Inequalities 4C

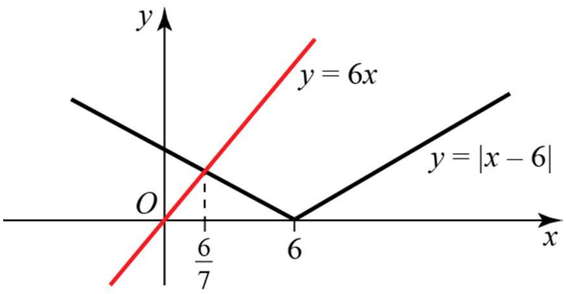
1 a The critical values are given by

$$x-6=6x$$
$$-5x=6$$
$$x=-1.2$$

Or

$$-(x-6)=6x$$
$$-7x=-6$$
$$x=\frac{6}{7}$$

Sketching $y=|x-6|$ and $y=6x$ gives



From the sketch, only $x=\frac{6}{7}$ is a valid critical value.

The solution is when the v-shaped graph is above the line

So the solution is $x<\frac{6}{7}$

1 b The critical values are given by

$x-3=x^2$

$x^2-x+3=0$

This has no solution as the discriminant is negative

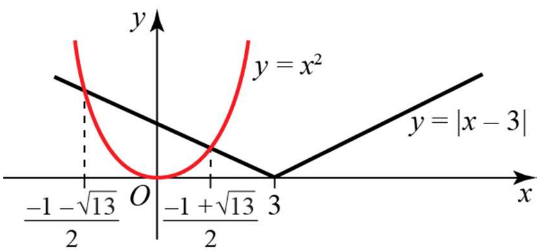
Or

$-(x-3)=x^2$

$x^2+x-3=0$

$x=\dfrac{-1\pm\sqrt{1+12}}{2}=\dfrac{-1\pm\sqrt{13}}{2}$ using the quadratic formula

Sketching $y=|x-3|$ and $y=x^2$ gives



The solution is when $y=|x-3|$ is above $y=x^2$

So the solution is: $\dfrac{-1-\sqrt{13}}{2}<x<\dfrac{-1+\sqrt{13}}{2}$

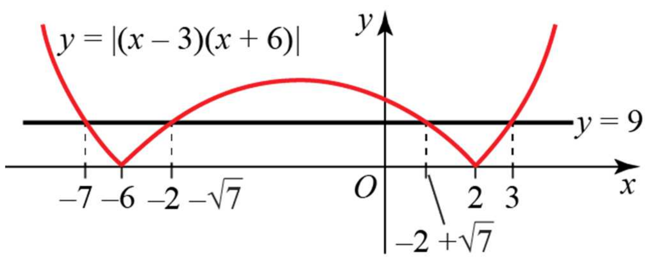
1 c The critical values are given by

$$(x-2)(x+6)=9$$
$$x^2+4x-12=9$$
$$x^2+4x-21=0$$
$$(x+7)(x-3)=0$$
$$x=-7 \text{ or } 3$$

Or

$$-(x^2+4x-12)=9$$
$$x^2+4x-3=0$$
$$x=\frac{-4\pm\sqrt{16+12}}{2}=\frac{-4\pm2\sqrt{7}}{2} \qquad \text{using the quadratic formula}$$
$$x=-2\pm\sqrt{7}$$

Sketching $y=|(x-3)(x+6)|$ and $y=9$ gives



The solution is when $y=|(x-3)(x+6)|$ is below the line $y=9$

So the solution is: $-7<x<-2-\sqrt{7}$ or $-2+\sqrt{7}<x<3$

1 d The critical values are given by

$2x+1=3$

$2x=2$

$x=1$

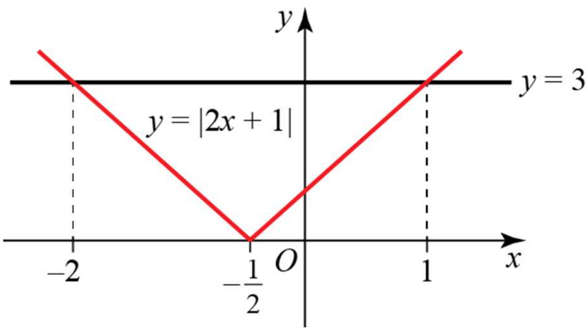
Or

$-(2x+1)=3$

$-2x=4$

$x=-2$

Sketching $y=|2x+1|$ and $y=3$ gives



The solution is when the v-shaped graph is above or on the line

So the solution is $x \leqslant -2$ or $x \geqslant 1$

1 e Rearranging gives $|2x| > 3 - x$

The critical values are given by

$2x = 3 - x$

$3x = 3$

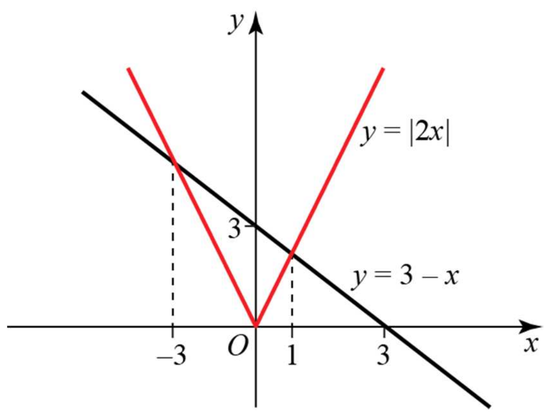
$x = 1$

Or

$-(2x) = 3 - x$

$x = -3$

Sketching $y = |2x|$ and $y = 3 - x$ gives



The solution is when the v-shaped graph is above the line

So the solution is $x < -3$ or $x > 1$

1 f Rearranging and simplifying gives

$\frac{x+3}{|x|+1} < 2$

$x+3 < 2|x|+2$ because $|x| + 1$ is always positive

$x+1 < |2x|$

The critical values are given by

$x+1 = 2x$

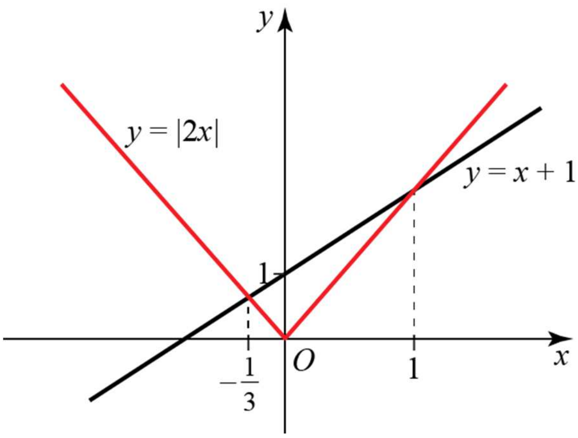
$x = 1$

Or

$x+1 = -(2x)$

$x = -\frac{1}{3}$

Sketching $y = x+1$ and $y = |2x|$ gives



The solution is when the v-shaped graph is above the line

So the solution is $x < -\frac{1}{3}$ or $x > 1$

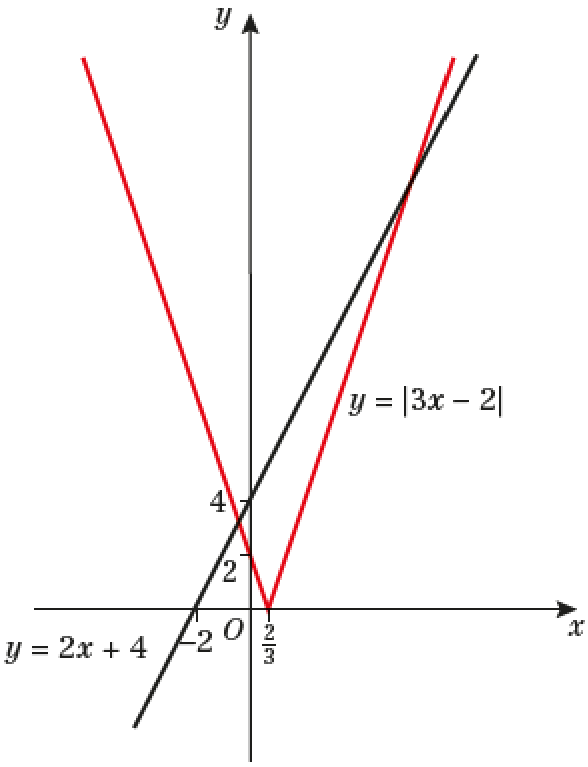
2 a $y = |3x - 2|$

This has a v-shaped graph with a minimum at $(\frac{2}{3}, 0)$. It crosses the y-axis at (0, 2)

$y = 2x + 4$

The graph is a straight line with a positive gradient that passes through (–2, 0) and (0, 4).

So the sketch of both functions is:



b The critical values are given by

$3x - 2 = 2x + 4$

$x = 6$

Or

$-(3x - 2) = 2x + 4$

$5x = -2$

$x = -\frac{2}{5}$

The solution is when the v-shaped graph is on or below the line

So the solution in set notation is $\left\{x: -\frac{2}{5} \leqslant x \leqslant 6\right\}$

3 a $y = |x^2 - 4|$

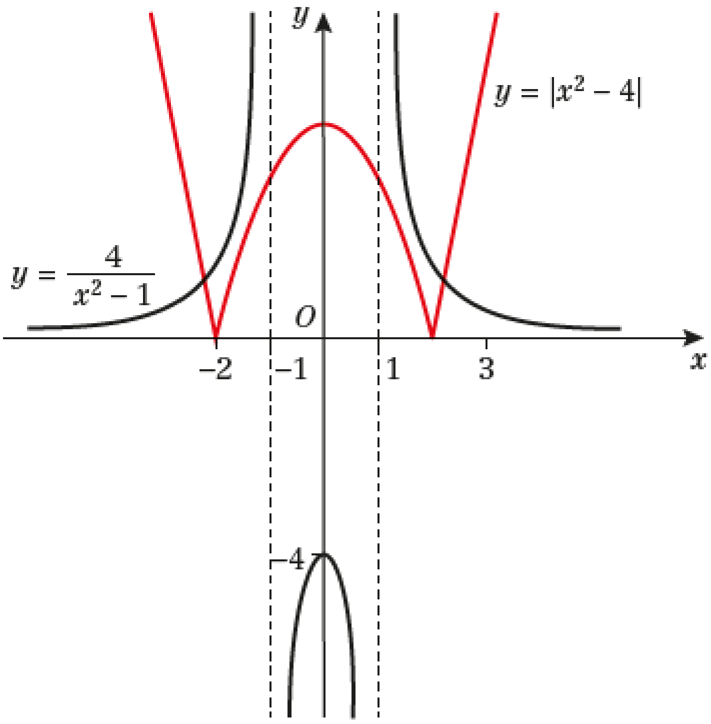
$y = x^2 - 4$ is a quadratic with a positive x^2 coefficient with a minimum at (0, –4)

So the graph of the modulus of this function has the section of $y = x^2 - 4$ below the x-axis reflected in that axis and it touches the x-axis at (–2, 0) and (2, 0)

$$y = \frac{4}{x^2 - 1}$$

The graph has vertical asymptotes at $x = -1$ and $x = 1$. When $-1 < x < 1$, $y < 0$ and there is a local maximum at (0, –4). The graph does not cut the coordinate axes.

So the sketch of both functions is:



3 b The critical values are given by

$$x^2 - 4 = \frac{4}{x^2 - 1}$$
$$(x^2 - 4)(x^2 - 1) = 4$$
$$x^4 - 5x^2 + 4 = 4$$
$$x^2(x^2 - 5) = 0$$
$$x = 0, \pm\sqrt{5}$$

Or

$$-(x^2 - 4) = \frac{4}{x^2 - 1}$$
$$(x^2 - 4)(x^2 - 1) = -4$$
$$x^4 - 5x^2 + 8 = 0$$

Since the determinant of this equation is less than zero $(b^2 - 4ac = 5^2 - 4 \times 8 = -7)$, there are no roots to this equation.

From the sketch in part **a**, the solution is when the red graph is on or below the black graph.

So the solution is $-\sqrt{5} \leqslant x < -1$ or $1 < x \leqslant \sqrt{5}$

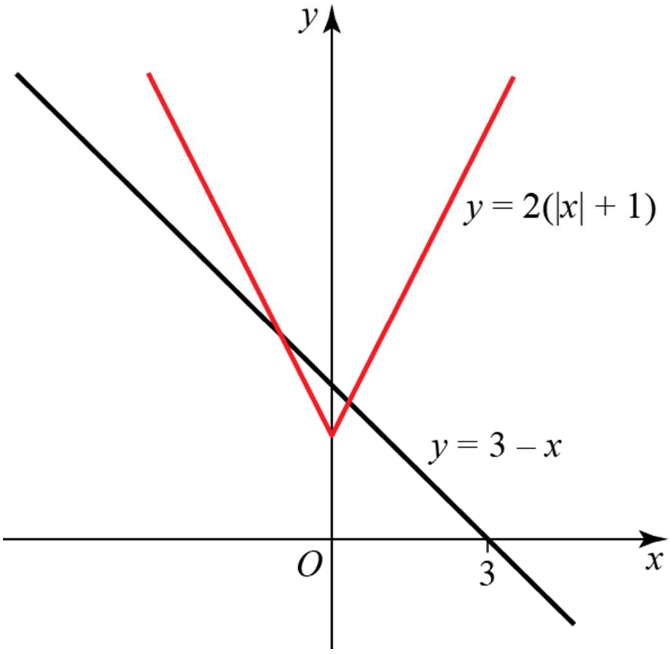
In set notation this is $\{x: -\sqrt{5} \leqslant x < -1\} \cup \{x: 1 < x \leqslant \sqrt{5}\}$

4 Rearranging and simplifying gives

$$\frac{3-x}{|x|+1} > 2$$

$3-x > 2(|x|+1)$ because $|x| + 1$ is always positive

Sketching the graphs of $y = 3-x$ and $y = 2(|x|+1)$ gives:



The critical values are given by

$$3-x = 2x+2$$
$$3x = 1$$
$$x = \frac{1}{3}$$

Or

$$3-x = -2x+2$$
$$x = -1$$

The solution is when the line is above the v-shaped graph

So the solution in set notation is $\left\{x:\ -1 < x < \frac{1}{3}\right\}$

5 $y = 1 - x$

The graph is a straight line with a negative gradient that passes through (0, 1) and (1, 0).

$y = \dfrac{x}{x+2}$

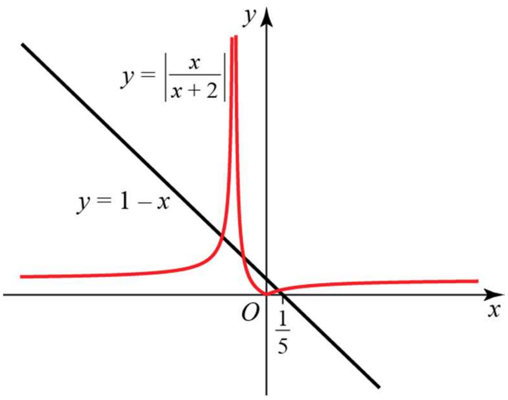
$y = \dfrac{x}{x+2} = \dfrac{x+2-2}{x+2} = 1 - \dfrac{2}{x+2}$ rearranging to see how the curve behaves as $x \to \infty$

The curve is a reciprocal graph. There is a horizontal asymptote at $y = 1$ (as $x \to \pm\infty, y \to 1$) and a vertical asymptote at $x = -2$ (as $x \to -2, y \to \pm\infty$). The graph crosses the axes at (0, 0).

When $-2 < x < 0, y < 0$.

So $y = \left|\dfrac{x}{x+2}\right|$ is the graph of $y = \dfrac{x}{x+2}$ but with the section $-2 < x < 0$ reflected in the x-axis

So the sketch of both curves is:



The critical values are given by

$$\frac{x}{x+2} = 1 - x$$
$$x = (1-x)(x+2)$$
$$x = 2 - x - x^2$$
$$x^2 + 2x - 2 = 0$$
$$x = \frac{-2 \pm \sqrt{4+8}}{2} = \frac{-2 \pm \sqrt{12}}{2} = -1 \pm \sqrt{3}$$ using the quadratic formula

Or

$$-\left(\frac{x}{x+2}\right) = 1 - x$$
$$x = (x-1)(x+2)$$
$$x = x^2 + x - 2$$
$$x^2 = 2$$
$$x = \pm\sqrt{2}$$

From the sketch note that only $-\sqrt{2}$ is a valid critical value.

The solution is when the line is above the curve

So the solution in set notation is $\{x: x < -1-\sqrt{3}\} \cup \{x: -\sqrt{2} < x < -1+\sqrt{3}\}$

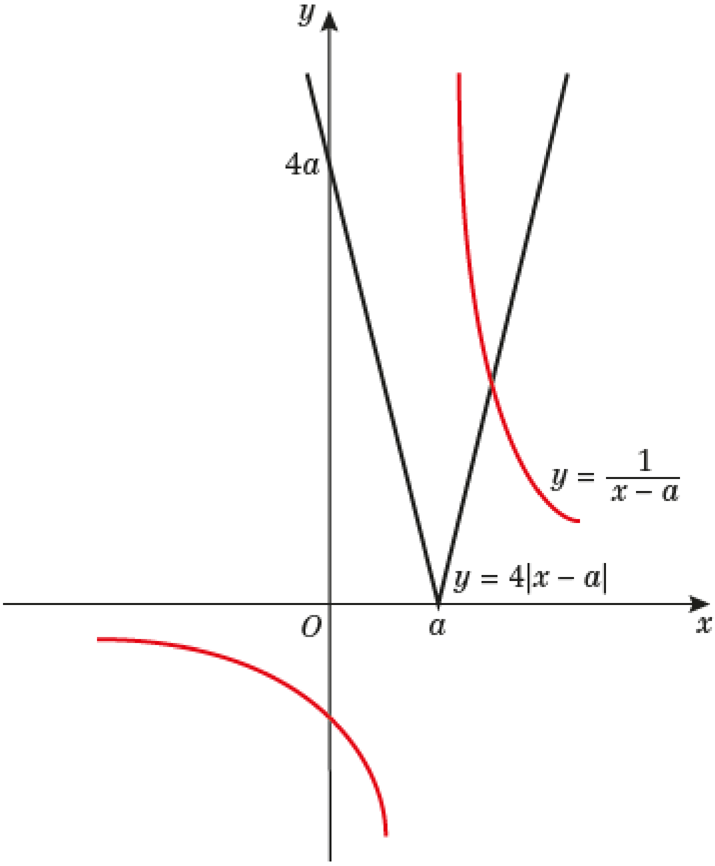
6 a $y = \dfrac{1}{x-a}$

The curve is a reciprocal graph. There is a horizontal asymptote at $y = 0$ (as $x \to \pm\infty$, $y \to 0$) and a vertical asymptote at $x = a$ (as $x \to a$, $y \to \pm\infty$). The graph crosses the y-axis at $(0, -\frac{1}{a})$.

$y = 4|x-a|$

The graph is two line segments meeting at $(a, 0)$. The graph cuts the y-axis at $(0, 4a)$.

So the sketch of both curves is:



Note the sketch assumes $a > 0$. If $a < 0$, a similar sketch is obtained but with the right-hand branches of both curves cutting the y-axis.

6 b $\frac{1}{x-a} = 4(x-a)$

$\frac{1}{4} = (x-a)^2$

$\pm\frac{1}{2} = x-a$

$x = a \pm \frac{1}{2}$

Only this case needs to be considered because the right-hand branch of V has the intersection.

From the sketch in part **a**, only $x = a + \frac{1}{2}$ is a valid critical value.

The solution is when the v-shaped graph is above the curve

So the solution is $x < a$ or $x > a + \frac{1}{2}$

7 Rearranging

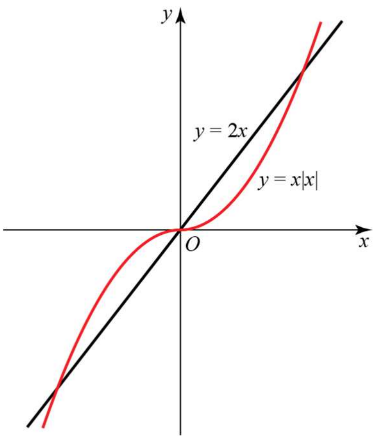
$$\frac{4x}{|x|+2} < x$$

$4x < x(|x|+2)$ as $(|x|+2)$ is always positive

$4x < x|x| + 2x$

$2x < x|x|$

The sketch of both curves is:



The critical values are given by

$$\begin{aligned} 2x &= x^2 \\ x^2 - 2x &= 0 \\ x(x-2) &= 0 \\ x &= 0, 2 \end{aligned}$$

Or

$$\begin{aligned} 2x &= -x^2 \\ x^2 + 2x &= 0 \\ x(x+2) &= 0 \\ x &= 0, -2 \end{aligned}$$

The solution is when the line is below the curve
So the solution is $-2 < x < 0$ or $x > 2$

8 a The student has not checked whether the all the critical values are valid, i.e. that the values the student has calculated actually correspond to intersections of the graphs.

b $y = x^2 + x - 8$

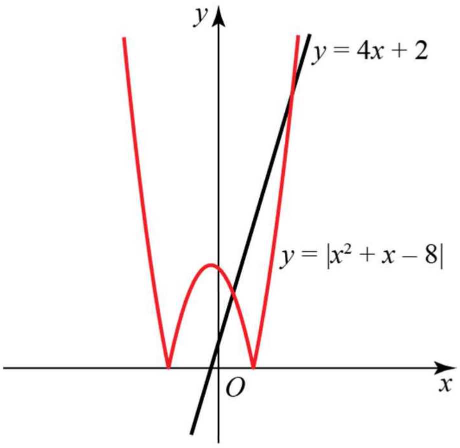
The curve is a quadratic graph with a positive x^2 coefficient, so it is a parabola and it has a minimum at (1, 0). The graph crosses the y-axis at (0, –8).

So the graph of $y = |x^2 + x - 8|$ will be the graph of $y = x^2 + x - 8$ but with the section of the latter curve that is below the x-axis reflected in the x-axis.

$y = 4x + 2$

The graph is a straight line with a positive gradient that passes through (0, 2) and $(-\frac{1}{2}, 0)$.

The sketch of both curves is:



The critical values are given by

$$x^2 + x - 8 = 4x + 2$$
$$x^2 - 3x - 10 = 0$$
$$(x-5)(x+2) = 0$$
$$x = -2, 5$$

From the sketch, only $x = 5$ is a valid critical value.

Or

$$-x^2 - x + 8 = 4x + 2$$
$$x^2 + 5x - 6 = 0$$
$$(x+6)(x-1) = 0$$
$$x = -6, 1$$

From the sketch, only $x = 1$ is a valid critical value.

The solution is when the line is below the curve

So the solution is $1 < x < 5$

Challenge

a If $(x+1)$ is a factor then $f(-1) = 0$ by the factor theorem.

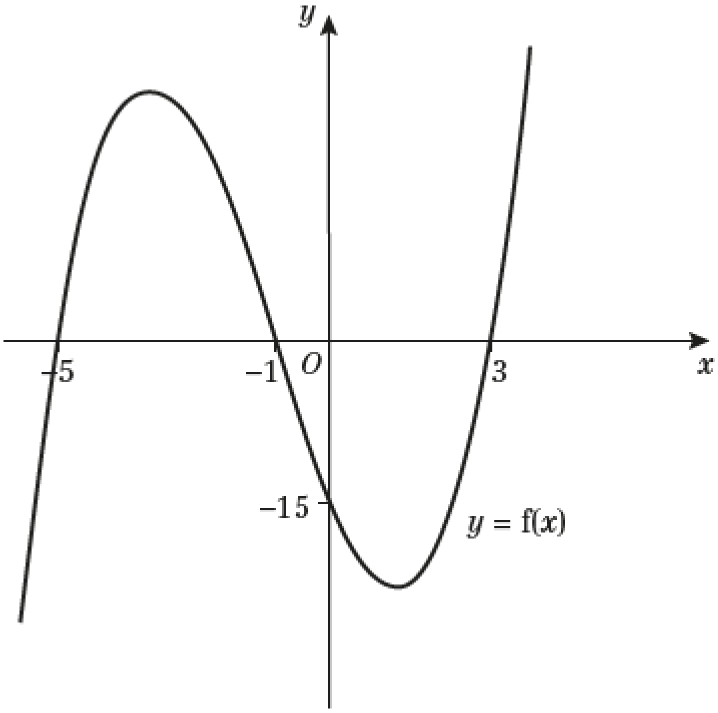
$$f(-1) = (-1)^3 + 3(-1)^2 - 13(-1) - 15$$
$$= -1 + 3 + 13 - 15 = 16 - 16 = 0$$

b Since $(x+1)$ is a factor, f(x) can be written as

$$f(x) = x^3 + 3x^2 - 13x - 15$$
$$= (x+1)(x^2 + 2x - 15)$$
$$= (x+1)(x+5)(x-3)$$

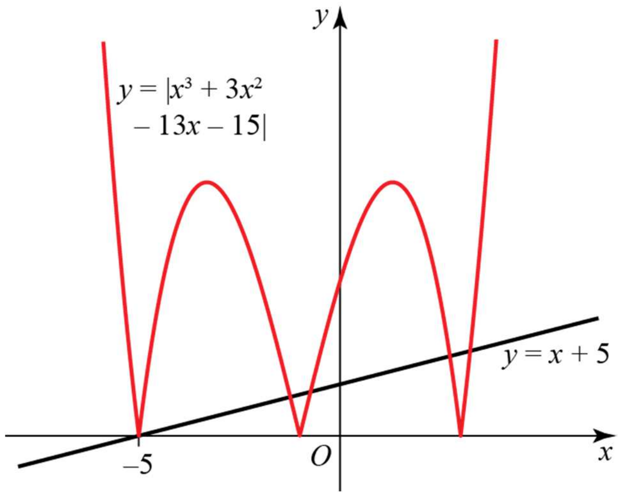
The graph of f(x) is a cubic, with a positive x^3 coefficient, so as $x \to \infty, y \to \infty$ and as $x \to -\infty, y \to -\infty$ and it cuts the x-axis at (–5, 0), (–1, 0) and (3, 0).

So the sketch of $y = f(x)$ is:



Challenge

c Using the sketch in part **b**, the sketch of $y=|x^3+3x^2-13x-15|$ with $y=x+5$ is



The critical values are given by

$$(x+1)(x+5)(x-3)=x+5$$
$$(x+5)\big((x+1)(x-3)-1\big)=0$$
$$(x+5)(x^2-2x-4)=0$$
$$x=-5,\ \frac{2\pm\sqrt{20}}{2}$$
$$x=-5,\ 1\pm\sqrt{5}$$

Or

$$-(x+1)(x+5)(x-3)=x+5$$
$$(x+5)\big((x+1)(x-3)+1\big)=0$$
$$(x+5)(x^2-2x-2)=0$$
$$x=-5,\ \frac{2\pm\sqrt{12}}{2}$$
$$x=-5,\ 1\pm\sqrt{3}$$

The solution is when the line is on or above the curve

So the solution is $x=-1,\ 1-\sqrt{5}\leqslant x\leqslant 1-\sqrt{3},\ 1+\sqrt{3}\leqslant x\leqslant 1+\sqrt{5}$

It can be written in set notation as

$$\{x\colon x=-5\}\cup\left\{x\colon 1-\sqrt{5}\leqslant x\leqslant 1-\sqrt{3}\right\}\cup\left\{x\colon 1+\sqrt{3}\leqslant x\leqslant 1+\sqrt{5}\right\}$$