Reducible differential equations 9A

1 a $z = \frac{y}{x} \Rightarrow y = xz$

$\therefore \quad \frac{dy}{dx} = z + x\frac{dz}{dx}$

Use the given substitution to express $\frac{dy}{dx}$ in terms of z, x and $\frac{dz}{dx}$

Substitute into the equation:

$$\frac{dy}{dx} = \frac{y}{x} + \frac{x}{y}$$

$$\therefore \quad z + x\frac{dz}{dx} = z + \frac{1}{z}$$

$$\therefore \quad x\frac{dz}{dx} = \frac{1}{z}$$

Separate the variables:

Then $\int z\,dz = \int \frac{1}{x}\,dx$

$$\therefore \quad \frac{z^2}{2} = \ln x + c$$

$$\therefore \quad \frac{y^2}{2x^2} = \ln x + c, \text{ as } z = \frac{y}{x}$$

$$\therefore \quad y^2 = 2x^2(\ln x + c)$$

1 b As $z=\dfrac{y}{x'}$ $y=zx$ and $\dfrac{\mathrm{d}y}{\mathrm{d}x}=z+x\dfrac{\mathrm{d}z}{\mathrm{d}x}$

$$\therefore \frac{\mathrm{d}y}{\mathrm{d}x}=\frac{y}{x}+\frac{x^2}{y^2} \Rightarrow z+x\frac{\mathrm{d}z}{\mathrm{d}x}=z+\frac{1}{z^2}$$

$$\therefore \qquad x\frac{\mathrm{d}z}{\mathrm{d}x}=\frac{1}{z^2}$$

Separate the variables:

Then $\int z^2\,\mathrm{d}z=\int \frac{1}{x}\mathrm{d}x$

$$\therefore \qquad \frac{z^3}{3}=\ln x+c$$

But $z=\dfrac{y}{x}$

$$\therefore \frac{y^3}{3x^3}=\ln x+c$$

$$\therefore \quad y^3=3x^3(\ln x+c)$$

c As $z=\dfrac{y}{x}$, $y=zx$ and $\dfrac{\mathrm{d}y}{\mathrm{d}x}=z+x\dfrac{\mathrm{d}z}{\mathrm{d}x}$

$$\therefore \frac{\mathrm{d}y}{\mathrm{d}x}=\frac{y}{x}+\frac{y^2}{x^2} \Rightarrow z+x\frac{\mathrm{d}z}{\mathrm{d}x}=z+z^2$$

$$\therefore \qquad x\frac{\mathrm{d}z}{\mathrm{d}x}=z^2$$

Separate the variables:

$$\therefore \int \frac{1}{z^2}\mathrm{d}z=\int \frac{1}{x}\,\mathrm{d}x$$

$$\therefore \quad -\frac{1}{z}=\ln x+c$$

$$\therefore \quad z=\frac{-1}{\ln x+c}$$

But $z=\dfrac{y}{x}$

$$\therefore \quad y=\frac{-x}{\ln x+c}$$

1 d $z=\dfrac{y}{x}\Rightarrow y=zx$ and $\dfrac{\mathrm{d}y}{\mathrm{d}x}=z+x\dfrac{\mathrm{d}z}{\mathrm{d}x}$

$$\therefore \frac{\mathrm{d}y}{\mathrm{d}x}=\frac{x^3+4y^3}{3xy^2}\Rightarrow z+x\frac{\mathrm{d}z}{\mathrm{d}x}=\frac{x^3+4z^3x^3}{3x\,z^2x^2}$$

$$\therefore \qquad x\frac{\mathrm{d}z}{\mathrm{d}x}=\frac{1+4z^3}{3z^2}-z$$

$$=\frac{1+z^3}{3z^2}$$

Separate the variables:

$$\therefore \int\frac{3z^2}{1+z^3}\,\mathrm{d}z=\int\frac{1}{x}\,\mathrm{d}x$$

$\therefore \ln(1+z^3)=\ln x+\ln A$, where A is constant

$\therefore \ln(1+z^3)=\ln Ax$

So $\quad 1+z^3=Ax$

And $\quad z^3=Ax-1$. But $z=\dfrac{y}{x}$

$$\therefore \qquad \frac{y^3}{x^3}=Ax-1$$

$\therefore \qquad y^3=x^3(Ax-1)$, where A is a positive constant.

2 a Given $z=y^{-2}$, $y=z^{-\frac{1}{2}}$ and $\dfrac{\mathrm{d}y}{\mathrm{d}x}=-\dfrac{1}{2}z^{-\frac{3}{2}}\dfrac{\mathrm{d}z}{\mathrm{d}x}$

Find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ in terms of $\dfrac{\mathrm{d}z}{\mathrm{d}x}$ and z

$$\frac{\mathrm{d}y}{\mathrm{d}x}+\left(\frac{1}{2}\tan x\right)y=-(2\sec x)y^3$$

So $\Rightarrow -\dfrac{1}{2}z^{-\frac{3}{2}}\dfrac{\mathrm{d}z}{\mathrm{d}x}+\left(\dfrac{1}{2}\tan x\right)z^{-\frac{1}{2}}=2\sec xz^{-\frac{3}{2}}$

$$\therefore \frac{\mathrm{d}z}{\mathrm{d}x}-z\tan x=4\sec x$$

2 b We wish to solve

$$\frac{dz}{dx} - z\tan x = 4\sec x \quad *$$

This is a first order equation which can be solved by using an integrating factor.

The integrating factor is $e^{-\int \tan x \, dx} = e^{\ln \cos x}$

$= \cos x$

The equation that you obtain needs an integrating factor to solve it.

Multiply the equation $*$ by $\cos x$

Then $\cos x \times \frac{dz}{dx} - z\sin x = 4$

$\therefore \quad \frac{d}{dx}(z\cos x) = 4$

$\therefore \quad z\cos x = \int 4\,dx$

$= 4x + c$

$z = \frac{4x+c}{\cos x}$

As $y = z^{-\frac{1}{2}}$, $y = \sqrt{\frac{\cos x}{4x+c}}$

3 a Given that $z = x^{\frac{1}{2}}$, $x = z^2$ and $\frac{dx}{dt} = 2z\frac{dz}{dt}$

So the equation $\frac{dx}{dt} + t^2 x = t^2 x^{\frac{1}{2}}$ becomes

$$2z\frac{dz}{dt} + t^2 z^2 = t^2 z$$

Divide through by $2z$: $\frac{dz}{dt} + \frac{1}{2}t^2 z = \frac{1}{2}t^2$

3 b We wish to solve

$$\frac{dz}{dt}+\frac{1}{2}t^2z=\frac{1}{2}t^2$$

The integrating factor is $e^{\int \frac{1}{2}t^2\,dt}=e^{\frac{1}{6}t^3}$

$$\therefore\ e^{\frac{1}{6}t^3}\frac{dz}{dt}+\frac{1}{2}t^2e^{\frac{1}{6}t^3}z=\frac{1}{2}t^2e^{\frac{1}{6}t^3}$$

$$\therefore\ \frac{d}{dt}\left(ze^{\frac{1}{6}t^3}\right)=\frac{1}{2}t^2e^{\frac{1}{6}t^3}$$

$$\therefore\ ze^{\frac{1}{6}t^3}=\int\frac{1}{2}t^2e^{\frac{1}{6}t^3}\,dt$$

$$=e^{\frac{1}{6}t^3}+c$$

$$\therefore\ z=1+ce^{-\frac{1}{6}t^3}$$

But $x=z^2$ $\therefore$ $x=\left(1+ce^{-\frac{1}{6}t^3}\right)^2$

4 a Let $z=y^{-1}$, then $y=z^{-1}$ and $\frac{dy}{dx}=-z^{-2}\frac{dz}{dx}$

So $\frac{dy}{dx}-\frac{1}{x}y=\frac{(x+1)^3}{x}y^2$ becomes

$$-z^{-2}\frac{dz}{dx}-\frac{1}{x}z^{-1}=\frac{(x+1)^3}{x}z^{-2}$$

Multiply through by $-z^2$: $\frac{dz}{dx}+\frac{1}{x}z=-\frac{(x+1)^3}{x}$

4 b We wish to solve

$$\frac{dz}{dx}+\frac{1}{x}z=-\frac{(x+1)^3}{x}$$

The integrating factor is $e^{\int P\,dx}=e^{\int \frac{1}{x}dx}=e^{\int \ln x}=x$

$$\therefore \quad x\frac{dz}{dx}+z=-(x+1)^3$$

$$\text{i.e. } \frac{d}{dx}(xz)=-(x+1)^3$$

$$\therefore \quad xz=-\int(x+1)^3dx$$

$$=-\frac{1}{4}(x+1)^4+c$$

$$\therefore \quad z=-\frac{1}{4x}(x+1)^4+\frac{c}{x}$$

$$\therefore \quad y=\frac{4x}{4c-(x+1)^4}$$

5 a Given that $z=y^2$, and so $y=z^{\frac{1}{2}}$ and $\frac{dy}{dx}=\frac{1}{2}z^{-\frac{1}{2}}\frac{dz}{dx}$

The equation $2(1+x^2)\frac{dy}{dx}+2xy=\frac{1}{y}$ becomes

$$2(1+x^2)\times\frac{1}{2}z^{-\frac{1}{2}}\frac{dz}{dx}+2xz^{\frac{1}{2}}=z^{-\frac{1}{2}}$$

Multiply the equation by $\frac{z^{\frac{1}{2}}}{1+x^2}$

Then $\frac{dz}{dx}+\frac{2x}{1+x^2}z=\frac{1}{1+x^2}$

5 b The integrating factor is $e^{\int \frac{2x}{1+x^2}dx} = e^{\ln(1+x^2)} = 1+x^2$

$$\therefore (1+x^2)\frac{dz}{dx} + 2xz = 1$$

$$\therefore \quad \frac{d}{dx}[(1+x^2)z] = 1$$

$$\therefore \quad (1+x^2)z = \int 1\, dx$$

$$= x + c$$

$$\therefore \quad z = \frac{x+c}{(1+x^2)}$$

As $y = z^{\frac{1}{2}}$, $\quad y = \sqrt{\frac{x+c}{(1+x^2)}}$

c When $x = 0,\ y = 2$ $\quad \therefore 2 = \sqrt{c} \Rightarrow c = 4$

$$\therefore y = \sqrt{\frac{x+4}{1+x^2}}$$

6 Given $z = y^{-(n-1)}$

$$\therefore \quad y = z^{-\frac{1}{(n-1)}}$$

$$\frac{dy}{dx} = \frac{-1}{n-1} z^{-\frac{1}{n-1}-1}\frac{dz}{dx}$$

$$= \frac{-1}{n-1} z^{-\frac{n}{n-1}}\frac{dz}{dx}$$

$\therefore \frac{dy}{dx} + Py = Qy^n$ becomes

$$\frac{-1}{n-1} z^{-\frac{n}{n-1}}\frac{dz}{dx} + P z^{-\frac{1}{n-1}} = Q z^{-\frac{n}{n-1}}$$

Multiply each term by $-(n-1)z^{\frac{n}{n-1}}$

Then $\frac{dz}{dx} - P(n-1)z^{\frac{n}{n-1}} z^{-\frac{1}{n-1}} = -Q(n-1)z^{\frac{n}{n-1}} z^{-\frac{n}{n-1}}$

i.e. $$\frac{dz}{dz} - P(n-1)z = -Q(n-1)$$

7 a Given $u = y + 2x$ and so $y = u - 2x$ and $\frac{dy}{dx} = \frac{du}{dx} - 2$

$\therefore$ the differential equation $\frac{dy}{dx} = -\frac{(1+2y+4x)}{1+y+2x}$ becomes

Rearrange the given substitution to give y in terms of u and x, and $\frac{dy}{dx}$ in terms of $\frac{du}{dx}$

$$\frac{du}{dx} - 2 = -\frac{1+2u}{1+u}$$

$$\therefore \quad \frac{du}{dx} = \frac{-(1+2u)+2(1+u)}{1+u}$$

$$\therefore \quad \frac{du}{dx} = \frac{1}{1+u}$$

b Separate the variables

$$\int (1+u)\,du = \int 1 \times dx$$

$$\therefore \quad u + \frac{u^2}{2} = x + c, \text{ where } c \text{ is constant}$$

And $\quad (y+2x) + \frac{(y+2x)^2}{2} = x + c$

$$2y + 4x + y^2 + 4xy + 4x^2 = 2x + 2c$$

i.e. $4x^2 + 4xy + y^2 + 2y + 2x = k$, where $k = 2c$

Challenge

Substitute $y = \frac{1}{v}, \frac{dy}{dx} = -\frac{1}{v^2}\frac{dv}{dx}$

Differential equation becomes

$$x^2\left(-\frac{1}{v^2}\frac{dv}{dx}\right) - \frac{x}{v} = \frac{1}{v^2}$$

$$\Rightarrow x\frac{dv}{dx} + v = -\frac{1}{x}$$

Integrate both sides to get $xv = -\ln x + C$

Substitute $v = \frac{1}{y}$ to get $y = \frac{-x}{\text{In}x + C}$