Numerical Methods 8B

1 a $(x_0, y_0) = (2, 2)$

$h = 0.25$

$\left(\frac{dy}{dx}\right)_0 = 2^3 - 2^2 = 4$

$y_1 = y_0 + h\left(\frac{dy}{dx}\right)_0$

$y_1 = 2 + 0.25 \times 4$

$y_1 = 3$

b $(x_1, y_1) = (2.25, 3)$

$\left(\frac{dy}{dx}\right)_1 = 2.25^3 - 3^2 = 2.39062\ldots$

$y_2 = y_0 + 2h\left(\frac{dy}{dx}\right)_1$

$y_2 = 2 + 2 \times 0.25 \times 2.39062\ldots$

$y_2 = 3.19531\ldots$

Therefore, at $x = 2.5$, $y \approx 3.195$ (3 d.p.)

2 $h = 0.1$

$(x_0, y_0) = (1, 1)$

$\left(\frac{dy}{dx}\right)_0 = \ln 1 + 3 \times 1 = 3$

$y_1 = y_0 + h\left(\frac{dy}{dx}\right)_0$

$y_1 = 1 + 0.1 \times 3$

$y_1 = 1.3$

$(x_1, y_1) = (1.1, 1.3)$

$\left(\frac{dy}{dx}\right)_1 = \ln 1.1 + 3 \times 1.3 = 3.99531\ldots$

$y_2 = y_0 + 2h\left(\frac{dy}{dx}\right)_1$

$y_2 = 1 + 2 \times 0.1 \times 3.99531\ldots$

$y_2 = 1.79906\ldots$

$(x_2, y_2) = (1.2, 1.79906\ldots)$

$\left(\frac{dy}{dx}\right)_2 = \ln 1.2 + 3 \times 1.79906\ldots = 5.57951\ldots$

$y_3 = y_1 + 2h\left(\frac{dy}{dx}\right)_2$

$y_3 = 1.3 + 2 \times 0.1 \times 5.57951\ldots$

$y_3 = 2.41590\ldots$

$(x_3, y_3) = (1.3, 2.41590\ldots)$

$\left(\frac{dy}{dx}\right)_3 = \ln 1.3 + 3 \times 2.41590\ldots = 1.79906\ldots$

$y_4 = y_2 + 2h\left(\frac{dy}{dx}\right)_3$

$y_4 = 1.79906\ldots + 2 \times 0.1 \times 1.79906\ldots$

$y_4 = 3.30108\ldots$

$(x_4, y_4) = (1.4, 3.30108\ldots)$

$\left(\frac{dy}{dx}\right)_4 = \ln 1.4 + 3 \times 3.30108\ldots = 10.23970\ldots$

$y_5 = y_3 + 2h\left(\frac{dy}{dx}\right)_4$

$y_5 = 2.41590\ldots + 2 \times 0.1 \times 10.23970\ldots$

$y_5 = 4.4638$

Therefore, at $x = 1.5$, $y \approx 4.464$ (3 d.p.)

3 a $(x_0, y_0) = (1, 2)$

$h = 0.2$

$\left(\frac{dy}{dx}\right)_0 = \sin(1 \times 2) = 0.90930\ldots$

$y_1 = y_0 + h\left(\frac{dy}{dx}\right)_0$

$y_1 = 2 + 0.2 \times 0.90930$

$y_1 = 2.18186\ldots$

So $y_1 \approx 2.1819$ (5 s.f.)

b $(x_1, y_1) = (1.2, 2.18186\ldots)$

$\left(\frac{dy}{dx}\right)_1 = \sin(1.2 \times 2.18186\ldots) = 0.49979\ldots$

$y_2 = y_0 + 2h\left(\frac{dy}{dx}\right)_1$

$y_2 = 2 + 2 \times 0.2 \times 0.49979\ldots$

$y_2 = 2.19992\ldots$

$(x_2, y_2) = (1.4, 2.19992\ldots)$

$\left(\frac{dy}{dx}\right)_2 = \sin(1.4 \times 2.19992\ldots) = 0.06167\ldots$

$y_3 = y_1 + 2h\left(\frac{dy}{dx}\right)_2$

$y_3 = 2.18186\ldots + 2 \times 0.2 \times 0.06167\ldots$

$y_3 = 2.20653\ldots$

$(x_3, y_3) = (1.6, 2.20653\ldots)$

$\left(\frac{dy}{dx}\right)_3 = \sin(1.6 \times 2.20653\ldots) = -0.37912\ldots$

$y_4 = y_2 + 2h\left(\frac{dy}{dx}\right)_3$

$y_4 = 2.19992\ldots + 2 \times 0.2 \times -0.37912\ldots$

$y_4 = 2.04827\ldots$

$(x_4, y_4) = (1.8, 2.04827\ldots)$

$\left(\frac{dy}{dx}\right)_4 = \sin(1.8 \times 2.04827\ldots) = -0.51866\ldots$

$y_5 = y_3 + 2h\left(\frac{dy}{dx}\right)_4$

$y_5 = 2.20653\ldots + 2 \times 0.2 \times -0.51866\ldots$

$y_5 = 1.99906$

Therefore, at $x = 2$, $y \approx 1.999$ (4 s.f.)

4 $(t_0, P_0) = (0, 700)$

$h = 0.5$ $\quad \left(\frac{dP}{dt}\right)_0 = 3 \times 700 - 0.002 \times 700^2$

$-100\cos(0.6 \times 0)$

$= 1020$

$P_1 = P_0 + h\left(\frac{dP}{dx}\right)_0$

$P_1 = 700 + 0.5 \times 1020$

$P_1 = 1210$

$(t_1, P_1) = (0.5, 1210)$

$\left(\frac{dP}{dt}\right)_1 = 3 \times 1210 - 0.002 \times 1210^2$

$-100\cos(0.6 \times 0.5)$

$= 606.27\ldots$

$P_2 = P_0 + 2h\left(\frac{dP}{dt}\right)_1$

$P_2 = 700 + 2 \times 0.5 \times 606.27\ldots$

$P_2 = 1306.27\ldots$

$(t_2, P_2) = (1, 1306.27\ldots)$

$\left(\frac{dP}{dt}\right)_2 = 3 \times 1306.27\ldots - 0.002 \times (1306.27\ldots)^2$

$-100\cos(0.6 \times 1)$

$= 423.60\ldots$

$P_3 = P_1 + 2h\left(\frac{dP}{dt}\right)_2$

$P_3 = 1210 + 2 \times 0.5 \times 423.60\ldots$

$P_3 = 1633.60\ldots$

$(t_3, P_3) = (1.5, 1633.60\ldots)$

$\left(\frac{dP}{dt}\right)_3 = 3 \times 1633.60\ldots - 0.002 \times (1633.60\ldots)^2$

$-100\cos(0.6 \times 1.5)$

$= -498.66$

$P_4 = P_2 + 2h\left(\frac{dP}{dt}\right)_3$

$P_4 = 1306.27\ldots + 2 \times 0.5 \times -498.66\ldots$

$P_4 = 807.60\ldots$

Therefore, at $t = 2$ months, $P \approx 810$ rabbits (rounded to the nearest 10 rabbits).

5 $(x_0, v_0) = (10, 12)$

$h = 0.5$

$\left(\frac{dv}{dx}\right)_0 = \frac{1.5\times10-24.8}{12} - 0.003\times12 = -0.85\ldots$

$v_1 = v_0 + h\left(\frac{dv}{dx}\right)_0$

$v_1 = 12 + 0.5\times -0.85\ldots$

$v_1 = 11.57\ldots$

$(x, v_1) = (10.5, 11.57\ldots)$

$\left(\frac{dv}{dx}\right)_1 = \frac{1.5\times10.5-24.8}{11.57\ldots} - 0.003\times11.57...$

$= -0.82\ldots$

$v_2 = v_0 + 2h\left(\frac{dv}{dx}\right)_1$

$v_2 = 12 + 2\times0.5\times -0.82\ldots = 11.18\ldots$

$(x_2, v_2) = (11, 11.18\ldots)$

$\left(\frac{dv}{dx}\right)_2 = \frac{1.5\times110-24.8}{11.18\ldots} - 0.003\times11.18\ldots$

$= -0.78\ldots$

$v_3 = v_1 + 2h\left(\frac{dv}{dx}\right)_2$

$v_3 = 11.57\ldots + 2\times0.5\times -0.78\ldots = 10.80\ldots$

Therefore, at $x = 11.5$, $v \approx 10.8$ (3 s.f.)

6 a $(x_0, y_0) = (1, 1)$

$h = 0.1$

$\left(\frac{dy}{dx}\right)_0 = 1^2 - 1 + 1 - 2\times1 = -1$

$y_1 = y_0 + h\left(\frac{dy}{dx}\right)_0$

$y_1 = 1 + 0.1\times -1$

$y_1 = 0.9$

b $(x_1, y_1) = (1.1, 0.9)$

$\left(\frac{dy}{dx}\right)_1 = 1.1^2 - 1.1 + 1 - 2\times0.9 = -0.69$

$y_2 = y_0 + 2h\left(\frac{dy}{dx}\right)_1$

$y_2 = 1 + 2\times0.1\times -0.69$

$y_2 = 0.862$

Therefore, at $x = 1.2$, $y \approx 0.862$ (3 d.p.)

6 c $\frac{dy}{dx} = x^2 - x + 1 - 2y$

$\frac{dy}{dx} + 2y = x^2 - x + 1$

Integrating factor is $e^{\int 2dx} = e^{2x}$

Multiplying both sides by the integrating factor:

$\left(\frac{dy}{dx} + 2y\right)e^{2x} = (x^2 - x + 1)e^{2x}$

$\frac{d(e^{2x}y)}{dx} = (x^2 - x + 1)e^{2x}$

$e^{2x}y = \int (x^2 - x + 1)e^{2x}dx$

$e^{2x}y = \int x^2e^{2x}dx - \int xe^{2x}dx + \int e^{2x}dx$

$e^{2x}y = x^2\frac{e^{2x}}{2} - \int 2x\frac{e^{2x}}{2}dx - \int xe^{2x}dx + \frac{e^{2x}}{2} + c$

$e^{2x}y = x^2\frac{e^{2x}}{2} - \left(2x\frac{e^{2x}}{2} - 2\frac{e^{2x}}{4}\right) + \frac{e^{2x}}{2} + c$

$e^{2x}y = x^2\frac{e^{2x}}{2} - xe^{2x} + e^{2x} + c$

Substituting $x = 1$ and $y = 1$ into the equation gives

$e^2 = \frac{e^2}{2} - e^2 + e^2 + c$

$c = \frac{e^2}{2}$

Therefore,

$e^{2x}y = x^2\frac{e^{2x}}{2} - xe^{2x} + e^{2x} + \frac{e^2}{2}$

So $y = \frac{x^2}{2} - x + 1 + \frac{e^{2-2x}}{2}$

d When $x = 1.2$

$y = \frac{1.2^2}{2} - 1.2 + 1 + \frac{e^{2-2\times1.2}}{2} = 0.85516\ldots$

Percentage error

$= \frac{|0.85516\ldots - 0.862|}{0.85516\ldots}\times100 = 0.79985\ldots\%$

Therefore, the percentage error is 0.80% (2 s.f.)

Challenge

$(x_0, y_0) = (0,0)$
$h = 0.5$
$\left(\frac{dy}{dx}\right)_0 = \frac{2}{0-1} + 0 = -2$
$y_1 = y_0 + h\left(\frac{dy}{dx}\right)_0$
$y_1 = 0 + 0.5 \times -2$
$y_1 = -1$
$(x_1, y_1) = (0.5, -1)$
$\left(\frac{dy}{dx}\right)_1 = \frac{2}{0.5-1} - 1 = -5$
$y_2 = y_0 + 2h\left(\frac{dy}{dx}\right)_1$
$y_2 = 0 + 2 \times 0.5 \times -5$
$y_2 = -5$

However, this estimate is not valid since $\frac{dy}{dx}$ is undefined at $x = 1$

The curve has a vertical asymptote at $x = 1$