac{(-4\mathbf{i}-7\mathbf{j}+4\mathbf{k}).(-\mathbf{i}-2\mathbf{j}-2\mathbf{k})}{\sqrt{(-4)^2+(-7)^2+4^2}\sqrt{(-1)^2+(-2)^2+(-2)^2}}=\frac{4+14-8}{\sqrt{81}\times\sqrt{9}}=\frac{10}{9\times3}=\frac{10}{27}$$

Let α be the acute angle between the line and the plane. As $\cos\theta>0, \theta$ is acute and so $\theta+\alpha=90°$

$$\cos\theta=\frac{10}{27}\Rightarrow\theta=68.3° \quad (3\text{ s.f.}), \text{ so } \alpha=90°-68.3°=21.7° \quad (3\text{ s.f.})$$

5 The shortest distance between two skew lines $\mathbf{r}=\mathbf{a}+\lambda\mathbf{b}$ and $\mathbf{r}=\mathbf{c}+\mu\mathbf{d}$ is $\left|\frac{(\mathbf{a}-\mathbf{c}).(\mathbf{b}\times\mathbf{d})}{|\mathbf{b}\times\mathbf{d}|}\right|$

In this case: $\mathbf{a}-\mathbf{c}=\mathbf{i}-(3\mathbf{i}-\mathbf{j}+\mathbf{k})=-2\mathbf{i}+\mathbf{j}-\mathbf{k}$

$$\mathbf{b}\times\mathbf{d}=(-3\mathbf{i}-12\mathbf{j}+11\mathbf{k})\times(2\mathbf{i}+6\mathbf{j}-5\mathbf{k})=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ -3 & -12 & 11\\ 2 & 6 & -5\end{vmatrix}=-6\mathbf{i}+7\mathbf{j}+6\mathbf{k}$$

$$\text{Shortest distance}=\left|\frac{(-2\mathbf{i}+\mathbf{j}-\mathbf{k}).(-6\mathbf{i}+7\mathbf{j}+6\mathbf{k})}{\sqrt{(-6)^2+7^2+6^2}}\right|=\left|\frac{12+7-6}{\sqrt{121}}\right|=\frac{13}{11}$$

6 **a** The line lies in the plane if it is perpendicular to the normal to the plane and the line and plane have a common point. The normal to the plane is $\mathbf{i}+\mathbf{j}-\mathbf{k}$ and the line is in the direction $-\mathbf{i}+2\mathbf{j}+\mathbf{k}$ and $(\mathbf{i}+\mathbf{j}-\mathbf{k}).(-\mathbf{i}+2\mathbf{j}+\mathbf{k})=-1+2-1=0$

So the direction of the line is perpendicular to the normal of the plane.

The line $\mathbf{r}=2\mathbf{i}+3\mathbf{j}+\mathbf{k}+\lambda(-\mathbf{i}+2\mathbf{j}+\mathbf{k})$ passes through the point (2, 3, 1). The point (2, 3, 1) also lies on the plane as $(2\mathbf{i}+3\mathbf{j}+\mathbf{k}).(\mathbf{i}+\mathbf{j}-\mathbf{k})=2+3-1=4$

So the line and plane have a point in common.

The line is perpendicular to the normal to the plane and has a common point with the plane, therefore it lies in the plane.

b The line $\mathbf{r}=-\mathbf{i}+2\mathbf{j}+4\mathbf{k}+\lambda(-\mathbf{i}+2\mathbf{j}+4\mathbf{k})$ is parallel to the plane as its direction $(-\mathbf{i}+2\mathbf{j}+\mathbf{k})$ is the same as the line in part a, which lies in the plane. The point (– 1, 2, 4) lies on the line. It does not lie on the plane as $(-\mathbf{i}+2\mathbf{j}+4\mathbf{k}).(\mathbf{i}+\mathbf{j}-\mathbf{k})=-1+2-4=-3\neq4$

So this line is parallel to the plane Π but lies on the plane $\mathbf{r}.(\mathbf{i}+\mathbf{j}-\mathbf{k})=-3$

The distance between the two planes is $\frac{4-(-3)}{|\mathbf{i}+\mathbf{j}-\mathbf{k}|}=\frac{7}{\sqrt{3}}=\frac{7\sqrt{3}}{3}$

So the shortest distance from the line to the plane is $\frac{7\sqrt{3}}{3}=4.04 \quad (3\text{ s.f.})$

7 a $\overrightarrow{AB} = -\mathbf{i} - \mathbf{j} - 5\mathbf{k}$ and $\overrightarrow{AC} = 2\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & -1 & -5 \\ 2 & 4 & -2 \end{vmatrix} = 22\mathbf{i} - 12\mathbf{j} - 2\mathbf{k}$$

Using $\mathbf{r.n} = \mathbf{a.n}$, with $\mathbf{n} = 22\mathbf{i} - 12\mathbf{j} - \mathbf{k}$ and $\mathbf{a} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$, this gives the equation of the plane as:

$\mathbf{r}.(22\mathbf{i} - 12\mathbf{j} - 2\mathbf{k}) = (\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}).(22\mathbf{i} - 12\mathbf{j} - 2\mathbf{k}) = 22 - 24 - 6 = -8$

In Cartesian form this may be written as $22x - 12y - 2z = -8$

Simplifying by dividing by –2 gives $-11x + 6y + z = 4$

b The volume (V) of the tetrahedron is given by $\frac{1}{6}\overrightarrow{AD}.\overrightarrow{AB} \times \overrightarrow{AC}$

$\overrightarrow{AD} = 4\mathbf{i} - 4\mathbf{j} + \mathbf{k}$, so

$$V = \frac{1}{6}(4\mathbf{i} - 4\mathbf{j} + \mathbf{k}).(22\mathbf{i} - 12\mathbf{j} - 2\mathbf{k}) = \frac{88 + 48 - 2}{6} = \frac{134}{6} = \frac{67}{3}$$

c $\overrightarrow{DC} = -2\mathbf{i} + 8\mathbf{j} - 3\mathbf{k}$

The normal to the plane from D to E is $-11\mathbf{i} + 6\mathbf{j} + \mathbf{k}$, so $\overrightarrow{DE} = k(-11\mathbf{i} + 6\mathbf{j} + \mathbf{k})$

Let the angle CDE be θ, then

$$\cos\theta = \frac{\overrightarrow{DC}.\overrightarrow{DE}}{\left|\overrightarrow{DC}\right|\left|\overrightarrow{DE}\right|} = \frac{k(-2\mathbf{i} + 8\mathbf{j} - 3\mathbf{k}).(-11\mathbf{i} + 6\mathbf{j} + \mathbf{k})}{k\sqrt{(-2)^2 + 8^2 + (-3)^2}\sqrt{(-11)^2 + 6^2 + 1^2}}$$

$$= \frac{22 + 48 - 3}{\sqrt{77}\sqrt{158}} = 0.6074 \quad \text{(4 d.p.)}$$

So $\theta = 0.918$ radians (3 d.p.)

8 a If the lines intersect, then there is λ and μ such that $\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + \lambda\begin{pmatrix} 3 \\ 3 \\ -2 \end{pmatrix} = \begin{pmatrix} a \\ 4 \\ -4 \end{pmatrix} + \mu\begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix}$

This gives three equations:

$-1 + 3\lambda = a + 2\mu$ **(1)**

$3\lambda = 4 + \mu$ **(2)**

$1 - 2\lambda = -4 - 3\mu$ **(3)**

Multiplying equation **(2)** by 3 and adding to equation **(3)** gives:

$9\lambda + 1 - 2\lambda = 12 + 3\mu - 4 - 3\mu \Rightarrow 7\lambda = 7 \Rightarrow \lambda = 1$

Substituting in equation **(2)** gives: $3 = 4 + \mu \Rightarrow \mu = -1$

Substituting values for λ and μ in equation **(1)** gives: $2 = a - 2 \Rightarrow a = 4$

8 b A normal to the plane containing the lines L_1 and L_2 is $\mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 3 & -2 \\ 2 & 1 & -3 \end{vmatrix} = -7\mathbf{i}+5\mathbf{j}-3\mathbf{k}$

Using $\mathbf{r.n}=\mathbf{a.n}$, with $\mathbf{n}=-7\mathbf{i}+5\mathbf{j}-3\mathbf{k}$ and $\mathbf{a}=-\mathbf{i}+\mathbf{k}$, this gives the equation of the plane as:

$\mathbf{r}.(-7\mathbf{i}+5\mathbf{j}-3\mathbf{k}) = (-\mathbf{i}+\mathbf{k}).(-7\mathbf{i}+5\mathbf{j}-3\mathbf{k}) = 7-3=4$

In Cartesian form this may be written as $-7x+5y-3z=4$

So, in the form required by the question, the equation is $-7x+5y-3z-4=0$

c The shortest distance between two skew lines $\mathbf{r}=\mathbf{a}+\lambda\mathbf{b}$ and $\mathbf{r}=\mathbf{c}+\mu\mathbf{d}$ is $\left|\dfrac{(\mathbf{a}-\mathbf{c}).(\mathbf{b}\times\mathbf{d})}{|\mathbf{b}\times\mathbf{d}|}\right|$

So when $a=1$, $\mathbf{a}-\mathbf{c}=-2\mathbf{i}-4\mathbf{j}+5\mathbf{k}$ and from part **b**, $\mathbf{b}\times\mathbf{d}=-7\mathbf{i}+5\mathbf{j}-3\mathbf{k}$

Let the shortest distance between L_1 and L_2 be d, then

$$d = \left|\frac{(-2\mathbf{i}-4\mathbf{j}+5\mathbf{k}).(-7\mathbf{i}+5\mathbf{j}-3\mathbf{k})}{\sqrt{(-7)^2+5^2+(-3)^2}}\right| = \left|\frac{14-20-15}{\sqrt{83}}\right| = \frac{21}{\sqrt{83}} = 2.31 \quad (3 \text{ s.f.})$$

9 a The direction perpendicular to the plane is given by $\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 2 & -2 \\ 0 & 2 & -1 \end{vmatrix} = 2\mathbf{i}-\mathbf{j}-2\mathbf{k}$

$|2\mathbf{i}-\mathbf{j}-2\mathbf{k}| = \sqrt{2^2+(-1)^2+(-2)^2} = 3$

So a unit vector perpendicular to the plane is $\frac{1}{3}(2\mathbf{i}-\mathbf{j}-2\mathbf{k})$, i.e. $\begin{pmatrix} \frac{2}{3} \\ -\frac{1}{3} \\ -\frac{2}{3} \end{pmatrix}$

The vector in the opposite direction, i.e. $\begin{pmatrix} -\frac{2}{3} \\ \frac{1}{3} \\ \frac{2}{3} \end{pmatrix}$, is also a unit vector perpendicular to the plane

b The direction of the line l is $\begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix} - \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \\ 1 \end{pmatrix} = \mathbf{i}+5\mathbf{j}+\mathbf{k}$

Let θ be the angle between the normal to the plane Π and the line l. So if θ is acute, $\alpha = 90° - \theta$ and if θ is obtuse, $\alpha = \theta - 90°$

Using the scalar product:

$$\cos\theta = \frac{(2\mathbf{i}-\mathbf{j}-2\mathbf{k}).(\mathbf{i}+5\mathbf{j}+\mathbf{k})}{\sqrt{2^2+(-1)^2+(-2)^2}\sqrt{1^2+5^2+1^2}} = -\frac{5}{3\sqrt{27}} = -\frac{5}{9\sqrt{3}}$$

$\Rightarrow \theta = 108.7°$ (1 d.p.)

So $\alpha = \theta - 90° = 18.7° = 19°$ (to the nearest degree)

9 c As $\frac{1}{3}(2\mathbf{i}-\mathbf{j}-2\mathbf{k})$ is the unit normal to the plane, then the perpendicular distance from A to Π is

$$\left|(\mathbf{i}+5\mathbf{j}+\mathbf{k}).\frac{1}{3}(2\mathbf{i}-\mathbf{j}-2\mathbf{k})\right|=\left|\frac{1}{3}(2-5-2)\right|=\frac{5}{3}=1.67 \quad (2 \text{ d.p.})$$

Alternatively, by geometry, if d is the distance between point A and the point at which the line l meets the plane Π and p is the perpendicular distance from A to Π then $p=d\sin\alpha$

$$\sin\alpha=\sin(\theta-90°)=-\cos\theta=\frac{5}{9\sqrt{3}}$$

$$d=|\mathbf{i}+5\mathbf{j}+\mathbf{k}|=\sqrt{27}$$

$$\text{So } p=\frac{5\sqrt{27}}{9\sqrt{3}}=\frac{5\times3\sqrt{3}}{9\sqrt{3}}=\frac{5}{3}=1.67 \quad (3 \text{ s.f.})$$

10 a Given a point with coordinates (α,β,γ) and a plane with equation $ax+by+cz=d$ the perpendicular distance from the point to the plane is $\frac{|a\alpha+b\beta+c\gamma-d|}{\sqrt{a^2+b^2+c^2}}$

So in this case the perpendicular distance from the point (3, –3, 2) to the plane Π_1 is

$$\frac{|2\times3+(-1)\times(-3)+3\times2-1|}{\sqrt{2^2+(-1)^2+3^2}}=\frac{14}{\sqrt{14}}=\sqrt{14}=3.74 \quad (3 \text{ s.f.})$$

b The vector equation of plane Π_1 is $\mathbf{r}.(2\mathbf{i}-\mathbf{y}+3\mathbf{k})=1$

So the normal to the plane Π_1 is $\mathbf{n}_1=2\mathbf{i}-\mathbf{y}+3\mathbf{k}$

Lines in the direction of $-2\mathbf{i}-4\mathbf{j}$ and $-\mathbf{i}+4\mathbf{j}+3\mathbf{k}$ lie in plane Π_2

So a normal to the plane Π_2 is $\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ -2 & -4 & 0\\ -1 & 4 & 3\end{vmatrix}=-12\mathbf{i}+6\mathbf{j}-12\mathbf{k}$

Dividing this vector by 6 also gives a normal to the plane Π_2: $\mathbf{n}_2=-2\mathbf{i}+\mathbf{j}-2\mathbf{k}$

Let the acute angle between the two planes be θ, then θ is also the angle between the respective normal to the planes, so

$$\cos\theta=\left|\frac{\mathbf{n}_1.\mathbf{n}_2}{|\mathbf{n}_1||\mathbf{n}_2|}\right|=\left|\frac{(2\mathbf{i}-\mathbf{j}+3\mathbf{k}).(-2\mathbf{i}+\mathbf{j}-2\mathbf{k})}{\sqrt{2^2+(-1)^2+3^2}\sqrt{(-2)^2+1^2+(-2)^2}}\right|=\left|\frac{-4-1-6}{\sqrt{14}\sqrt{9}}\right|=\frac{11}{3\sqrt{14}}$$

$$\Rightarrow\theta=0.201 \quad (3 \text{ s.f.})$$

10 c The direction of the line of intersection is perpendicular to the normal of each plane.

Hence the direction is $\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 3 \\ 2 & -1 & 2 \end{vmatrix} = \mathbf{i} + 2\mathbf{j}$

To find a point on both planes, use the Cartesian equations of the planes.
Find the Cartesian equation of plane Π_2 by solving

$x = -2\lambda - \mu$ **(1)**

$y = -4\lambda + 4\mu$ **(2)**

$z = 3\mu$ **(3)**

Multiply equation **(1)** by 2 and subtract from equation **(2)**: $y - 2x = 6\mu$ **(4)**
Multiply equation **(3)** by 2 and subtract from equation **(4)**: $y - 2x - 2z = 0$

So the Cartesian equations are

$\Pi_1:\ 2x - y + 3z = 1$

$\Pi_2:\ -2x + y - 2x = 0$

Setting $y = 0$ and solving gives $z = 1$ and $x = -1$, so $(-1, 0, 1)$ is a point on the line of intersection

Hence a vector equation for the line of intersection is $r = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + t\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$

11 First find two non-parallel directions in the transformed plane. These are obtained by applying the transformation to the two directions in Π_1 giving:

$\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -3 \\ 0 & 2 & 1 \end{pmatrix}\begin{pmatrix} -1 \\ -2 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ -2 \\ -4 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -3 \\ 0 & 2 & 1 \end{pmatrix}\begin{pmatrix} 4 \\ -2 \\ 3 \end{pmatrix} = \begin{pmatrix} 10 \\ -11 \\ -1 \end{pmatrix}$

Hence a normal to Π_2 is given by $\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & -2 & -4 \\ 10 & -11 & -1 \end{vmatrix} = -42\mathbf{i} - 41\mathbf{j} + 31\mathbf{k}$

Find a point on Π_2 by transforming the point given on Π_1 giving $\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -3 \\ 0 & 2 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 8 \\ -8 \\ 5 \end{pmatrix}$

Using $\mathbf{r.n} = \mathbf{a.n}$, with $\mathbf{n} = -42\mathbf{i} - 41\mathbf{j} + 31\mathbf{k}$ and $\mathbf{a} = 8\mathbf{i} - 8\mathbf{j} + 5\mathbf{k}$, this gives the equation of Π_2 as
$\mathbf{r}.(-42\mathbf{i} - 41\mathbf{j} + 31\mathbf{k}) = (8\mathbf{i} - 8\mathbf{j} + 5\mathbf{k}).(-42\mathbf{i} - 41\mathbf{j} + 31\mathbf{k}) = -336 + 328 + 155 = 147$

So $\mathbf{r}.(-42\mathbf{i} - 41\mathbf{j} + 31\mathbf{k}) = 147$ or, in column vector format, $\mathbf{r}.\begin{pmatrix} -42 \\ -41 \\ 31 \end{pmatrix} = 147$

12 The shape enclosed by the four planes will be a tetrahedron, each face corresponds to one of the plane and each vertex of the tetrahedron is where three planes intersect. To find the volume of the tetrahedron, find the coordinates of each of the vertices and hence the vectors of three edges joining at a vertex.

Let A be the intersection of Π_1, Π_2 and Π_3. So solving:

$2x - y + 3z = 1$ **(1)**

$x + y - 3z = 2$ **(2)**

$3x - 2y - z = 4$ **(3)**

Equation **(1)** + **(2)** gives: $3x = 3 \Rightarrow x = 1$

Substituting for x and multiplying equation **(2)** by 2 and adding to equation **(3)** gives:

$5 - 7z = 8 \Rightarrow z = -\frac{3}{7}$

Substituting for x and z in equation **(2)** gives: $1 + y + \frac{9}{7} = 2 \Rightarrow y = -\frac{2}{7}$

So point A is $\left(1, -\frac{2}{7}, -\frac{3}{7}\right)$

Let B be the intersection of Π_1, Π_2 and Π_4

B's coordinates are found by solving:

$2x - y + 3z = 1$

$x + y - 3z = 2$

$x + y = 0$

This gives $x = 1$, $y = -1$ and $z = -\frac{2}{3}$, so point B is $\left(1, -1, -\frac{2}{3}\right)$

Let C be the intersection of Π_1, Π_3 and Π_4

C's coordinates are found by solving:

$2x - y + 3z = 1$

$3x - 2y - z = 4$

$x + y = 0$

$x = \frac{13}{18}$, $y = -\frac{13}{18}$ and $z = -\frac{7}{18}$, so point C is $\left(\frac{13}{18}, -\frac{13}{18}, -\frac{7}{18}\right)$

Let D be the intersection of Π_2, Π_3 and Π_4

D's coordinates are found by solving:

$x + y - 3z = 2$

$3x - 2y - z = 4$

$x + y = 0$

$x = \frac{2}{3}$, $y = -\frac{2}{3}$ and $z = -\frac{2}{3}$, so point D is $\left(\frac{2}{3}, -\frac{2}{3}, -\frac{2}{3}\right)$

12 (continued)

Using the coordinates calculated above, find $\overrightarrow{AB}$, $\overrightarrow{AC}$ and $\overrightarrow{AD}$

$$\overrightarrow{AB} = \left(\mathbf{i} - \mathbf{j} - \frac{2}{3}\mathbf{k}\right) - \left(\mathbf{i} - \frac{2}{7}\mathbf{j} - \frac{3}{7}\mathbf{k}\right) = -\frac{5}{7}\mathbf{j} - \frac{5}{21}\mathbf{k}$$

$$\overrightarrow{AC} = \left(\frac{13}{18}\mathbf{i} - \frac{13}{18}\mathbf{j} - \frac{7}{18}\mathbf{k}\right) - \left(\mathbf{i} - \frac{2}{7}\mathbf{j} - \frac{3}{7}\mathbf{k}\right) = -\frac{5}{18}\mathbf{i} - \frac{55}{126}\mathbf{j} + \frac{5}{126}\mathbf{k}$$

$$\overrightarrow{AD} = \left(\frac{2}{3}\mathbf{i} - \frac{2}{3}\mathbf{j} - \frac{2}{3}\mathbf{k}\right) - \left(\mathbf{i} - \frac{2}{7}\mathbf{j} - \frac{3}{7}\mathbf{k}\right) = -\frac{1}{3}\mathbf{i} - \frac{8}{21}\mathbf{j} - \frac{5}{21}\mathbf{k}$$

Evaluate $\overrightarrow{AC} \times \overrightarrow{AD}$

$$\overrightarrow{AC} \times \overrightarrow{AD} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\frac{5}{18} & -\frac{55}{126} & \frac{5}{126} \\ -\frac{1}{3} & -\frac{8}{21} & -\frac{5}{21} \end{vmatrix} = \frac{315}{2646}\mathbf{i} - \frac{30}{378}\mathbf{j} - \frac{15}{378}\mathbf{k}$$

Hence the volume of the space enclosed by the four planes is given by

$$\frac{1}{6}\left|\overrightarrow{AB}.(\overrightarrow{AC} \times \overrightarrow{AD})\right| = \frac{1}{6}\left|\left(-\frac{5}{7}\mathbf{j} - \frac{5}{21}\mathbf{k}\right).\left(\frac{315}{2646}\mathbf{i} - \frac{30}{378}\mathbf{j} - \frac{15}{378}\mathbf{k}\right)\right|$$

$$= \frac{1}{6}\left(\frac{150}{2646} + \frac{75}{7938}\right) = \frac{1}{6}\left(\frac{150}{2646} + \frac{25}{2646}\right)$$

$$= \frac{175}{6 \times 2646} = \frac{25}{6 \times 378} = \frac{25}{2268}$$

Challenge

a Suppose the point (x, y, z) lies in the plane and transforms to (x_1, y_1, z_1)

Applying the transformation gives:

$$\begin{pmatrix} 2 & -1 & 2 \\ 2 & 2 & -1 \\ -1 & 2 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2x - y + 2z \\ 2x + 2y - z \\ -x + 2y + 2z \end{pmatrix}$$

So $x_1 = 2x - y + 2z$, $y_1 = 2x + y - z$, $z_1 = -x + 2y + 2z$

Summing the coordinates of the transformed point gives

$$x_1 + y_1 + z_1 = 2x - y + 2z + 2x + 2y - z - x + 2y + 2z$$
$$= 3x + 3y + 3z$$
$$= 3(x + y + z) = 0 \quad \text{as } (x, y, z) \text{ lies in the plane}$$

So the transformed point lies on the original plane, and therefore the plane is invariant under the transformation.

Challenge

b Now suppose a point on the plane is invariant, then it must map to itself. So

$$2x - y + 2z = x$$
$$2x + 2y - z = y$$
$$-x + 2y + 2z = z$$

Which simplifies to

$$x - y + 2z = 0$$
$$2x + y - z = 0$$
$$-x + 2y + z = 0$$

Note that the origin is clearly a solution to this system of equations.

But the matrix $\mathbf{M} = \begin{pmatrix} 1 & -1 & 2 \\ 2 & 1 & -1 \\ -1 & 2 & 1 \end{pmatrix}$ is non-singular as det $\mathbf{M}$ = $1\times 3 - (-1)\times 1 + 2\times 5 = 14 \neq 0$

As the matrix has a non-zero determinant, the system must have a unique solution, so the origin (0, 0, 0) must be the only solution to these equations.