Taylor Series 6B

1 a We can evaluate the limit directly since there are no singularities:

$$\lim_{x\to 0}\frac{7+x}{5-x}=\frac{7+0}{5-0}=\frac{7}{5}$$

b Again, there are no singularities, so:

$$\lim_{x\to 0}\frac{3-2x}{x+2}=\frac{3-2\cdot 0}{0+2}=\frac{3}{2}$$

c Here we should divide through by x in the numerator and denominator and then use the fact that

$$\lim_{x\to\infty}\tfrac{1}{x}=0:\lim_{x\to\infty}\frac{4-2x}{2+x}=\lim_{x\to\infty}\frac{\frac{4}{x}-2}{\frac{2}{x}+1}=\frac{0-2}{0+1}=-2$$

d Again, dividing through by x, we find:

$$\lim_{x\to\infty}\frac{4x+1}{3+2x}=\lim_{x\to\infty}\frac{4+\frac{1}{x}}{\frac{3}{x}+2}=\frac{4+0}{0+2}=2$$

2 a We make use of the Taylor expansion about $x = 0$:

$$\sin 4x=4x-\frac{4^3}{3!}x^3+\cdots\Rightarrow\frac{\sin 4x}{x}=4-\frac{4^3}{3!}x^2+\cdots$$

$$\Rightarrow\lim_{x\to 0}\frac{\sin 4x}{x}=4$$

b Here we use $\cos x=1-\frac{1}{2!}x^2+\frac{1}{4!}x^4+\cdots$ to see that:

$$\cos x-1=-\frac{1}{2}x^2+\cdots\Rightarrow\frac{\cos x-1}{x^2}=\frac{1}{2}+\cdots$$

$$\Rightarrow\lim_{x\to 0}\frac{\cos x-1}{x^2}=-\frac{1}{2}$$

c We use $e^{3x}=1+3x+\frac{9}{2!}x^2+\cdots$ to see:

$$e^{3x}-1=3x+\frac{9}{2!}x^2+\cdots\Rightarrow\frac{x}{e^{3x}-1}=\frac{1}{3+\frac{9}{2!}x+\cdots}$$

$$\Rightarrow\lim_{x\to 0}\frac{x}{e^{3x}-1}=\frac{1}{3}$$

d We use $\arctan 4x=4x-\frac{4^3}{3}x^3+\cdots$ to see:

$$\frac{x}{\arctan 4x}=\frac{1}{4-\frac{4^3}{3}x^2+\cdots}$$

$$\Rightarrow\lim_{x\to 0}\frac{x}{\arctan 4x}=\frac{1}{4}$$

3 a We use the Taylor series for $\sin x$ about $x = \pi$:

$$\sin x = \cos\pi(x-\pi) - \frac{1}{6}\cos\pi(x-\pi)^3 + \ldots$$

$$\Rightarrow \sin x = -(x-\pi) + \frac{1}{6}(x-\pi)^3 + \ldots$$

Then we find:

$$\lim_{x\to\pi}\frac{x-\pi}{\sin x} = \lim_{x\to\pi}\frac{1}{-1+\frac{1}{6}(x-\pi)^2} = -1$$

b Here we look for a Taylor expansion about $x = 2$ for $\sin(x^2-4)$:

$$\frac{d}{dx}\sin(x^2-4) = 2x\cos(x^2-4)$$

$$\Rightarrow \sin(x^2-4) = \sin(2^2-4)$$

$$+ 2\cdot 2\cdot\cos(2^2-4)(x-2) + (x-2)^2 + \ldots$$

$$\Rightarrow \sin(x^2-4) = 4(x-2) + (x-2)^2 \ldots$$

So we find that:

$$\lim_{x\to 2}\frac{\sin(x^2-4)}{x-2} = \lim_{x\to 2}\left(4 + (x-2) + \ldots\right) = 4$$

4 a We make use of the Taylor series for $\ln(1+x)$ about $x = 0$ to deduce that:

$$\ln(1+x^2) = x^2 - \frac{x^4}{2} + \ldots$$

Then we see:

$$\lim_{x\to 0}\frac{\ln(1+x^2)}{x^2} = \lim_{x\to 0}\left(1 - \frac{x^2}{2} + \ldots\right) = 1$$

b Now we want the Taylor series for $\ln x$ about $x = 1$ given by: $\ln x = (x-1) - \frac{1}{2}(x-1)^2 + \ldots$

Then we note that:

$$(x-1) = (\sqrt{x}-1)(\sqrt{x}+1)$$

$$\Rightarrow \frac{\ln x}{\sqrt{x}-1} = \sqrt{x}+1 - \frac{1}{2}(\sqrt{x}+1)(x-1)^2 + \ldots\ldots$$

And observe that in the limit as $x \to 1$, the second and remaining terms do go to zero, so:

$$\lim_{x\to 1}\frac{\ln x}{\sqrt{x}-1} = \sqrt{1}+1 = 2$$

4 c We calculate the leading order terms in the denominator and numerator:

$$e^x - e^{-x} - 2x = \left(1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \ldots\right)$$
$$-\left(1 - x + \frac{1}{2}x^2 - \frac{1}{6}x^3 + \ldots\right) - 2x = \frac{1}{3}x^3 + \ldots$$

$$x^2 - x\ln(1+x) = x^2 - x\left(x - \frac{1}{2}x^2 + \ldots\right)$$
$$= \frac{1}{2}x^3 + \ldots$$

Then we can take the limit after dividing through by x^3 to obtain:

$$\lim_{x\to 0}\frac{e^x - e^{-x} - 2x}{x^2 - x\ln(1+x)} = \lim_{x\to 0}\frac{\frac{1}{3}x^3 + \ldots}{\frac{1}{2}x^3 + \ldots}$$
$$= \lim_{x\to 0}\frac{\frac{1}{3} + \ldots}{\frac{1}{2} + \ldots} = \frac{2}{3}$$

d We perform a similar procedure:

$$e^{x^2}\sin x - x = \left(1 + x^2 + \ldots\right)\left(x - \frac{1}{3!}x^3 + \ldots\right) - x$$
$$= x - x + x^3 - \frac{1}{3!}x^3 + \ldots = \frac{5}{6}x^3 + \ldots$$
$$x\ln(1+x^2) = x\left(x^2 - \frac{1}{2}x^4 + \ldots\right) = x^3 - \frac{1}{2}x^5 + \ldots$$

Then in the limit we have that:

$$\lim_{x\to 0}\frac{e^{x^2}\sin x - x}{x\ln(1+x^2)} = \lim_{x\to 0}\frac{\frac{5}{6}x^3 + \ldots}{x^3 - \ldots} = \frac{5}{6}$$

5 a The first few leading order terms in the Taylor series for $\sin x$ and e^{-x} are:

$$\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \ldots$$
$$e^{-x} = 1 - x + \frac{1}{2!}x^2 - \frac{1}{3!}x^3 + \ldots$$

which can be found easily by differentiating the two functions using:

$$\frac{d}{dx}\sin x = \cos x,\ \frac{d}{dx}\cos x = -\sin x,\ \frac{d}{dx}e^{-x} = -e^{-x}$$

5 b Now we compute:

$$\frac{1}{\sin x}-\frac{1}{1-e^{-x}}=\frac{1-e^{-x}-\sin x}{\sin x(1-e^{-x})}$$

Plugging in the Taylor expansions:

$$1-e^{-x}-\sin x=1-\left(1-x+\frac{1}{2!}x^2+\ldots\right)-(x+\ldots)$$
$$=-\frac{1}{2}x^2+\ldots$$
$$\sin x(1-e^{-x})=(x+\ldots)\cdot(1-(1-x+\ldots))$$
$$=x^2+\ldots$$

Then we can take the limit:

$$\lim_{x\to 0}\left(\frac{1}{\sin x}-\frac{1}{1-e^{-x}}\right)=\lim_{x\to 0}\frac{1-e^{-x}-\sin x}{\sin x(1-e^{-x})}$$
$$\Rightarrow \lim_{x\to 0}\left(\frac{1}{\sin x}-\frac{1}{1-e^{-x}}\right)=\lim_{x\to 0}\frac{-\frac{1}{2}x^2+\ldots}{x^2+\ldots}=-\frac{1}{2}$$

6 a Using:

$$\frac{\mathrm{d}}{\mathrm{d}x}x^{\frac{1}{2}}=\frac{1}{2}x^{-\frac{1}{2}},\ \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{2}x^{-\frac{1}{2}}\right)=-\frac{1}{4}x^{-\frac{3}{2}},\ldots$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\ln x=\frac{1}{x},\ \frac{\mathrm{d}}{\mathrm{d}x}\frac{1}{x}=-\frac{1}{x^2},\ \frac{\mathrm{d}}{\mathrm{d}x}\left(-\frac{1}{x^2}\right)=\frac{2}{x^3},\ldots$$

we find that the first few leading order terms in the Taylor series for $\ln x$ and $\sqrt{x}$ about $x=1$ are given by:

$$\ln x=(x-1)-\frac{1}{2!}(x-1)^2+\frac{2}{3!}(x-1)^3+\ldots$$
$$\Rightarrow \ln x=(x-1)-\frac{1}{2}(x-1)^2+\frac{1}{3}(x-1)^3+\ldots$$
$$\sqrt{x}=1+\frac{1}{2}(x-1)-\frac{1}{2!}\cdot\frac{1}{4}(x-1)^2+\frac{1}{3!}\cdot\frac{3}{8}(x-1)^3+\ldots$$
$$\Rightarrow \sqrt{x}=1+\frac{1}{2}(x-1)-\frac{1}{8}(x-1)^2+\frac{1}{16}(x-1)^3+\ldots$$

b We have that:

$$\frac{1}{\sqrt{x}}-\frac{1}{\ln x-1}=\frac{-1+\ln x-\sqrt{x}}{(\ln x-1)\sqrt{x}}$$
$$=\frac{-1+(x-1)-\frac{1}{2}(x-1)^2+\ldots-\left(1+\frac{1}{2}(x-1)-\frac{1}{8}(x-1)^2+\ldots\right)}{\left(-1+(x-1)-\frac{1}{2}(x-1)^2+\ldots\right)\left(1+\frac{1}{2}(x-1)-\frac{1}{8}(x-1)^2-\ldots\right)\cdots}$$
$$=\frac{-2+\frac{1}{2}(x-1)+\ldots}{-1+\frac{1}{2}(x-1)+\ldots}\to\frac{-2}{-1}=2,\text{as } x\to 1$$

7 a We use $\frac{d}{dx}(\sinh x) = \cosh x$ as well as $\frac{d}{dx}(\cosh x) = \sinh x$ to write down:

$\sinh x = x + \frac{1}{6}x^3 + \frac{1}{120}x^5 + \ldots$

where we use $\sinh 0 = 0, \cosh 0 = 1$

b Note that $\operatorname{cosech} x = \frac{1}{\sinh x}$, so

$$\lim_{x\to 0}(2x \operatorname{cosech} 3x) = \lim_{x\to 0}\frac{2x}{\sinh 3x}$$

$$= \lim_{x\to 0}\frac{2x}{3x + \frac{1}{3!}(3x)^3 + \ldots} = \lim_{x\to 0}\frac{2}{3 + \frac{1}{2}(9x^2) + \ldots}$$

$$\Rightarrow \lim_{x\to 0}(2x \operatorname{cosech} 3x) = \frac{2}{3}$$

8 a We compute the necessary derivatives:

$$\frac{d}{dx}\sqrt{1+4x} = 2(1+4x)^{-\frac{1}{2}}$$

$$\frac{d}{dx}2(1+4x)^{-\frac{1}{2}} = -4(1+4x)^{-\frac{3}{2}}$$

$$\Rightarrow \sqrt{1+4x} = 3 + \frac{2}{3}(x-2) - \frac{2}{27}(x-2)^2 + \ldots$$

b To find the limit we can find a finite Taylor series for $x^4 - 2x^2 - 8$ such that

$x^4 - 2x^2 - 8$

$= 24(x-2) + 22(x-2)^2 + 8(x-2)^3 + (x-2)^4$

Then writing $\sqrt{1+4x} - 3 = \frac{2}{3}(x-2) - \frac{2}{27}(x-2)^2 + \ldots$

The limit is given by:

$$\lim_{x\to 2}\frac{\sqrt{1+4x} - 3}{x^4 - 2x^2 - 8} = \lim_{x\to 2}\frac{\frac{2}{3} - \frac{2}{27}(x-2) + \ldots}{24 + 22(x-2) + \ldots} = \frac{1}{36}$$

Challenge

a We do the required differentiation:

$$\frac{d}{dy}\sqrt{1+5y} = \frac{5}{2}(1+5y)^{-\frac{1}{2}}$$

$$\frac{d}{dy}\left(\frac{5}{2}(1+5y)^{-\frac{1}{2}}\right) = -\frac{25}{4}(1+5y)^{-\frac{3}{2}}$$

$$\frac{d}{dy}\left(-\frac{25}{4}(1+5y)^{-\frac{3}{2}}\right) = \frac{375}{8}(1+5y)^{-\frac{5}{2}}$$

$$\Rightarrow \sqrt{1+5y} = 1 + \frac{5}{2}y - \frac{25}{8}y^2 + \frac{125}{16}y^3 + \ldots$$

Challenge

b Now we make the substitution $y = \frac{1}{x}$ in the expression $\sqrt{x^2 + 5x} - x$ and note that taking the limit as $x \to \infty$ is the same as taking the limit as $y \to 0$

$$\sqrt{x^2 + 5x} - x = \frac{1}{y}\sqrt{1+5y} - \frac{1}{y}$$

Then: $$= \frac{1}{y}\left(\left(1 + \frac{5}{2}y - \frac{25}{8}y^2 + \ldots\right) - 1\right) = \frac{5}{2} - \frac{25}{8}y + \ldots$$

$$\Rightarrow \lim_{x\to\infty}\sqrt{x^2 + 5x} - x = \lim_{y\to 0}\left(\frac{5}{2} - \frac{25}{8}y + \ldots\right) = \frac{5}{2}$$