Numerical Methods 8C

1 a $(x_0, y_0) = (2, 4)$

$h = 0.1$

$\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)_0 = 1$

$\frac{y_1 - y_0}{h} = \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)_0$

$y_1 = 4 + 0.1 \times 1$

$y_1 = 4.1$

$(x_1, y_1) = (2.1, 4.1)$

$\left(\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right)_1 = 2.1 + 4.1 - 1 = 5.2$

$\frac{y_2 - 2y_1 + y_0}{h^2} = \left(\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right)_1$

$y_2 = 2 \times 4.1 - 4 + 0.1^2 \times 5.2$

$y_2 = 4.252$

$(x_2, y_2) = (2.2, 4.252)$

$\left(\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right)_2 = 2.2 + 4.252 - 1 = 5.452$

$\frac{y_3 - 2y_2 + y_1}{h^2} = \left(\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right)_2$

$y_3 = 2 \times 4.252 - 4.1 + 0.1^2 \times 5.452$

$y_3 = 4.45852$

1 b $(x_0, y_0) = (1,1)$

$h = 0.2$

$\left(\frac{dy}{dx}\right)_0 = 2$

$\frac{y_1 - y_0}{h} = \left(\frac{dy}{dx}\right)_0$

$y_1 = 1 + 0.2 \times 1$

$y_1 = 1.4$

$(x_1, y_1) = (1.2, 1.4)$

$\left(\frac{d^2y}{dx^2}\right)_1 = 1.2^2 + 1.4^2 = 3.4$

$\frac{y_2 - 2y_1 + y_0}{h^2} = \left(\frac{d^2y}{dx^2}\right)_1$

$y_2 = 2 \times 1.4 - 1 + 0.2^2 \times 3.4$

$y_2 = 1.936$

$(x_2, y_2) = (1.4, 1.936)$

$\left(\frac{d^2y}{dx^2}\right)_2 = 1.4^2 + 1.936^2 = 5.708096$

$\frac{y_3 - 2y_2 + y_1}{h^2} = \left(\frac{d^2y}{dx^2}\right)_2$

$y_3 = 2 \times 1.936 - 1.4 + 0.2^2 \times 5.708096$

$y_3 = 2.70032384$

$y_3 \approx 2.700324 \ (6 \text{ d.p.})$

1 c $(x_0, y_0) = (2,1)$

$h = 0.1$

$\left(\frac{dy}{dx}\right)_0 = 1$

$\frac{y_1 - y_0}{h} = \left(\frac{dy}{dx}\right)_0$

$y_1 = 1 + 0.1 \times 1$

$y_1 = 1.1$

$(x_1, y_1) = (2.1, 1.1)$

$\left(\frac{d^2y}{dx^2}\right)_1 = 2 \times 2.1 \times 1.1 - 1.1^2 + 1 = 4.41$

$\frac{y_2 - 2y_1 + y_0}{h^2} = \left(\frac{d^2y}{dx^2}\right)_1$

$y_2 = 2 \times 1.1 - 1 + 0.1^2 \times 4.41$

$y_2 = 1.2441$

$(x_2, y_2) = (2.2, 1.2441)$

$\left(\frac{d^2y}{dx^2}\right)_2 = 2 \times 2.2 \times 1.2441 - 1.2441^2 + 1 = 4.926\ldots$

$\frac{y_3 - 2y_2 + y_1}{h^2} = \left(\frac{d^2y}{dx^2}\right)_2$

$y_3 = 2 \times 1.2441 - 1.1 + 0.1^2 \times 4.926\ldots$

$y_3 = 1.43746255\ldots$

$y_3 \approx 1.437463 \ (6 \text{ d.p.})$

1 d $(x_0, y_0) = (3, 2)$

$h = 0.05$

$\left(\frac{dy}{dx}\right)_0 = 2$

$\frac{y_1 - y_0}{h} = \left(\frac{dy}{dx}\right)_0$

$y_1 = 2 + 0.05 \times 2$

$y_1 = 2.1$

$(x_1, y_1) = (3.05, 2.1)$

$\left(\frac{d^2y}{dx^2}\right)_1 = \sin(3.05 \times 2.1) - 2 = -1.878\ldots$

$\frac{y_2 - 2y_1 + y_0}{h^2} = \left(\frac{d^2y}{dx^2}\right)_1$

$y_2 = 2 \times 2.1 - 2 + 0.05^2 \times -1.878\ldots$

$y_2 = 2.19530378 \approx 2.195304$ (6 d.p.)

$(x_2, y_2) = (3.1, 2.195\ldots)$

$\left(\frac{d^2y}{dx^2}\right)_2 = \sin(3.1 \times 2.195\ldots) - 2 = -1.501\ldots$

$\frac{y_3 - 2y_2 + y_1}{h^2} = \left(\frac{d^2y}{dx^2}\right)_2$

$y_3 = 2 \times 2.195\ldots - 2.1 + 0.05^2 \times -1.501\ldots$

$y_3 = 2.28685466\ldots$

$y_3 \approx 2.286855$ (6 d.p.)

2 a $(x_0, y_0) = (1, 2)$

$h = 0.1$

$\left(\frac{dy}{dx}\right)_0 = 0.5$

$\left(\frac{d^2y}{dx^2}\right)_0 = 1 + 2 - 0.5 = 2.5$

$\frac{y_1 - y_{-1}}{2h} = \left(\frac{dy}{dx}\right)_0$

$y_1 - y_{-1} = 2 \times 0.1 \times 0.5$

$y_1 - y_{-1} = 0.1$

$\frac{y_1 - 2y_0 + y_{-1}}{h^2} = \left(\frac{d^2y}{dx^2}\right)_0$

$y_1 + y_{-1} = 2 \times 2 + 0.1^2 \times 2.5$

$y_1 + y_{-1} = 4.025$

Adding the two equations gives

$2y_1 = 0.1 + 4.025 = 4.125$

$y_1 = 2.0625$

b $(x_0, y_0) = (2, 3)$

$h = 0.05$

$\left(\frac{dy}{dx}\right)_0 = 2$

$\left(\frac{d^2y}{dx^2}\right)_0 = 3 \times 2^2 - 2 \times \sin 3$

$= 11.71775998\ldots$

$\frac{y_1 - y_{-1}}{2h} = \left(\frac{dy}{dx}\right)_0$

$y_1 - y_{-1} = 2 \times 0.05 \times 2 = 0.2$

$\frac{y_1 - 2y_0 + y_{-1}}{h^2} = \left(\frac{d^2y}{dx^2}\right)_0$

$y_1 + y_{-1} = 2 \times 3 + 0.05^2 \times 11.71775998\ldots$

$y_1 + y_{-1} = 6.02929440\ldots$

Adding the two equations gives

$2y_1 = 0.2 + 6.02929440\ldots = 6.22929440\ldots$

$y_1 = 3.11464720\ldots$

$y_1 = 3.114647$ (6 d.p.)

2 c $(x_0, y_0) = (3,1)$

$h = 0.1$

$\left(\frac{dy}{dx}\right)_0 = 1$

$\left(\frac{d^2y}{dx^2}\right)_0 = 3\times3\times1-1 = 8$

$\frac{y_1 - y_{-1}}{2h} = \left(\frac{dy}{dx}\right)_0$

$y_1 - y_{-1} = 2\times0.1\times1$

$y_1 - y_{-1} = 0.2$

$\frac{y_1 - 2y_0 + y_{-1}}{h^2} = \left(\frac{d^2y}{dx^2}\right)_0$

$y_1 + y_{-1} = 2\times1 + 0.1^2\times8$

$y_1 + y_{-1} = 2.08$

Adding the two equations gives

$2y_1 = 0.2 + 2.08 = 2.28$

$y_1 = 1.14$

d $(x_0, y_0) = (0,1.5)$

$h = 0.2$

$\left(\frac{dy}{dx}\right)_0 = 0.8$

$\left(\frac{d^2y}{dx^2}\right)_0 = \sin 0 - 2\times0\times1.5\times0.8 = 0$

$\frac{y_1 - y_{-1}}{2h} = \left(\frac{dy}{dx}\right)_0$

$y_1 - y_{-1} = 2\times0.2\times0.8$

$y_1 - y_{-1} = 0.32$

$\frac{y_1 - 2y_0 + y_{-1}}{h^2} = \left(\frac{d^2y}{dx^2}\right)_0$

$y_1 + y_{-1} = 2\times1.5 + 0.8^2\times0$

$y_1 + y_{-1} = 3$

Adding the two equations gives

$2y_1 = 0.32 + 3 = 3.32$

$y_1 = 1.66$

3 a $(x_0, y_0) = (1,1)$

$\left(\frac{dy}{dx}\right)_0 = -1$

$h = 0.1$

$\frac{y_1 - y_0}{h} = \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)_0$

$y_1 = 1 + 0.1 \times -1$

$y_1 = 0.9$

b $(x_1, y_1) = (1.1, 0.9)$

$\left(\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right)_1 = 1.1^3 - 0.9^2 = 0.521$

$\frac{y_2 - 2y_1 + y_0}{h^2} = \left(\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right)_1$

$y_2 = 2 \times 0.9 - 1 + 0.1^2 \times 0.521$

$y_2 = 0.8052 (4 d.p.)$

$(x_2, y_2) = (1.2, 0.80521)$

$\left(\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right)_2 = 1.2^3 - 0.80521^2 = 1.07963\ldots$

$\frac{y_3 - 2y_2 + y_1}{h^2} = \left(\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right)_2$

$y_3 = 2 \times 0.80521 - 0.9 + 0.1^2 \times 1.07963$

$y_3 = 0.72122\ldots$

$y_3 \approx 0.7212 \ (4 \text{ d.p.})$

4 $(x_0, y_0) = (2,1)$

$\left(\frac{dy}{dx}\right)_0 = 0.6$

$h = 0.2$

$\frac{y_1 - y_0}{h} = \left(\frac{dy}{dx}\right)_0$

$y_1 = 1 + 0.2 \times 0.6$

$y_1 = 1.12$

$(x_1, y_1) = (2.2, 1.12)$

$\left(\frac{d^2y}{dx^2}\right)_1 = 3\sin 2.2 - 1.12^2 + 1$

$= 2.1710892\ldots$

$\frac{y_2 - 2y_1 + y_0}{h^2} = \left(\frac{d^2y}{dx^2}\right)_1$

$y_2 = 2 \times 1.12 - 1 + 0.2^2 \times 2.17109\ldots$

$y_2 = 1.3268436\ldots \approx 1.326844 (6d.p.)$

$(x_2, y_2) = (2.4, 1.32684\ldots)$

$\left(\frac{d^2y}{dx^2}\right)_2 = 3\sin 2.4 - (1.32684\ldots)^2 + 1$

$= 1.2658757\ldots$

$\frac{y_3 - 2y_2 + y_1}{h^2} = \left(\frac{d^2y}{dx^2}\right)_2$

$y_3 = 2 \times 1.3268436\ldots - 1.12 + 0.2^2 \times 1.2658757\ldots$

$y_3 = 1.58432216\ldots$

$y_3 \approx 1.584322 \ (6 \text{ d.p.})$

5 a $(x_0, y_0) = (2,0)$

$\left(\frac{dy}{dx}\right)_0 = 3$

$\left(\frac{d^2y}{dx^2}\right)_0 = \frac{2^2 - 0}{3 \times 2} \times 3 = 2$

5 b $h = 0.1$

$$\frac{y_1 - y_{-1}}{2h} = \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)_0$$

$$y_1 - y_{-1} = 2 \times 0.1 \times 3$$

$$y_1 - y_{-1} = 0.6$$

$$\frac{y_1 - 2y_0 + y_{-1}}{h^2} = \left(\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right)_0$$

$$y_1 + y_{-1} = 2 \times 0 + 0.1^2 \times 2$$

$$y_1 + y_{-1} = 0.02$$

Adding the two equations gives

$$2y_1 = 0.6 + 0.02 = 0.62$$

$$y_1 = 0.31$$

So at $x = 2.1$, $y \approx 0.31$

6 $(x_0, y_0) = (1, -3)$

$$\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)_0 = -0.5$$

$$\left(\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right)_0 = \sin(1 \times -3) - (-0.5) = 0.35888\ldots$$

$h = 0.05$

$$\frac{y_1 - y_{-1}}{2h} = \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)_0$$

$$y_1 - y_{-1} = 2 \times 0.05 \times -0.5$$

$$y_1 - y_{-1} = -0.05$$

$$\frac{y_1 - 2y_0 + y_{-1}}{h^2} = \left(\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right)_0$$

$$y_1 + y_{-1} = 2 \times -3 + 0.05^2 \times 0.35888\ldots$$

$$y_1 + y_{-1} = -5.99910\ldots$$

Adding the two equations gives

$$2y_1 = -0.05 - 5.99910\ldots = -6.04910\ldots$$

$$y_1 = -3.02455\ldots$$

So at $x = 1.05$, $y \approx -3.02455$ (5 d.p.)