Reducible differential equations 9B

First express $x\frac{dy}{dx}$ as $\frac{dy}{du}$ and $x\frac{d^2y}{dx^2}$ as $\frac{d^2y}{du^2}-\frac{dy}{du}$

1 a $x^2\frac{d^2y}{dx^2}+6x\frac{dy}{dx}+4y=0$ $*$

As $x=e^u$, $\frac{dx}{du}=e^u=x$

Form the chain rule $\frac{dy}{du}=\frac{dy}{dx}\times\frac{dx}{du}$

$\therefore \quad \frac{dy}{du}=x\frac{dy}{dx}$ **(1)**

Also
$$\frac{d^2y}{du^2}=\frac{d}{du}\left(x\frac{dy}{dx}\right)$$
$$=\frac{dx}{du}\times\frac{dy}{dx}+x\frac{d^2y}{dx^2}\times\frac{dx}{du}$$
$$=\frac{dy}{du}+x^2\frac{d^2y}{dx^2}$$

$\therefore \quad x^2\frac{d^2y}{dx^2}=\frac{d^2y}{du^2}-\frac{dy}{du}$ **(2)**

Use the results **(1)** and **(2)** to change the variable in $*$

$$\therefore \frac{d^2y}{du^2}-\frac{dy}{du}+6\frac{dy}{du}+4y=0$$
i.e. $\frac{d^2y}{du^2}+5\frac{dy}{du}+4y=0$

This has auxiliary equation

$$m^2+5m+4=0$$
$\therefore (m+4)(m+1)=0$
i.e. $m=-4$ or -1

$\therefore$ The solution of the differential equation † is

$$y=Ae^{-4u}+Be^{-u}$$

But $e^u=x$

$\therefore \quad e^{-u}=x^{-1}=\frac{1}{x}$

and $e^{-4u}=x^{-4}=\frac{1}{x^4}$

$\therefore \quad y=\frac{A}{x^4}+\frac{B}{x}$

1 b $x^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+5x\frac{\mathrm{d}y}{\mathrm{d}x}+4y=0$ *

As $x=\mathrm{e}^u$, $x\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\mathrm{d}y}{\mathrm{d}u}$ and $x^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=\frac{\mathrm{d}^2y}{\mathrm{d}u^2}-\frac{\mathrm{d}y}{\mathrm{d}u}$

(See solution to question 1 for proof this.)

Use these results to change the variable in *

$\therefore \quad \frac{\mathrm{d}^2y}{\mathrm{d}u^2}-\frac{\mathrm{d}y}{\mathrm{d}u}+5\frac{\mathrm{d}y}{\mathrm{d}u}+4y=-0.$

$\therefore \quad \frac{\mathrm{d}^2y}{\mathrm{d}u^2}+4\frac{\mathrm{d}y}{\mathrm{d}u}+4y=0 \quad \dagger$

This has auxiliary equation

$m^2+4m+4=0$

$\therefore \quad (m+2)^2=0$

$\therefore \quad m=-2$ only

The solution of the differential equation † is thus

$$y=(A+Bu)\mathrm{e}^{-2u}$$

As $x=\mathrm{e}^u$ $\therefore$ $\mathrm{e}^{-2u}=x^{-2}=\frac{1}{x^2}$

and $\quad u=\ln x$

$\therefore \quad y=(A+B\ln x)\times\frac{1}{x^2}$

Use $x\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\mathrm{d}y}{\mathrm{d}u}$ and $x^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=\frac{\mathrm{d}^2y}{\mathrm{d}u^2}-\frac{\mathrm{d}y}{\mathrm{d}u}$

Ensure that you can prove these two results.

1 c $x^2\frac{d^2y}{dx^2}+6x\frac{dy}{dx}+6y=0$ *

As $x=e^u$, $x\frac{dy}{dx}=\frac{dy}{du}$ and $x^2\frac{d^2y}{dx^2}=\frac{d^2y}{du^2}-\frac{dy}{du}$

(See solution to question 1 for proof of this.)

Use these results to change the variable in *

$\therefore \frac{d^2y}{du^2}-\frac{dy}{du}+6\frac{dy}{du}+6y=0$

$\therefore \quad \frac{d^2y}{du^2}+5\frac{dy}{du}+6y=0 \quad \dagger$

This has auxiliary equation

$m^2+5m+6=0$

$\therefore \quad (m+2)(m+3)=0$

$\therefore \quad m=-2 \text{ or } -3$

The solution of the differential equation † is thus

$y=Ae^{-2u}+Be^{-3u}$

As $x=e^u$, $e^{-2u}=x^{-2}=\frac{1}{x^2}$

and $e^{-3u}=x^{-3}=\frac{1}{x^3}$

$\therefore \quad y=\frac{A}{x^2}+\frac{B}{x^3}$

Use $x\frac{dy}{dx}=\frac{dy}{du}$ and $x^2\frac{d^2y}{dx^2}=\frac{d^2y}{du^2}-\frac{dy}{du}$

d $x^2\frac{d^2y}{dx^2}+4\frac{dy}{dx}-28y=0$ *

As $x=e^u$, $x\frac{dy}{dx}=\frac{dy}{du}$ and $x^2\frac{d^2y}{dx^2}=\frac{d^2y}{du^2}-\frac{dy}{du}$

Substitute these results into equation *

$\therefore \frac{d^2y}{du^2}-\frac{dy}{du}+4\frac{dy}{du}-28y=0$

$\therefore \quad \frac{d^2y}{du^2}+3\frac{dy}{du}-28y=0 \quad \dagger$

This has auxiliary equation:

$m^2+3m-28=0$

$\therefore \quad (m+7)(m-4)=0$

$\therefore \quad m=-7 \text{ or } 4$

$\therefore y=Ae^{-7u}+Be^{4u}$ is the solution to †

As $x=e^u$, $\therefore \; e^{-7u}=\frac{1}{x^7}$

and $e^{4u}=x^4$

$\therefore \quad y=\frac{A}{x^7}+Bx^4$

Use $x\frac{dy}{dx}=\frac{dy}{du}$ and $x^2\frac{d^2y}{dx^2}=\frac{d^2y}{du^2}-\frac{dy}{du}$

1 e $x^2\frac{d^2y}{dx^2}-4x\frac{dy}{dx}-14y=0$ *

Use $x\frac{dy}{dx}=\frac{dy}{du}$ and $x^2\frac{d^2y}{dx^2}=\frac{d^2y}{du^2}-\frac{dy}{du}$

As $x=e^u$, $x\frac{dy}{dx}=\frac{dy}{du}$ and $x^2\frac{d^2y}{dx^2}=\frac{d^2y}{du^2}-\frac{dy}{du}$

Substituting these results into $*$ gives

$$\frac{d^2y}{du^2}-\frac{dy}{du}-4\frac{dy}{du}-14y=0$$

i.e. $\frac{d^2y}{du^2}-5\frac{dy}{du}-14y=0$ †

This has auxiliary equation:

$$m^2-5m-14=0$$

i.e. $(m-7)(m+2)=0$

$\therefore$ $m=7$ or -2

$\therefore$ The solution of the differential equation † is

$$y=Ae^{7u}+Be^{-2u}$$

But $x=e^u$, $\therefore$ $e^{7u}=x^7$

and $e^{-2u}=x^{-2}=\frac{1}{x^2}$

$\therefore$ $y=Ax^7+\frac{B}{x^2}$

f $x^2\frac{d^2y}{dx^2}+3x\frac{dy}{dx}+2y=0$ *

Use $x\frac{dy}{dx}=\frac{dy}{du}$ and $x^2\frac{d^2y}{dx^2}=\frac{d^2y}{du^2}-\frac{dy}{du}$

A proof of these results is given in the book in Section 5.6

As $x=e^u$, $x\frac{dy}{dx}=\frac{dy}{du}$ and $x^2\frac{d^2y}{dx^2}=\frac{d^2y}{du^2}-\frac{dy}{du}$

Substitute these results into $*$ to give:

$$\frac{d^2y}{du^2}-\frac{dy}{du}+3\frac{dy}{du}+2y=0$$

i.e. $\frac{d^2y}{du^2}+2\frac{dy}{du}+2y=0$ †

This has auxiliary equation:

$$m^2+2m+2=0$$

$\therefore$ $m=\frac{-2\pm\sqrt{4-8}}{2}$

$=-1\pm i$

The solution of the differential equation † is thus

$$y=e^{-u}[A\cos u+B\sin u]$$

As $x=e^u$, $e^{-u}=x^{-1}=\frac{1}{x}$

and $u=\ln x$

$\therefore$ $y=\frac{1}{x}[A\cos\ln x+B\sin\ln x]$

2 a $y = \frac{z}{x}$ implies $xy = z$

$\therefore \quad x\frac{\mathrm{d}y}{\mathrm{d}x} + y = \frac{\mathrm{d}z}{\mathrm{d}x}$

Also $x\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + \frac{\mathrm{d}y}{\mathrm{d}x} + \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}^2z}{\mathrm{d}x^2}$

$\therefore$ The equation $x\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + (2-4x)\frac{\mathrm{d}y}{\mathrm{d}x} - 4y = 0$

becomes $\frac{\mathrm{d}^2z}{\mathrm{d}x^2} - 4\left(\frac{\mathrm{d}z}{\mathrm{d}x} - y\right) - 4y = 0$

i.e. $\frac{\mathrm{d}^2z}{\mathrm{d}x^2} - 4\frac{\mathrm{d}z}{\mathrm{d}x} = 0 \quad *$

as required.

Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ and $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ in terms of $\frac{\mathrm{d}z}{\mathrm{d}x}$ and $\frac{\mathrm{d}^2z}{\mathrm{d}x^2}$

b $m^2 - 4m = 0$

$\Rightarrow m(m-4) = 0$

$\Rightarrow m = 0$ or 4

So general solution is

$z = A + Be^{4x}$

c $yx = A + Be^{4x}$

$\Rightarrow y = \frac{A}{x} + \frac{B}{x}e^{4x}$

3 a $y = \frac{z}{x^2}$ implies $z = yx^2$ or $x^2y = z$

$\therefore \quad x^2\frac{\mathrm{d}y}{\mathrm{d}x} + 2xy = \frac{\mathrm{d}z}{\mathrm{d}x} \qquad$ **(1)**

Also $x^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 2x\frac{\mathrm{d}y}{\mathrm{d}x} + 2x\frac{\mathrm{d}y}{\mathrm{d}x} + 2y = \frac{\mathrm{d}^2z}{\mathrm{d}x^2} \qquad$ **(2)**

Express $\frac{\mathrm{d}z}{\mathrm{d}x}$ and $\frac{\mathrm{d}^2z}{\mathrm{d}x^2}$ in terms of $\frac{\mathrm{d}y}{\mathrm{d}x}$ and $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ respectively.

The differential equation:

$x^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 2x(x+2)\frac{\mathrm{d}y}{\mathrm{d}x} + 2(x+1)^2y = \mathrm{e}^{-x}$ can be written

$\left(x^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 4x\frac{\mathrm{d}y}{\mathrm{d}x} + 2y\right) + \left(2x^2\frac{\mathrm{d}y}{\mathrm{d}x} + 4xy\right) + 2x^2y = \mathrm{e}^{-x}$

Using the results **(1)** and **(2)**

$\frac{\mathrm{d}^2z}{\mathrm{d}x^2} + 2\frac{\mathrm{d}z}{\mathrm{d}x} + 2z = \mathrm{e}^{-x} \quad \dagger$

as required.

3 b This has auxiliary equation as

$m^2+2m+2=0$

$\therefore \quad m=\dfrac{-2\pm\sqrt{4-8}}{2}$

$m=-1\pm\mathrm{i}$

$\therefore\ z=\mathrm{e}^{-x}(A\cos x+B\sin x)$ is the complementary function

A particular integral of † is $z=\lambda\mathrm{e}^{-x}$

$\therefore\ \dfrac{\mathrm{d}z}{\mathrm{d}x}=-\lambda\mathrm{e}^{-x}$ and $\dfrac{\mathrm{d}^2z}{\mathrm{d}x^2}=\lambda\mathrm{e}^{-x}$

Substituting into †

$(\lambda-2\lambda+2\lambda)\mathrm{e}^{-x}=\mathrm{e}^{-x}$

$\therefore \quad \lambda=1$

So $z=\mathrm{e}^{-x}$ is a particular integral.

$\therefore$ The general solution of † is

$z=\mathrm{e}^{-x}(A\cos x+B\sin x+1)$

c Now $z=x^2y$

$\therefore\ y=\dfrac{\mathrm{e}^{-x}}{x^2}(A\cos x+B\sin x+1)$ is the general solution of the given differential equation.

4 a $z=\sin x$ implies $\dfrac{\mathrm{d}z}{\mathrm{d}x}=\cos x$

Find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ in terms of $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ and find $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}$ in terms of $\dfrac{\mathrm{d}^2y}{\mathrm{d}z^2}$ and $\dfrac{\mathrm{d}y}{\mathrm{d}z}$

$\therefore\ \dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{\mathrm{d}y}{\mathrm{d}z}\times\cos x$

and $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}=\dfrac{\mathrm{d}^2y}{\mathrm{d}z^2}\cos^2x-\dfrac{\mathrm{d}y}{\mathrm{d}z}\sin x$

$\therefore$ The equation $\cos x\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}+\sin x\dfrac{\mathrm{d}y}{\mathrm{d}x}-2y\cos^3x=2\cos^5x$

becomes $\cos^3x\dfrac{\mathrm{d}^2y}{\mathrm{d}z^2}-\cos x\sin x\dfrac{\mathrm{d}y}{\mathrm{d}z}+\cos x\sin x\dfrac{\mathrm{d}y}{\mathrm{d}z}-2y\cos^3x=2\cos^5x$

$\therefore$ Divide by $\cos^3x$ gives:

$\dfrac{\mathrm{d}^2y}{\mathrm{d}z^2}-2y=2\cos^2x$

$=2(1-z^2)$ † $\quad$ [as $\cos^2x=1-\sin^2x=1-z^2$]

4 b First solve $\frac{d^2y}{dz^2} - 2y = 0$

This has auxiliary equation

$m^2 - 2 = 0$

$\therefore \quad m = \pm\sqrt{2}$

$\therefore$ The complementary function is $y = Ae^{\sqrt{2}z} + Be^{-\sqrt{2}z}$

Let $y = \lambda z^2 + \mu z + v$ be a particular integral of the differential equation †

Then $\frac{dy}{dz} = 2\lambda z + \mu$ and $\frac{d^2y}{dz^2} = 2\lambda$

Substitute into †

Then $2\lambda - 2(\lambda z^2 + \mu z + v) = 2(1 - z^2)$

Compare coefficients of z^2: $-2\lambda = -2 \quad \therefore \quad \lambda = 1$

Compare coefficients of z: $-2\mu = 0 \quad \therefore \quad \mu = 0$

Compare constants: $\quad 2\lambda - 2v = 2 \quad \therefore \quad v = 0$

$\therefore z^2$ is the particular integral.

$\therefore$ The general solution of † is

$y = Ae^{\sqrt{2}z} + Be^{-\sqrt{2}z} + z^2.$

But $z = \sin x$

$\therefore \quad y = Ae^{\sqrt{2}\sin x} + Be^{-\sqrt{2}\sin x} + \sin^2 x$

5 a $x = ut, \frac{dx}{dt} = u + t\frac{du}{dt}, \frac{d^2x}{dt^2} = 2\frac{du}{dt} + t\frac{d^2u}{dt^2}$

So the differential equation becomes

$$t^2\left(2\frac{du}{dt} + t\frac{d^2u}{dt^2}\right) - 2t\left(u + t\frac{du}{dt}\right) = -2\left(1 - 2t^2\right)ut$$

which rearranges to give
$$t^3\left(\frac{d^2u}{dt^2} - 4u\right) = 0$$
$$\Rightarrow \frac{d^2u}{dt^2} - 4u = 0$$

b $m^2 - 4 = 0$

$\Rightarrow m = \pm 2$

So general solution is

$u = Ae^{2t} + Be^{-2t}$

$\Rightarrow x = t\left(Ae^{2t} + Be^{-2t}\right)$

5 c $\frac{dx}{dt} = Ae^{2t} + Be^{-2t} + t\left(2Ae^{2t} + -2Be^{-2t}\right)$

$x = 2$ at $t = 1 \Rightarrow Ae^2 + Be^{-2} = 2$

$\frac{dx}{dt} = 1$ at $t = 1 \Rightarrow Ae^2 + Be^{-2} + 2Ae^2 - 2Be^{-2} = 1$

$\Rightarrow 3Ae^2 - Be^{-2} = 1$

Adding the equations we obtain

$4Ae^2 = 3$

$\Rightarrow A = \frac{3}{4e^2}$

and then $B = \frac{5}{4e^{-2}}$

so the particular solution is

$x = t\left(\frac{3}{4e^2}e^{2t} + \frac{5}{4e^{-2}}e^{-2t}\right)$

Challenge

$u = \frac{dy}{dx} \Rightarrow \frac{du}{dx} = \frac{d^2y}{dx^2}$

So the equation becomes

$x\frac{du}{dx} + u = 12x$

$\Rightarrow \frac{du}{dx} + \frac{1}{x}u = 12$

So the integrating factor is $e^{\int \frac{1}{x}dx} = e^{\ln x} = x$

This gives $\frac{d}{dx}(xu) = 12x$

$\Rightarrow xu = 6x^2 + A$

$\Rightarrow u = 6x + \frac{A}{x}$

$\Rightarrow y = 3x^2 + A\ln x + B$