Mixed exercise 3

1 a Parametric equations: $\cos\theta = \frac{x}{4}$ and $\sin\theta = \frac{x}{9}$

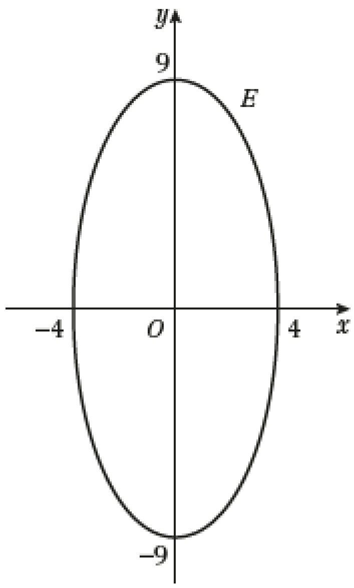
$\cos^2\theta + \sin^2\theta \equiv 1$

Substituting the values for $\cos\theta$ and $\sin\theta$ in the equation for ellipse E

gives the Cartesian equation: $\frac{x^2}{4^2} + \frac{y^2}{9^2} = 1$

b Comparing with the equation in its standard form, $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $a = 4$ and $b = 9$

So a sketch of E is:



c Use the chain rule to find the gradient: $\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{9\cos\theta}{-4\sin\theta}$

The gradient of the normal is $\frac{4\sin\theta}{9\cos\theta}$

Use $y - y_0 = m(x - x_0)$ to get the equation: $y - 9\sin\theta = \frac{4\sin\theta}{9\cos\theta}(x - 4\cos\theta)$

$\Rightarrow 9y\cos\theta - 81\sin\theta\cos\theta = 4\sin\theta(x - 4\cos\theta)$

So the equation of the normal is $4x\sin\theta - 9y\cos\theta = -65\sin\theta\cos\theta$

2 a Parametric equations: $x = \pm 2\cosh t$ and $y = 5\sinh t$

$\cosh^2 t - \sinh^2 t \equiv 1$

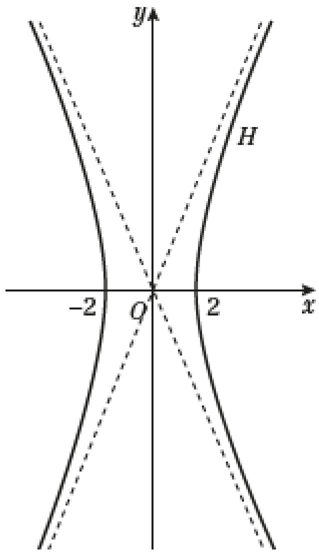
Substituting the values for $\cosh t$ and $\sinh t$ in the equation for the hyperbola H

gives the Cartesian equation: $\frac{x^2}{2^2} - \frac{y^2}{5^2} = 1$

2 b Comparing with the equation in its standard form, $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, $a=2$ and $b=5$

So the asymptotes are the lines $y=\pm\frac{5}{2}x$ and the hyperbola cuts the x-axis at ±2

A sketch is the following:



c Use the chain rule to find the gradient: $\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\frac{5\cosh t}{2\sinh t}$

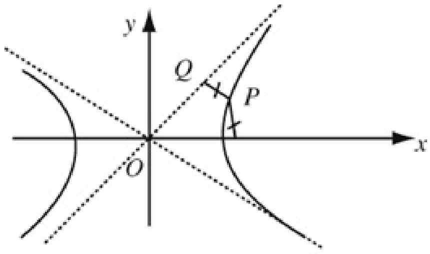
Use $y-y_0=m(x-x_0)$ to get the equation: $y-5\sinh t=\frac{5\cosh t}{2\sinh t}(x-2\cosh t)$

$$\Rightarrow 2y\sinh t+10=5x\cosh t$$

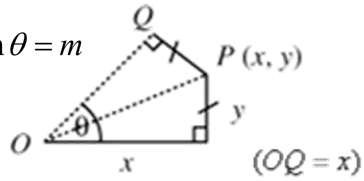
3 a Asymptotes are $y=\pm\frac{b}{a}x$ so $m=\frac{b}{a}$ $\Rightarrow b=am$

Substituting in the equation for the hyperbola: $\frac{x^2}{a^2}-\frac{y^2}{a^2m^2}=1$

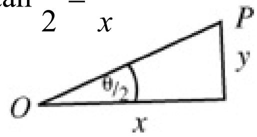
3 b Let Q be the point on the asymptote.



$\tan\theta = m$



$\tan\dfrac{\theta}{2} = \dfrac{y}{x}$



Using $\tan\theta = \dfrac{2\tan\left(\frac{\theta}{2}\right)}{1-\tan^2\left(\frac{\theta}{2}\right)} \Rightarrow m = \dfrac{2\left(\frac{y}{x}\right)}{1-\left(\frac{y^2}{x^2}\right)} = \dfrac{2xy}{(x^2-y^2)}$ **(1)**

But P lies on the hyperbola so from part **a**, $x^2m^2 - y^2 = a^2m^2$

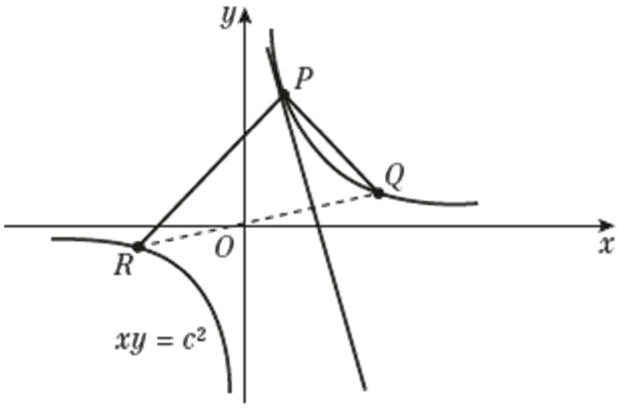
So $\quad m^2 = \dfrac{y^2}{x^2-a^2}$ **(2)**

Using $\mathbf{(1)}^2$ and **(2)** $\quad \dfrac{4x^2\cancel{y^2}}{(x^2-y^2)^2} = \dfrac{\cancel{y^2}}{x^2-a^2}$

So $\;4x^2(x^2-a^2) = (x^2-y^2)^2$

4 a Gradient of chord $PQ = \dfrac{\frac{c}{p}-\frac{c}{q}}{cp-cq} = \dfrac{\cancel{c}(q-p)}{pq\cancel{c}(p-q)} = \dfrac{-1}{pq}$

b



$P\left(cp, \dfrac{c}{p}\right)$; $Q\left(cq, \dfrac{c}{q}\right)$; $R\left(cr, \dfrac{c}{r}\right)$

Gradient of $PQ = -\dfrac{1}{pq}$

Gradient of $PR = -\dfrac{1}{pr}$

If $Q\hat{P}R = 90° \Rightarrow -\dfrac{1}{pq} \times -\dfrac{1}{pr} = -1 \quad \Rightarrow -1 = p^2qr$ **(1)**

To find gradient of tangent at P, let $q \to p$ for chord PQ

Gradient of tangent at P is $-\dfrac{1}{p^2}$

Gradient of chord RQ is $-\dfrac{1}{qr}$

$$\frac{-1}{qr} \times -\frac{1}{p^2} = \frac{1}{p^2qr}$$

But from **(1)** $p^2qr = -1$

So gradient of tangent at P × gradient of $QR = -1$

Therefore the tangent at P is perpendicular to chord QR.

5 a $y = ct^{-1},\ x = ct \Rightarrow \frac{dy}{dx} = \frac{-ct^{-2}}{c} = -\frac{1}{t^2}$

Equation of tangent is: $y - \frac{c}{t} = -\frac{1}{t^2}(x - ct)$

$\Rightarrow yt^2 - ct = -x + ct \quad \text{or} \quad t^2y + x = 2ct$

b Let $S\left(cs, \frac{c}{s}\right)$ and $T\left(ct, \frac{c}{t}\right)$ be two points on the hyperbola $xy = 16\ (c = 4)$

So tangent at S is $s^2y + x = 2cs$

Using the equations for the tangent at T and S, intersection of tangents is:

$$(t^2 - s^2)y = 2c(t - s)$$

$$y = \frac{2c}{t+s}$$

So $x = 2ct - \frac{2ct^2}{t+s} = \frac{2cts}{t+s}$

When $c = 4$ the point of intersection is $\left(\frac{8ts}{t+s}, \frac{8}{t+s}\right)$

The tangents intersect at $(-3, 3)$:

$x = -3$: $-3 = \frac{8ts}{t+s} \Rightarrow -3(t+s) = 8ts$

$y = 3$: $3 = \frac{8}{t+s} \Rightarrow 3(t+s) = 8$

So $ts = -1 \Rightarrow t = -\frac{1}{s}$

Substituting for t: $3\left(s - \frac{1}{s}\right) = 8 \Rightarrow 3s^2 - 8s - 3 = 0$

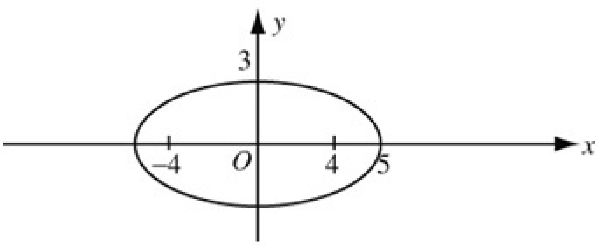
Factorising: $(3s+1)(s-3) = 0$

So $s = -\frac{1}{3}$ or $s = 3$; $t = 3$ or $t = -\frac{1}{3}$

The points S and T are $\left(-\frac{4}{3}, -12\right)$ and $\left(12, \frac{4}{3}\right)$

6 a $9x^2 + 25y^2 = 225 \Rightarrow \frac{x^2}{25} + \frac{y^2}{9} = 1$

$\therefore a = 5,\ b = 3$



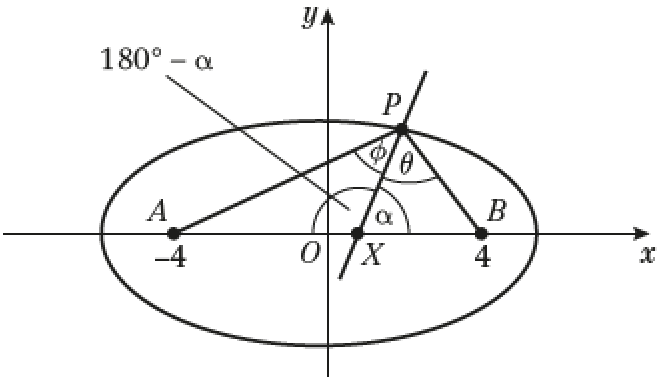
$b^2 = a^2(1 - e^2) \Rightarrow 9 = 25(1 - e^2) \quad \Rightarrow e^2 = \frac{16}{25} \Rightarrow e = \frac{4}{5}$

The foci are at $\left(\pm\frac{a}{e}, 0\right)$, which is $(\pm 4, 0)$, so A and B are the foci.

From the properties of an ellipse, $PS + PS' = 2a = 10$

So $PA + PB = 10$

6 b



Normal at P is: $5x\sin t - 3y\cos t = 16\cos t\sin t$

X is when $y = 0$, i.e. $x = \dfrac{16}{5}\cos t$

$PB^2 = (5\cos t - 4)^2 + (3\sin t)^2 = 25\cos^2 t - 40\cos t + 16 + 9\sin^2 t$

$= 16\cos^2 t - 40\cos t + 25 = (4\cos t - 5)^2$

$\Rightarrow PB = 5 - 4\cos t$ and $PA = 10 - PB = 5 + 4\cos t$

$AX = 4 + \dfrac{16}{5}\cos t,\ BX = 4 - \dfrac{16}{5}\cos t$

If PX bisects angle APB, then angle θ = angle ϕ

Consider sine rule on ΔPAX: $\sin\phi = \dfrac{\sin(180° - \alpha)AX}{AP} = \dfrac{\sin\alpha(4 + \frac{16}{5}\cos t)}{5 + 4\cos t} = \dfrac{4}{5}\sin\alpha$

Consider sine rule on ΔPBX: $\sin\theta = \dfrac{\sin\alpha BX}{PB} = \dfrac{\sin\alpha(4 - \frac{16}{5}\cos t)}{5 - 4\cos t} = \dfrac{4}{5}\sin\alpha$

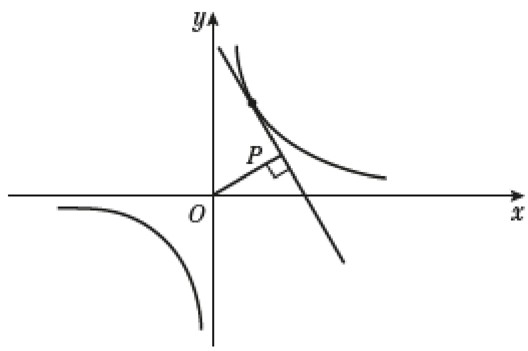
So $\sin\theta = \sin\phi$ and since both angles are acute, the normal bisects APB.

7 a $y = ct^{-1},\ x = ct \ \Rightarrow \dfrac{dy}{dx} = \dfrac{-ct^{-2}}{c} = -\dfrac{1}{t^2}$

Equation of tangent is: $y - \dfrac{c}{t} = -\dfrac{1}{t^2}(x - ct)$

$\Rightarrow yt^2 - ct = -x + ct$ or $t^2 y + x = 2ct$

b Gradient of tangent is $-\dfrac{1}{t^2}$, so gradient of OP is t^2



Equation of OP is $y = t^2 x$

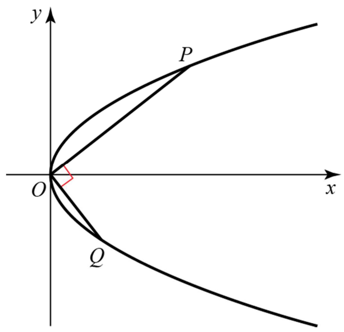
Equation of tangent is $t^2 y = 2ct - x$

Solving $t^4 x = 2ct - x$

$\Rightarrow \quad x = \dfrac{2ct}{1+t^4},\ y = \dfrac{2ct^3}{1+t^4}$

$$x^2 + y^2 = \frac{4c^2t^2 + 4c^2t^6}{(1+t^4)^2} = \frac{4c^2t^2(1+t^4)}{(1+t^4)^2}$$

$$\left.\begin{aligned}\Rightarrow (x^2+y^2)^2 &= \frac{16c^4t^4}{(1+t^4)^2}\\ xy &= \frac{4c^2t^4}{(1+t^4)^2}\end{aligned}\right\} \Rightarrow (x^2+y^2)^2 = 4c^2xy$$



8 **a** Gradient $OP = \frac{2ap}{ap^2} = \frac{2}{p}$, gradient of $OQ = \frac{2}{q}$

Since OP and OQ are perpendicular,

$\frac{4}{pq} = -1 \Rightarrow pq = -4$

b Normal at Q is $y + xq = aq^3 + 2aq$

c Normal at P is $y + xp = ap^3 + 2ap$

Solving the equations for the tangents simultaneously:

$x(q-p) = a(q^3 - p^3) + 2a(q-p)$

$x\,\cancel{(q-p)} = a\,\cancel{(q-p)}(q^2 + qp + p^2) + 2a\,\cancel{(q-p)}$

$x = a(q^2 + p^2 + qp + 2)$

$y = ap^3 + \cancel{2ap} - apq^2 - ap^3 - aqp^2 - \cancel{2ap} \Rightarrow y = -apq(q+p)$

But if $pq = -4$ then R is $(ap^2 + aq^2 - 2a, 4a(p+q))$

d Express $p^2 + q^2$ as $(p+q)^2 - 2pq$

Then $X = a((p+q)^2 - 2pq - 2) = a((p+q)^2 + 6)$

$Y = 4a(p+q) \Rightarrow p + q = \frac{Y}{4a}$

$\Rightarrow \quad X = a\left(\frac{Y^2}{16a^2} + 6\right)$

$X - 6a = \frac{Y^2}{16a} \Rightarrow Y^2 = 16aX - 96a^2$

9 $y = mx + c$ and $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$\Rightarrow b^2x^2 + a^2(mx+c)^2 = a^2b^2$

$b^2x^2 + a^2m^2x^2 + 2a^2mxc + a^2c^2 = a^2b^2$

$\Rightarrow x^2(b^2 + a^2m^2) + 2a^2mcx + a^2(c^2 - b^2) = 0$

For a tangent the discriminant = 0:

$4a^4m^2c^2 = 4(b^2 + a^2m^2)a^2(c^2 - b^2)$

$a^2m^2c^2 = b^2c^2 - b^4 + a^2m^2c^2 - a^2m^2b^2$

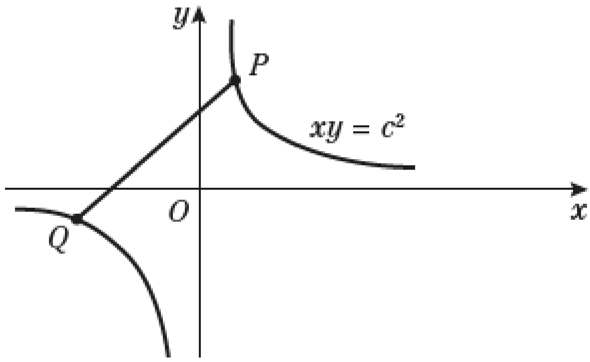
$a^2m^2b^2 + b^4 = b^2c^2$

$\Rightarrow \quad c^2 = a^2m^2 + b^2$

$c = \pm\sqrt{a^2m^2 + b^2}$

So the lines $y = mx \pm \sqrt{a^2m^2 + b^2}$ are tangents for all m.

10 Chord PQ has gradient $\dfrac{\frac{c}{p}-\frac{c}{q}}{cp-cq}=\dfrac{c(q-p)}{pqc(p-q)}=-\dfrac{1}{pq}$



If gradient $=1$, then $pq=-1$

Tangent at P is $p^2y+x=2cp$

Tangent at Q is $q^2y+x=2cq$

Intersection: $(p^2-q^2)y=2c(p-q)\Rightarrow y=\dfrac{2c}{p+q}$

$\Rightarrow x=2cp-\dfrac{2cp^2}{p+q}=\dfrac{2cpq}{p+q}$

So the point of intersection R is $\left(\dfrac{2cpq}{p+q},\dfrac{2c}{p+q}\right)$

But $pq=-1$, so R is $x=\dfrac{-2c}{p+q}$, $y=\dfrac{2c}{p+q}$, i.e. $y=-x$

The locus of R is the line $y=-x$

11 a Use the chain rule to find the gradient: $\dfrac{dy}{dx}=\dfrac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}=\dfrac{4\cos\theta}{-6\sin\theta}=-\dfrac{2\cos\theta}{3\sin\theta}$

Line l_1 has equation: $y-4\sin\theta=-\dfrac{2\cos\theta}{3\sin\theta}(x-6\cos\theta)$

$3y\sin\theta-12\sin^2\theta=-2x\cos\theta+12\cos^2\theta$

$2x\cos\theta+3y\sin\theta=12\cos^2\theta+12\sin^2\theta$

$2x\cos\theta+3y\sin\theta=12$

b $y=0$: $2x\cos\theta=12\Rightarrow x=\dfrac{6}{\cos\theta}$

$x=0$: $3y\sin\theta=12\Rightarrow y=\dfrac{4}{\sin\theta}$

So A has coordinates $\left(\dfrac{6}{\cos\theta},0\right)$ and B has coordinates $\left(0,\dfrac{4}{\sin\theta}\right)$

Midpoint of AB has coordinates $\left(\dfrac{3}{\cos\theta},\dfrac{2}{\sin\theta}\right)$, so $x=\dfrac{3}{\cos\theta}$ and $y=\dfrac{2}{\sin\theta}$

Using $\cos^2\theta+\sin^2\theta\equiv1$: $\left(\dfrac{3}{x}\right)^2+\left(\dfrac{2}{y}\right)^2=1$

The locus of the midpoint of AB is $\dfrac{9}{x^2}+\dfrac{4}{y^2}=1$

12 a Use the chain rule to find the gradient: $\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{5\cos\theta}{-13\sin\theta}$

An equation for the tangent line l_1 is : $y - 5\sin\theta = -\frac{5\cos\theta}{13\sin\theta}(x - 13\cos\theta)$

$$13y\sin\theta - 65\sin^2\theta = -5x\cos\theta + 65\cos^2\theta$$

$$13y\sin\theta + 5x\cos^2 = 65$$

b The point A has x-coordinate equal to 0, so its coordinates are $\left(0, \frac{5}{\sin\theta}\right)$

Line l_2 is perpendicular to l_1 so has gradient $\frac{13\sin\theta}{5\cos\theta}$

The equation of the line l_2 is given by: $y - \frac{5}{\sin\theta} = \frac{13\sin\theta}{5\cos\theta}x$

$$5y\sin\theta\cos\theta - 25\cos\theta = 13x\sin^2\theta$$

c Using $a = 13$ and $b = 5$, $b^2 = a^2(1-e^2) \Rightarrow 25 = 169(1-e^2) \Rightarrow e = \sqrt{1 - \frac{25}{169}}$

So the eccentricity of the ellipse is $e = \frac{12}{13}$

l_2 cuts the x-axis at $(-ae, 0)$, which is $(-12, 0)$.

Substitute this into the equation of l_2:

$$-25\cos\theta = -156\sin^2\theta$$

$$25\cos\theta = 156(1-\cos^2\theta)$$

$$156\cos^2\theta + 25\cos\theta - 156 = 0$$

The solutions of this equation are $\cos\theta = \frac{-25 \pm 313}{312}$

This gives either $\cos\theta = -\frac{338}{312}$, which can't be a cosine, or $\cos\theta = \frac{12}{13} = e$

13 a Use the chain rule to find the gradient: $\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{8\sec^2\theta}{4\sec\theta\tan\theta} = \frac{2}{\sin\theta}$

The gradient of the normal l_1 is $-\frac{\sin\theta}{2}$

Use $y - y_0 = m(x - x_0)$ to get the equation: $y - 8\tan\theta = -\frac{\sin\theta}{2}(x - 4\sec\theta)$

$$2y - 16\tan\theta = -x\sin\theta + 4\tan\theta$$

$$x\sin\theta + 2y = 20\tan\theta$$

13 b $y = 0$: $x\sin\theta = 20\tan\theta \Rightarrow x = 20\sec\theta$

$x = 0$: $2y = 20\tan\theta \Rightarrow y = 10\tan\theta$

So A has coordinates $(20\sec\theta, 0)$ and B has coordinates $(0, 10\tan\theta)$

Midpoint of AB has coordinates $(10\sec\theta, 5\tan\theta)$, so $x = 10\sec\theta$ and $y = 5\tan\theta$

Using $\sec^2\theta - \tan^2\theta \equiv 1$: $\quad \dfrac{x^2}{10^2} - \dfrac{y^2}{5^2} = 1$

The locus of the midpoint is $\dfrac{x^2}{100} - \dfrac{y^2}{25} = 1$, which is the equation of a hyperbola.

14 a Use the chain rule to find the gradient of the tangent: $\dfrac{dy}{dx} = \dfrac{\frac{dy}{dt}}{\frac{dx}{dt}} = \dfrac{b\cos t}{-a\sin t}$

The gradient of the normal is $\dfrac{a\sin t}{b\cos t}$

An equation for the line is: $y - b\sin t = \dfrac{a\sin t}{b\cos t}(x - a\cos t)$

$$by\cos t - b^2\sin t\cos t = ax\sin t - a^2\sin t\cos t$$

$$ax\sin t - by\cos t = (a^2 - b^2)\sin t\cos t$$

b $y = 0$: $ax\sin t = (a^2 - b^2)\sin t\cos t \Rightarrow x = \left(\dfrac{a^2 - b^2}{a}\right)\cos t$

$x = 0$: $by\cos t = -(a^2 - b^2)\sin t\cos t \Rightarrow y = -\left(\dfrac{a^2 - b^2}{b}\right)\sin t$

So M has coordinates $\left(\dfrac{a^2 - b^2}{a}\cos t, 0\right)$ and N has coordinates $\left(0, -\dfrac{a^2 - b^2}{b}\sin t\right)$

Midpoint of MN has coordinates: $\left(\dfrac{a^2 - b^2}{2a}\cos t, -\dfrac{a^2 - b^2}{2b}\sin t\right)$

$x = \dfrac{a^2 - b^2}{2a}\cos t \Rightarrow \cos t = \dfrac{2ax}{a^2 - b^2}$

$y = -\dfrac{a^2 - b^2}{2b}\sin t \Rightarrow \sin t = -\dfrac{2by}{a^2 - b^2}$

Using $\cos^2 t + \sin^2 t \equiv 1$: $\quad \dfrac{4a^2x^2}{(a^2 - b^2)^2} + \dfrac{4b^2y^2}{(a^2 - b^2)^2} = 1$

$$\Rightarrow \quad 4a^2x^2 + 4b^2y^2 = (a^2 - b^2)^2$$

The locus described by the midpoint of MN is $4a^2x^2 + 4b^2y^2 = (a^2 - b^2)^2$

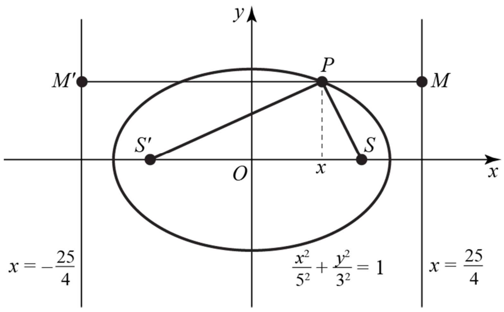
15 $a = 5$ and $b = 3$

Using $b^2 = (1-e^2)a^2$: $9 = 25(1-e^2) \Rightarrow e^2 = \frac{16}{25} \Rightarrow e = \frac{4}{5}$

The directrices of the ellipse are the lines of equation $x = \pm\frac{25}{4}$

P is the point (x, y).

By definition, for any point P on an ellipse, the ratio of the distance of P from the focus of the ellipse to the distance of P from the directrix is constant, called the eccentricity e.



Let M be the point on the directrix $x = \frac{25}{4}$ where $PS = ePM$

Let M' be the point on the directrix $x = -\frac{25}{4}$ where $PS' = ePM'$

PM and PM' are parallel to the x-axis.

$PM' = \frac{25}{4} + x$ and $PM = \frac{25}{4} - x$

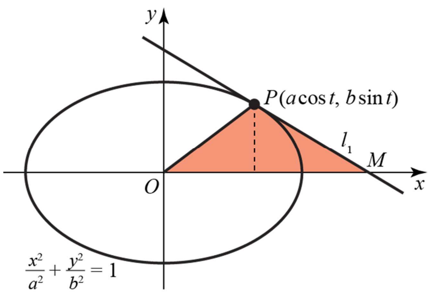
So $PS + PS' = ePM + ePM' = \frac{4}{5} \times \frac{50}{2} = 10$

16 $y = b\sin t,\ x = a\cos t \Rightarrow \frac{dy}{dx} = \frac{b\cos t}{-a\sin t}$

Equation of l_1 is: $y - b\sin t = -\frac{b\cos t}{a\sin t}(x - a\cos t)$

$$bx\cos t + ay\sin t = ab$$

l_1 meets the x-axis at $M = \left(\frac{a}{\cos t}, 0\right)$, so the area required is the area of the triangle OMP.



The height of the triangle is the length of the perpendicular from P to the x-axis

$= b\sin t$

Area of shaded triangle OMP

$= \frac{1}{2}$ base $\times$ height

$= \frac{1}{2} \times \frac{a}{\cos t} \times b\sin t = \frac{ab\tan t}{2}$

17 The parametric equations for the ellipse are $x = 6\cos\theta,\ y = 3\sin\theta,\ 0 \leqslant \theta < 2\pi$

Using the chain rule, the gradient of the tangent is $\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{\cos\theta}{2\sin\theta}$

At point $P\left(3, \frac{3\sqrt{3}}{2}\right)$: $3 = 6\cos\theta,\ \frac{3\sqrt{3}}{2} = 3\sin\theta \Rightarrow \tan\theta = \sqrt{3} \Rightarrow \theta = \frac{\pi}{3}$

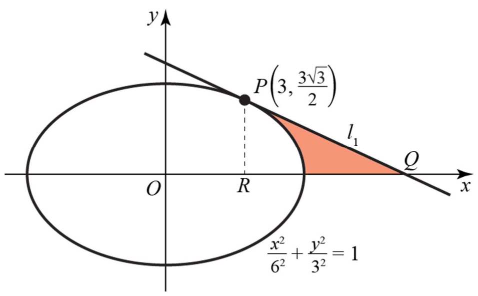
At P the gradient of the tangent is $-\frac{1}{2\tan\theta} = -\frac{1}{2\sqrt{3}}$

So the equation of line l_1 is $y - \frac{3\sqrt{3}}{2} = -\frac{1}{2\sqrt{3}}(x-3)$

By substituting $y = 0$, line l_1 meets the x-axis at $Q = (12, 0)$

Let $R(3, 0)$ be the projection of P on the x-axis.

Then the area of the triangle PQR is $\frac{1}{2} \times PQ \times QR = \frac{1}{2} \times \frac{3\sqrt{3}}{2}(12-3) = \frac{27\sqrt{3}}{4}$



You need to subtract the region of PQR which is contained in the ellipse.

In the first quadrant the ellipse is described by the function $y = 3\sqrt{1 - \frac{x^2}{36}}$

Since the ellipse meets the x-axis at $x = 6$, the area of the region is given by the integral $3\int_3^6 \sqrt{1 - \frac{x^2}{36}}\,\mathrm{d}x$

Solve this with the substitution $x = 6\sin\theta$:

$$3\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \sqrt{1-\sin^2\theta}\,(6\cos\theta)\,\mathrm{d}\theta = 18\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \cos^2\theta\,\mathrm{d}\theta$$

$$= 18\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{1+\cos 2\theta}{2}\,\mathrm{d}\theta$$

$$= 9 \times \frac{\pi}{3} + 9\left[\frac{\sin 2\theta}{2}\right]_{\frac{\pi}{6}}^{\frac{\pi}{2}}$$

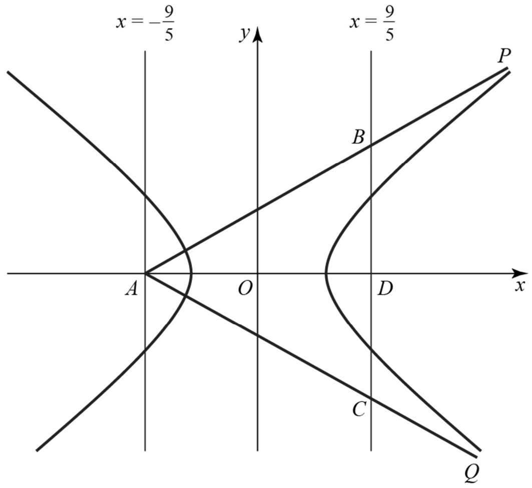
$$= 3\pi + 9\left(-\frac{\sqrt{3}}{4}\right) = 3\pi - \frac{9\sqrt{3}}{4}$$

So the area of the shaded region is $\frac{27\sqrt{3}}{4} - 3\pi + \frac{9\sqrt{3}}{4} = 9\sqrt{3} - 3\pi$

18 $a = 3$ and $b = 4$, so $b^2 = a^2(e^2 - 1) \Rightarrow e^2 = \frac{25}{9} \Rightarrow e = \frac{5}{3}$

Directrices are at $\pm\frac{a}{e}$ so they are $x = \pm\frac{9}{5}$

Let P and Q be on the right-hand side of the hyperbola. The tangents at P and Q meet the directrix $x = -\frac{9}{5}$ at A ($y = 0$) and the directrix $x = \frac{9}{5}$ at B and C. The point D is where the directrix $x = \frac{9}{5}$ crosses the x-axis.



The base of triangle ABC is the distance BC, while the height is the distance between the directrices.
The parametric equation for H is $x = 3\sec t,\ y = 4\tan t$

Use the chain rule to find the gradient: $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{4\sec t}{3\tan t}$

So the tangent to H has the equation: $y - 4\tan t = \frac{4\sec t}{3\tan t}(x - 3\sec t)$

$$3y\tan t - 12\tan^2 t = 4x\sec t - 12\sec^2 t$$

$$3y\tan t + 12(\sec^2 t - \tan^2 t) = 4x\sec t$$

$$3y\tan t + 12 = 4x\sec t$$

$$4x\sec t - 3y\tan t = 12$$

Tangents meet at A, which is $\left(-\frac{9}{5}, 0\right)$, so let $x = -\frac{9}{5},\ y = 0$

$\Rightarrow \frac{36}{5}\sec t = 12$ so $\cos t = \frac{3}{5} \Rightarrow t = \pm 0.927\ldots$

$\sec \pm 0.927\ldots = \frac{5}{3},\quad \tan \pm 0.927\ldots = \pm\frac{4}{3}$

The gradient of the tangent with positive slope is $\frac{4\sec t}{3\tan t} = \frac{4\times\frac{5}{3}}{3\times\frac{4}{3}} = \frac{5}{3}$

B is on the tangent, so $\frac{BD}{AD} = \frac{5}{3} \Rightarrow BD = \frac{5}{3}\times\frac{18}{5} = 6$

By symmetry, BD is half of BC, so the area of ABC is $BD \times AD = 6 \times \frac{18}{5} = \frac{108}{5}$

19 a The parametric equation of the hyperbola H is $x = \sec t,\ y = \tan t$

Use the chain rule to find the gradient: $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\sec t}{\tan t}$

So the tangent to H has the equation: $y - \tan t = \frac{\sec t}{\tan t}(x - \sec t)$

$$y\tan t - \tan^2 t = x\sec t - \sec^2 t$$

$$y\tan t + (\sec^2 t - \tan^2 t) = x\sec t$$

$$y\tan t + 1 = x\sec t$$

$$x\sec t - y\tan t = 1$$

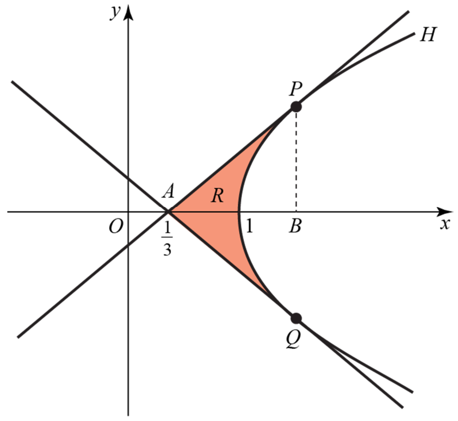
Tangents meet at $\left(\frac{1}{3}, 0\right)$, so let $x = \frac{1}{3},\ y = 0$

$\Rightarrow \frac{1}{3}\sec t = 1$ so $\sec t = 3$

Using $\sec^2 t - \tan^2 t \equiv 1$, $\tan^2 t = 8 \Rightarrow \tan t = \pm 2\sqrt{2}$

So P and Q are $\left(3, 2\sqrt{2}\right)$ and $\left(3, -2\sqrt{2}\right)$

19 b Let A be the point where PQ meets the x-axis, and B be the point where the tangents cross.



The area of the triangle ABP is $\frac{1}{2}\left(3-\frac{1}{3}\right)\times 2\sqrt{2}=\frac{8\sqrt{2}}{3}$

In the first Cartesian quadrant, and for $y>1$, the hyperbola can be seen as the graph of the function $y=\sqrt{1-x^2}$. This can be integrated: the integral $\int_1^3 \sqrt{1-x^2}\,dx$ can be solved by substituting $x=\cosh u$, as follows: $\int_0^{\text{arcosh}3} \sinh^2 u\,du = \int_0^{\text{arcosh}3} \frac{\cosh 2u-1}{2}\,du$

$$= -\frac{\text{arcosh}3}{2}+\left[\frac{\sinh 2u}{4}\right]_0^{\text{arcosh}3}$$

$$= -\frac{\text{arcosh}3}{2}+\frac{3\sinh \text{arcosh}3}{2}$$

$$= -\frac{\text{arcosh}3}{2}+3\sqrt{2}$$

The area of the shaded region is twice the area of the triangle minus twice the value of the integral, so it is $\frac{16\sqrt{2}}{3}+\text{arcosh}\,3-6\sqrt{2}=\text{arcosh}\,3-\frac{2}{3}\sqrt{2}$

Challenge

Let $P = (a\cos\theta, b\sin\theta)$

The focus has coordinates (ae, 0), so the distance PS^2 is:

$$PS^2 = (ae - a\cos\theta)^2 + b^2\sin^2\theta$$
$$= a^2e^2 - 2a^2e\cos\theta + a^2\cos^2\theta + b^2\sin^2\theta$$

The equation of the normal is $ax\sin\theta - by\cos\theta = (a^2 - b^2)\sin\theta\cos\theta$

This intersects the x-axis at $x = \dfrac{(a^2 - b^2)\cos\theta}{a} = ae^2\cos\theta$

Then $QS^2 = (ae^2\cos\theta - ae)^2$
$$= a^2e^4\cos^2\theta - 2a^2e^3\cos\theta + a^2e^2$$

If $QS = ePS$, then $QS^2 = e^2PS^2 \Rightarrow \dfrac{QS^2}{e^2} = PS^2$

$$\frac{QS^2}{e^2} = \frac{a^2e^4\cos^2\theta - 2a^2e^3\cos\theta + a^2e^2}{e^2} = a^2e^2\cos^2\theta - 2a^2e\cos\theta + a^2$$

Set $\dfrac{QS^2}{e^2} = PS^2$:

$$a^2e^2\cos^2\theta - 2a^2e\cos\theta + a^2 = a^2e^2 - 2a^2e\cos\theta + a^2\cos^2\theta + b^2\sin^2\theta$$
$$a^2e^2\cos^2\theta + a^2 = a^2e^2 + a^2\cos^2\theta + b^2\sin^2\theta$$

Use $b^2 = a^2(1 - e^2) \Rightarrow a^2e^2 = a^2 - b^2$:

$$(a^2 - b^2)\cos^2\theta + a^2 = a^2e^2 + a^2\cos^2\theta + b^2\sin^2\theta$$
$$a^2 - b^2\cos^2\theta = a^2e^2 + b^2\sin^2\theta$$
$$a^2 = a^2e^2 + b^2$$

The last equation is true as it is a rearrangement of the defining equation for eccentricity, so this is proved.