Review Exercise 1

1 $(-\mathbf{i}-\mathbf{j}+\mathbf{k})\times(-\mathbf{i}+\mathbf{j}-\mathbf{k})$

$$=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ -1 & -1 & 1\\ -1 & 1 & -1\end{vmatrix}$$

$$=\begin{vmatrix}-1 & 1\\ 1 & -1\end{vmatrix}\mathbf{i}-\begin{vmatrix}-1 & 1\\ -1 & -1\end{vmatrix}\mathbf{j}+\begin{vmatrix}-1 & -1\\ -1 & 1\end{vmatrix}\mathbf{k}$$

$$=\left((-1\times-1)-(1\times1)\right)\mathbf{i}-\left((-1\times-1)-(-1\times1)\right)\mathbf{j}+\left((-1\times1)-(-1\times-1)\right)\mathbf{k}$$

$$=(1-1)\mathbf{i}-(1-(-1))\mathbf{j}+(-1-1)\mathbf{k}=-2\mathbf{j}-2\mathbf{k}$$

Formulae for finding the vector product are given in the Edexcel formulae booklet which is provided for the examination. One form it gives is $\mathbf{a}\times\mathbf{b}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix}$ and that has been used here.

Hence

$$|(-\mathbf{i}-\mathbf{j}+\mathbf{k})\times(-\mathbf{i}+\mathbf{j}-\mathbf{k})|=\sqrt{\left((-2)^2+(-2)^2\right)}$$

$$=\sqrt{8}=2\sqrt{2}$$

You use the formula for the magnitude of a vector $|x\mathbf{i}+y\mathbf{j}+z\mathbf{k}|=\sqrt{(x^2+y^2+z^2)}$

2 a We do this with the determinant method: $\begin{vmatrix}\mathrm{i} & \mathrm{j} & \mathrm{k}\\ 2 & -1 & 3\\ k & 1 & 0\end{vmatrix}=\begin{pmatrix}0-3\\ 0+3k\\ 2+k\end{pmatrix}=\begin{pmatrix}-3\\ 3k\\ 2+k\end{pmatrix}$

b The magnitude of the vector product is given by $\sqrt{3^2+(3k)^2+(2+k)^2}$

Then, in order to minimise it, we need to minimise $9+9k^2+4+4k+k^2=10k^2+4k+13$

In quadratic expressions of the form ak^2+bk+c the minimum is realised by $k=-\frac{b}{2a}$, so in this case the least possible magnitude is obtained when $k=-\frac{1}{5}$

In particular, the value of this magnitude is $\sqrt{13-\frac{10}{25}}=\sqrt{\frac{315}{25}}=\frac{3\sqrt{35}}{5}$

3 $\overrightarrow{AB} = \mathbf{b} - \mathbf{a}, \quad \overrightarrow{AC} = \mathbf{c} - \mathbf{a}$

$\overrightarrow{AB} \times \overrightarrow{AC} = (\mathbf{b} - \mathbf{a}) \times (\mathbf{c} - \mathbf{a})$

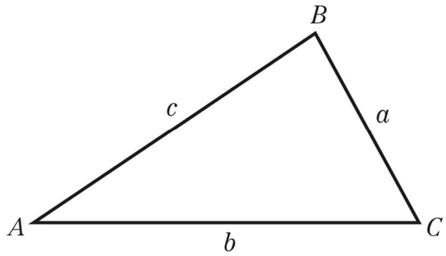
$= \mathbf{b} \times \mathbf{c} - \mathbf{b} \times \mathbf{a} - \mathbf{a} \times \mathbf{c} + \mathbf{a} \times \mathbf{a}$

As $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$, $\mathbf{c} \times \mathbf{a} = -\mathbf{a} \times \mathbf{c}$ and $\mathbf{a} \times \mathbf{a} = 0$

$\overrightarrow{AB} \times \overrightarrow{AC} = \mathbf{b} \times \mathbf{c} + \mathbf{a} \times \mathbf{b} + \mathbf{c} \times \mathbf{a}$

$= \mathbf{a} \times \mathbf{b} + \mathbf{b} \times \mathbf{c} + \mathbf{c} \times \mathbf{a}$

You multiply out the brackets using the usual rules of algebra. You must take care with the order in which the vectors are multiplied as the vector product is not commutative. For a vector product $\mathbf{x} \times \mathbf{y} = -\mathbf{y} \times \mathbf{x}$.



The area of the triangle, Δ, say, is given by

$\Delta = \frac{1}{2} bc \sin A$

$= \frac{1}{2} AC \times AB \sin A$

$= \frac{1}{2} \left| \overrightarrow{AB} \times \overrightarrow{AC} \right|$

$= \frac{1}{2} \left| \mathbf{a} \times \mathbf{b} + \mathbf{b} \times \mathbf{c} + \mathbf{c} \times \mathbf{a} \right|$, as required.

The magnitude of the vector product $\mathbf{a} \times \mathbf{b}$ is $|\mathbf{a}||\mathbf{b}| \sin \theta$, where θ is the angle between the vectors. The vector product can be interpreted as a vector with magnitude twice the area of the triangle which has the vectors as two of its sides.

4 a $\overrightarrow{AB} = (5\mathbf{i} + 8\mathbf{j} + 2\mathbf{k}) - (2\mathbf{i} + 7\mathbf{j} - \mathbf{k})$

$= 3\mathbf{i} + \mathbf{j} + 3\mathbf{k}$

$\overrightarrow{AC} = (6\mathbf{i} + 7\mathbf{j} + 4\mathbf{k}) - (2\mathbf{i} + 7\mathbf{j} - \mathbf{k})$

$= 4\mathbf{i} + 5\mathbf{k}$

$\overrightarrow{AB} \times \overrightarrow{AC} = (3\mathbf{i} + \mathbf{j} + 3\mathbf{k}) \times (4\mathbf{i} + 5\mathbf{k})$

$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 1 & 3 \\ 4 & 0 & 5 \end{vmatrix}$

$= \begin{vmatrix} 1 & 3 \\ 0 & 5 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 3 & 3 \\ 4 & 5 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 3 & 1 \\ 4 & 0 \end{vmatrix} \mathbf{k}$

$= 5\mathbf{i} - 3\mathbf{j} - 4\mathbf{k}$

$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$
It is important to get the vectors the right way round. It is a common error to use $\overrightarrow{AB} = \overrightarrow{OA} - \overrightarrow{OB}$ and obtain the negative of the correct answer.

b $\overrightarrow{AD} = (12\mathbf{i} + \mathbf{j} - 9\mathbf{k}) - (2\mathbf{i} + 7\mathbf{j} - \mathbf{k})$

$= 10\mathbf{i} - 6\mathbf{j} - 8\mathbf{k}$

$\overrightarrow{AD} \cdot (\overrightarrow{AB} \times \overrightarrow{AC}) = (10\mathbf{i} - 6\mathbf{j} - 8\mathbf{k}) \cdot (5\mathbf{i} - 3\mathbf{j} - 4\mathbf{k})$

$= 10 \times 5 + (-6) \times (-3) + (-8) \times (-4)$

$= 50 + 18 + 32 = 100$

$10\mathbf{i} - 6\mathbf{j} - 9\mathbf{k} = 2(5\mathbf{i} - 3\mathbf{j} - 4\mathbf{k})$ so $\overrightarrow{AD}$ and $\overrightarrow{AB} \times \overrightarrow{AC}$ are parallel. As the vector product is perpendicular to *AB* and *AC*, it follows that the line *AD* is perpendicular to the plane of the triangle *ABC*.

4 c The volume of the prism, P say, is given by

$P = \frac{1}{2}\overrightarrow{AD}\cdot(\overrightarrow{AB}\times\overrightarrow{AC}) = \frac{1}{2}\times 100 = 50$

In this case, the volume of the prism is the area of the triangle ABC, which is half the magnitude of $\overrightarrow{AB}\times\overrightarrow{AC}$, multiplied by the distance AD. (Even if the line AD is not perpendicular to the plane of the triangle ABC, the triple scalar product is still twice the volume of the prism.)

5 a $\overrightarrow{AC} = \begin{pmatrix}6\\4\\5\end{pmatrix} - \begin{pmatrix}3\\1\\2\end{pmatrix} = \begin{pmatrix}3\\3\\3\end{pmatrix}$

$\overrightarrow{AD} = \begin{pmatrix}-7\\6\\-3\end{pmatrix} - \begin{pmatrix}3\\1\\2\end{pmatrix} = \begin{pmatrix}-10\\5\\-5\end{pmatrix}$

$\overrightarrow{AC}\times\overrightarrow{AD} = \begin{pmatrix}3\\3\\3\end{pmatrix}\times\begin{pmatrix}-10\\5\\-5\end{pmatrix} = \begin{pmatrix}3\times(-5)-3\times 5\\3\times(-10)-3\times(-5)\\3\times 5-3\times(-10)\end{pmatrix}$

$= \begin{pmatrix}-30\\-15\\45\end{pmatrix}$

For writing vectors, you can use either the form with **i**s, **j**s and **k**s, or column vectors, which are used in this solution. Sometimes it may even be appropriate to use a mixture of the two. The form using **i**, **j** and **k** usually gives a more compact solution but many find column vectors quicker to write. The choice is entirely up to you and you may choose to vary it from question to question.

b $\mathbf{r} = \begin{pmatrix}3\\1\\2\end{pmatrix} + \lambda\begin{pmatrix}-2\\-1\\3\end{pmatrix}$

The vector $\begin{pmatrix}-30\\-15\\45\end{pmatrix}$ is perpendicular to both $\overrightarrow{AC}$ and $\overrightarrow{AD}$

This vector or any multiple of it may be used for the equation of the line.

c For B to lie on the line there must be a value of λ for which

$\begin{pmatrix}5\\2\\-1\end{pmatrix} = \begin{pmatrix}3\\1\\2\end{pmatrix} + \lambda\begin{pmatrix}-2\\-1\\3\end{pmatrix}$

Equating the x components of the vectors

$5 = 3 - 2\lambda \Rightarrow \lambda = -1$

Checking this value of λ for the other components

y component:

$1 + \lambda\times(-1) = 1 + (-1)\times(-1) = 2$, as required.

z component:

$2 + \lambda\times 3 = 2 + (-1)\times 3 = -1$, as required.

Hence, B lies on the line.

5 d $\overrightarrow{AB} = \begin{pmatrix} 5 \\ 2 \\ -1 \end{pmatrix} - \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix}$

$$\overrightarrow{AB} \cdot (\overrightarrow{AC} \times \overrightarrow{AD}) = \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix} \cdot \begin{pmatrix} -30 \\ -15 \\ 45 \end{pmatrix} = 2 \times (-30) + 1 \times (-15) + (-3) \times 45$$

$$= -60 - 15 - 135 = -210$$

The volume of the tetrahedron, V say, is given by

$$V = \frac{1}{6}\left|\overrightarrow{AB} \cdot (\overrightarrow{AC} \times \overrightarrow{AD})\right| = \frac{1}{6}|-210| = \frac{1}{6} \times 210 = 35$$

The volume of the tetrahedron is one sixth of the triple scalar product.

6 a $\overrightarrow{AB} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} - \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \\ 3 \end{pmatrix}$

$$\overrightarrow{AC} = \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix} - \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 4 \\ 2 \end{pmatrix}$$

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{pmatrix} -1 \\ 3 \\ 3 \end{pmatrix} \times \begin{pmatrix} 0 \\ 4 \\ 2 \end{pmatrix} = \begin{pmatrix} -6 \\ 2 \\ -4 \end{pmatrix}$$

b A vector equation of Π is $\mathbf{r} \cdot \begin{pmatrix} -6 \\ 2 \\ -4 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} -6 \\ 2 \\ -4 \end{pmatrix} = -12 - 2 = -14$

Once you have a vector $\mathbf{n}$ perpendicular to the plane, you can find a vector equation of the plane using $\mathbf{r.n} = \mathbf{a.n}$, where $\mathbf{a}$ is the position vector of any point on the plane. Here the position vector of A has been used but the position vectors of B and C would do just as well. As the scalar product is quite quickly worked out, it is a useful check to recalculate, using one of the other points. All should give the same answer, here –14

Let $\mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$$\mathbf{r} \cdot \begin{pmatrix} -6 \\ 2 \\ -4 \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} -6 \\ 2 \\ -4 \end{pmatrix} = -6x + 2y - 4z = -14$$

A Cartesian equation of Π is

$-6x + 2y - 4z = -14$

Divide throughout by –2

$3x - y + 2z = 7$, as required.

6 c A vector equation of the line l is

$$\mathbf{r} = \begin{pmatrix} 5 \\ 5 \\ 3 \end{pmatrix} + t\begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix}$$

The two vector forms of a straight line $(\mathbf{r}-\mathbf{a})\times\mathbf{b}=0$ and $\mathbf{r}=\mathbf{a}+t\mathbf{b}$ are equivalent and you can always interchange one with the other. Here

$$\mathbf{a} = \begin{pmatrix} 5 \\ 5 \\ 3 \end{pmatrix} \text{ and } \mathbf{b} = \begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix}$$

Parametric equations of l are

$x = 5+2t,\ y = 5-t,\ z = 3-2t$

Substituting the parametric equations into

$$3x - y + 2z = 7$$
$$3(5+2t)-(5-t)+2(3-2t) = 7$$
$$15+6t-5+t+6-4t = 7$$
$$3t = -9 \Rightarrow t = -3$$

The coordinates of T are

$$(5+2t,\ 5-t,\ 3-2t) = (5-6,\ 5+3,\ 3+6)$$
$$= (-1,\ 8,\ 9\ )$$

d $\overrightarrow{BT} = \overrightarrow{OT} - \overrightarrow{OB} = \begin{pmatrix} -1 \\ 8 \\ 9 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} -2 \\ 6 \\ 6 \end{pmatrix}$

When A, B and T lie on the same straight line, AB and BT are in the same direction. So you show that the vectors $\overrightarrow{AB}$ and $\overrightarrow{BT}$ are parallel.

From part **a**

$$\overrightarrow{AB} = \begin{pmatrix} -1 \\ 3 \\ 3 \end{pmatrix}$$

Hence

$\overrightarrow{AB} = \frac{1}{2}\overrightarrow{BT}$ and AB is parallel to BT.

Hence A, B and T lie in the same straight line.

Points which lie on the same straight line are called **collinear** points.

7 a Equating the x components

$2t = 1 - 2u$ **(1)**

Equating the y components

$1 + t = 1 + u \Rightarrow t = u$ **(2)**

Substituting **(2)** into **(1)**

$2u = 1 - 2u \Rightarrow u = \frac{1}{4}$

As $t = u$, $t = \frac{1}{4}$

Checking the z components

For l: $3 - t = 3 - \frac{1}{4} = \frac{11}{4}$

For m: $-1 + u = -1 + \frac{1}{4} = -\frac{3}{4}$

$\frac{11}{4} \neq -\frac{3}{4}$, so the lines do not intersect.

To show that two lines do not intersect, you find the values of the two parameters, here t and u, which make two of the components equal and then you show that, with these values, the third components are not equal.

b $\overrightarrow{OA} = \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix} + t_1 \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$

$\overrightarrow{OB} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} + u_1 \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} + u_1 \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix} - t_1 \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 - 2t_1 - 2u_1 \\ -t_1 + u_1 \\ -4 + t_1 + u_1 \end{pmatrix}$$

$$= (1 - 2t_1 - 2u_1)\mathbf{i} + (-t_1 + u_1)\mathbf{j} + (-4 + t_1 + u_1)\mathbf{k}$$

7 c If $\overrightarrow{AB}$ is perpendicular to l

$$\begin{pmatrix} 1-2t_1-2u_1 \\ -t_1+u_1 \\ -4+t_1+u_1 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} = 0$$

$$2-4t_1-4u_1-t_1+u_1+4-t_1-u_1=0$$

$$6t_1+4u_1=6 \qquad \textbf{(3)}$$

As $\overrightarrow{AB}$ is perpendicular to l, the scalar product of $\overrightarrow{AB}$ with the direction of l, which is $\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$, is zero.

This gives one equation in t_1 and u_1

Carrying out the same process with the direction of m, gives you a second equation in t_1 and u_1

You solve these simultaneous equations for t_1 and u_1 and use these values to find $\overrightarrow{AB}$. The magnitude of this vector is the length you have been asked to find.

If $\overrightarrow{AB}$ is perpendicular to m

$$\begin{pmatrix} 1-2t_1-2u_1 \\ -t_1+u_1 \\ -4+t_1+u_1 \end{pmatrix} \cdot \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} = 0$$

$$-2+4t_1+4u_1-t_1+u_1-4+t_1+u_1=0$$

$$4t_1+6u_1=6 \qquad \textbf{(4)}$$

$3\times\textbf{(4)}-2\times\textbf{(3)}$

$$10u_1=6 \Rightarrow u_1=\frac{3}{5}$$

Substituting $u_1=\frac{3}{5}$ into **(4)**

$$4t_1+\frac{18}{5}=6 \Rightarrow t_1=\frac{6-\frac{18}{5}}{4}=\frac{3}{5}$$

$$\overrightarrow{AB} = \begin{pmatrix} 1-2t_1-2u_1 \\ -t_1+u_1 \\ -4+t_1+u_1 \end{pmatrix} = \begin{pmatrix} 1-\frac{6}{5}-\frac{6}{5} \\ -\frac{3}{5}+\frac{3}{5} \\ -4+\frac{3}{5}+\frac{3}{5} \end{pmatrix} = \begin{pmatrix} -\frac{7}{5} \\ 0 \\ -\frac{14}{5} \end{pmatrix}$$

$$\left|\overrightarrow{AB}\right|^2 = \left(-\frac{7}{5}\right)^2 + \left(-\frac{14}{5}\right)^2 = \frac{245}{25} = \frac{49}{5}$$

The length of AB is given by

$$\left|\overrightarrow{AB}\right| = \sqrt{\left(\frac{49}{5}\right)} = \frac{7}{\sqrt{5}}, \text{ as required.}$$

This length is the shortest distance between the two skew lines. This question illustrates one of the methods by which this shortest distance can be found.

8 **a** Since the direction vector of the line is given by $\begin{pmatrix} -1-2 \\ 3-5 \\ 2-0 \end{pmatrix} = \begin{pmatrix} -3 \\ -2 \\ 2 \end{pmatrix}$, whose magnitude is $\sqrt{17}$,

the direction cosines l, m and n are easily found as follows:

$l = -\frac{3}{\sqrt{17}}$
$m = -\frac{2}{\sqrt{17}}$
$n = \frac{2}{\sqrt{17}}$

b In order to find a Cartesian equation, we use the standard formula: it is $\frac{x-2}{-3} = \frac{y-5}{-2} = \frac{z}{2}$

9 **a**

$$\begin{aligned} \overrightarrow{AB} &= -\mathbf{i} + 2\mathbf{j} - (3\mathbf{i} - \mathbf{j} + 4\mathbf{k}) \\ &= -4\mathbf{i} + 3\mathbf{j} - 4\mathbf{k} \\ \overrightarrow{AC} &= 5\mathbf{i} - 3\mathbf{j} + 7\mathbf{k} - (3\mathbf{i} - \mathbf{j} + 4\mathbf{k}) \\ &= 2\mathbf{i} - 2\mathbf{j} + 3\mathbf{k} \end{aligned}$$

$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$
It is important to get the vectors the right way round.
It is a common error to use $\overrightarrow{AB} = \overrightarrow{OA} - \overrightarrow{OB}$ and obtain the negative of the correct answer.

$$\begin{aligned} \overrightarrow{AB} \times \overrightarrow{AC} &= (-4\mathbf{i} + 3\mathbf{j} - 4\mathbf{k}) \times (2\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}) \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -4 & 3 & -4 \\ 2 & -2 & 3 \end{vmatrix} \\ &= \begin{vmatrix} 3 & -4 \\ -2 & 3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} -4 & -4 \\ 2 & 3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} -4 & 3 \\ 2 & -2 \end{vmatrix}\mathbf{k} \\ &= \mathbf{i} + 4\mathbf{j} + 2\mathbf{k} \end{aligned}$$

b An equation of Π is

$$\begin{aligned} \mathbf{r} \cdot (\mathbf{i} + 4\mathbf{j} + 2\mathbf{k}) &= (3\mathbf{i} - \mathbf{j} + 4\mathbf{k}) \cdot (\mathbf{i} + 4\mathbf{j} + 2\mathbf{k}) \\ &= 3 \times 1 + (-1) \times 4 + 4 \times 2 \\ &= 3 - 4 + 8 = 7 \end{aligned}$$

So $\mathbf{r} \cdot (\mathbf{i} + 4\mathbf{j} + 2\mathbf{k}) = 7$

Once you have a vector $\mathbf{n}$ perpendicular to the plane, you can find a vector equation of the plane using $\mathbf{r} \cdot \mathbf{n} = \mathbf{a} \cdot \mathbf{n}$, where $\mathbf{a}$ is the position vector of any point on the plane. Here the position vector of A has been used but the position vectors of B and C would do just as well. As the scalar product is quite quickly worked out, it is a useful check to recalculate, using one of the other points. All should give the same answer, here 7

c

$$\begin{aligned} \overrightarrow{AD} &= 5\mathbf{i} + 2\mathbf{j} + 3\mathbf{k} - (3\mathbf{i} - \mathbf{j} + 4\mathbf{k}) \\ &= 2\mathbf{i} + 3\mathbf{j} - \mathbf{k} \end{aligned}$$

$$\begin{aligned} \overrightarrow{AD} \cdot (\overrightarrow{AB} \times \overrightarrow{AC}) &= (2\mathbf{i} + 3\mathbf{j} - \mathbf{k}) \cdot (\mathbf{i} + 4\mathbf{j} + 2\mathbf{k}) \\ &= 2 \times 1 + 3 \times 4 + (-1) \times 2 \\ &= 2 + 12 - 2 = 12 \end{aligned}$$

The volume, V say, of the tetrahedron is given by

$$V = \frac{1}{6}\left|\overrightarrow{AD} \cdot (\overrightarrow{AB} \times \overrightarrow{AC})\right| = \frac{1}{6} \times 12 = 2$$

10 a $\mathbf{n}_1 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -4 & 1 & 3 \\ 0 & 1 & 2 \end{vmatrix}$

$= \begin{vmatrix} 1 & 3 \\ 1 & 2 \end{vmatrix}\mathbf{i} - \begin{vmatrix} -4 & 3 \\ 0 & 2 \end{vmatrix}\mathbf{j} + \begin{vmatrix} -4 & 1 \\ 0 & 1 \end{vmatrix}\mathbf{k}$

$= -\mathbf{i} + 8\mathbf{j} - 4\mathbf{k}$

If the equation of a plane is given to you in the form $\mathbf{r} = \mathbf{a} + u\mathbf{b} + v\mathbf{c}$, then you can find a normal to the plane by finding $\mathbf{b} \times \mathbf{c}$.

b $\mathbf{n}_2 = 3\mathbf{i} + \mathbf{j} - \mathbf{k}$

The Cartesian equation $3x + y - z = 3$ can be written in the vector form $\mathbf{r} \cdot (3\mathbf{i} + \mathbf{j} - \mathbf{k}) = 3$, where $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$
Comparison with the standard form, $\mathbf{r} \cdot \mathbf{n} = p$, gives you that $3\mathbf{i} + \mathbf{j} - \mathbf{k}$ is perpendicular to Π_2

c $\mathbf{n}_1 \times \mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 8 & -4 \\ 3 & 1 & -1 \end{vmatrix}$

$= \begin{vmatrix} 8 & -4 \\ 1 & -1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} -1 & -4 \\ 3 & -1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} -1 & 8 \\ 3 & 1 \end{vmatrix}\mathbf{k}$

$= -4\mathbf{i} - 13\mathbf{j} - 25\mathbf{k} = -1(4\mathbf{i} + 13\mathbf{j} + 25\mathbf{k})$

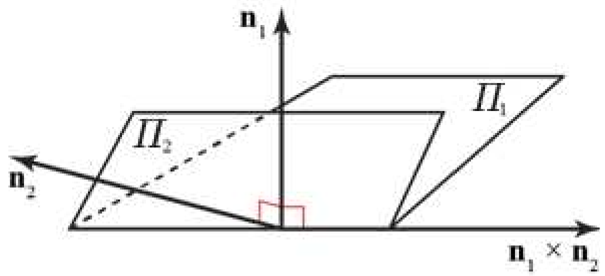
The scalar product $\mathbf{n}_1 \times \mathbf{n}_2$ is perpendicular to both $\mathbf{n}_1$ and $\mathbf{n}_2$

So to show that a vector, $\mathbf{r}$ say, is perpendicular to two other vectors, you can show that $\mathbf{r}$ is parallel to the vector product of the two other vectors. An alternative method is to show that the scalar product of $\mathbf{r}$ with each of the other two vectors is zero.

$\mathbf{n}_1 \times \mathbf{n}_2$ is perpendicular to the plane containing $\mathbf{n}_1$ and $\mathbf{n}_2$, and $4\mathbf{i} + 13\mathbf{j} + 25\mathbf{k}$ is a multiple of $\mathbf{n}_1 \times \mathbf{n}_2$

Hence $4\mathbf{i} + 13\mathbf{j} + 25\mathbf{k}$ is perpendicular to both $\mathbf{n}_1$ and $\mathbf{n}_2$

d $\mathbf{r} = \mathbf{i} + \mathbf{j} + \mathbf{k} + t(4\mathbf{i} + 13\mathbf{j} + 25\mathbf{k})$



This diagram illustrates that the line of intersection of the planes Π_1 and Π_2 lies in the direction of $\mathbf{n}_1 \times \mathbf{n}_2$
In this case, $4\mathbf{i} + 13\mathbf{j} + 25\mathbf{k} = -\mathbf{n}_1 \times \mathbf{n}_2$ and can be used as the direction of the line, as can any other multiple of this vector.

11 a $a(4\mathbf{i}+\mathbf{j}+2\mathbf{k})\cdot(\mathbf{i}-5\mathbf{j}+3\mathbf{k})=a(4\times1+1\times(-5)+2\times3)$

$=a(4-5+6)=5a$

Hence, A lies in the plane Π, as required.

For A to lie on the plane with equation $\mathbf{r.n}=5a$, when $\mathbf{r}$ is replaced by the position vector of A, $\mathbf{r.n}$ must give $5a$.

b $\overrightarrow{BA}=a(4\mathbf{i}+\mathbf{j}+2\mathbf{k})-a(2\mathbf{i}+11\mathbf{j}-4\mathbf{k})$

$=a(2\mathbf{i}-10\mathbf{j}+6\mathbf{k})$

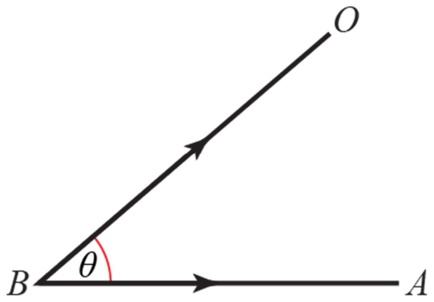
$\overrightarrow{BA}=2a(\mathbf{i}-5\mathbf{j}+3\mathbf{k})$

When a plane has an equation of the form $\mathbf{r.n}=p$, the vector $\mathbf{n}$ is perpendicular to the plane.

$\overrightarrow{BA}$ is parallel to the vector $\mathbf{i}-5\mathbf{j}+3\mathbf{k}$, which is perpendicular to the plane Π

Hence $\overrightarrow{BA}$ is perpendicular to the plane Π, as required.

c



The angle OBA is the angle between BO and BA. Both these line segments have a definite sense and so you must use the scalar product $\overrightarrow{BO}\cdot\overrightarrow{BA}$ to find θ

If you used $\overrightarrow{OB}\cdot\overrightarrow{BA}$, you would obtain the supplementary angle $(180^\circ-\theta)$, which is not the correct answer.

Let $\angle OAB=\theta$

$|\overrightarrow{BO}|=a\sqrt{((-2)^2+(-11)^2+4^2)}=a\sqrt{(141)}$

$|\overrightarrow{BA}|=a\sqrt{(2^2+(-10)^2+6^2)}=a\sqrt{(140)}$

$\overrightarrow{BO}\cdot\overrightarrow{BA}=a(-2\mathbf{i}-11\mathbf{j}+4\mathbf{k})\cdot a(2\mathbf{i}-10\mathbf{j}+6\mathbf{k})$

Finding the angle between two vectors using the scalar product is part of the C4 specification. Knowledge of the C4 specification is a pre-requisite of the FP3 specification.

$|\overrightarrow{BO}|\cdot|\overrightarrow{BA}|\cos\theta=a^2((-2)\times2+(-11)\times(-10)+4\times6)$

$a\sqrt{(141)}\times a\sqrt{(140)}\cos\theta=a^2(-4+110+24)$

$\cos\theta=\dfrac{130}{\sqrt{(141)}\sqrt{(140)}}=0.925272\ldots$

$\theta=22.3^\circ$ (to the nearest one tenth of a degree)

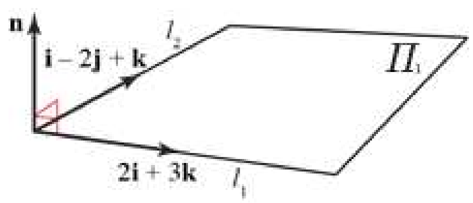
12 a A vector $\mathbf{n}$ perpendicular to l_1 and l_2 is given by

$\mathbf{n}=(2\mathbf{i}+3\mathbf{k})\times(\mathbf{i}-2\mathbf{j}+\mathbf{k})$

$=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 2 & 0 & 3\\ 1 & -2 & 1\end{vmatrix}$

$=\begin{vmatrix}0 & 3\\ -2 & 1\end{vmatrix}\mathbf{i}-\begin{vmatrix}2 & 3\\ 1 & 1\end{vmatrix}\mathbf{j}+\begin{vmatrix}2 & 0\\ 1 & -2\end{vmatrix}\mathbf{k}$

$=6\mathbf{i}+\mathbf{j}-4\mathbf{k}$



The vector $2\mathbf{i}+3\mathbf{k}$ is parallel to l_1 and $\mathbf{i}-2\mathbf{j}+\mathbf{k}$ is parallel to l_2

The vector product of two vectors is perpendicular to both vectors and so is perpendicular to the plane containing both lines.

12 b An equation for Π_1 has the form

$\mathbf{r}\cdot(6\mathbf{i}+\mathbf{j}-4\mathbf{k})=p$

$p=(\mathbf{i}+6\mathbf{j}-\mathbf{k})\cdot(6\mathbf{i}+\mathbf{j}-4\mathbf{k})$

$=6+6+4=16$

A vector equation of Π_1 is

$\mathbf{r}\cdot(6\mathbf{i}+\mathbf{j}-4\mathbf{k})=16$

A Cartesian equation of Π_1 is given by

$(x\mathbf{i}+y\mathbf{j}+z\mathbf{k})\cdot(6\mathbf{i}+\mathbf{j}-4\mathbf{k})=16$

$6x+y-4z=16$, as required.

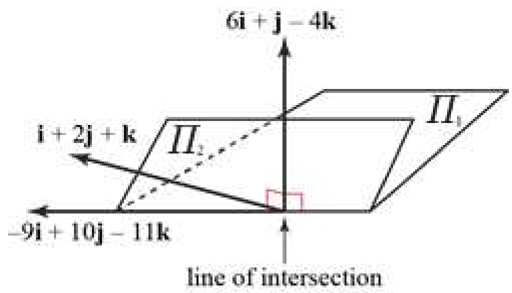
To obtain a Cartesian equation of a plane when you have a vector equation in the form **r.n** = p, replace **r** by $x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$ and work out the scalar product.

c The point with coordinates (3, p, 0) lies on l_1 and, hence, must lie on Π_1

Substituting (3, p, 0) into the result of part **b**

$18+p=16\Rightarrow p=-2$

d The line of intersection lies in the direction given by



$$(\mathbf{i}+2\mathbf{j}+\mathbf{k})\times(6\mathbf{i}+\mathbf{j}-4\mathbf{k})=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\1&2&1\\6&1&-4\end{vmatrix}$$

$$=\begin{vmatrix}2&1\\1&-4\end{vmatrix}\mathbf{i}-\begin{vmatrix}1&1\\6&-4\end{vmatrix}\mathbf{j}+\begin{vmatrix}1&2\\6&1\end{vmatrix}\mathbf{k}$$

$$=-9\mathbf{i}+10\mathbf{j}-11\mathbf{k}$$

To find one point that lies on both Π_1 and Π_2

$\Pi_1:\quad 6x+y-4z=16$ **(1)**

$\Pi_2:\quad x+2y+z=2$ **(2)**

(1) + 4 × **(2)** gives $10x+9y=24$

Choose $x=-3$, $y=6$

Substitute into **(2)**

$-3+12+z=2\Rightarrow z=-7$

You need to find just one point that is on both planes and there are infinitely many possibilities. Here you can choose any pair of values of x and y which fit this equation. A choice here has been made which gives whole numbers but you may find, for example, $y=0$, $x=2.4$ easier to see.

One point on the line is (–3, 6, –7)

An equation of the line is

$\left(\mathbf{r}-(-3\mathbf{i}+6\mathbf{j}-7\mathbf{k})\right)\times(-9\mathbf{i}+10\mathbf{j}-11\mathbf{k})=0$

The form $(\mathbf{r}-\mathbf{a})\times\mathbf{b}=0$, for the equation of a straight line, represents a line that passes through the point with position vector **a** and is parallel to the vector **b**.

13 a $\overrightarrow{RP} = \begin{pmatrix}-1\\3\\-2\end{pmatrix} - \begin{pmatrix}3\\0\\c\end{pmatrix} = \begin{pmatrix}-4\\3\\-2-c\end{pmatrix}$

$\overrightarrow{RQ} = \begin{pmatrix}4\\-1\\-1\end{pmatrix} - \begin{pmatrix}3\\0\\c\end{pmatrix} = \begin{pmatrix}1\\-1\\-1-c\end{pmatrix}$

$\overrightarrow{RP} \times \overrightarrow{RQ} = \begin{pmatrix}-4\\3\\-2-c\end{pmatrix} \times \begin{pmatrix}1\\-1\\-1-c\end{pmatrix}$

$= \begin{pmatrix}3(-1-c)-(2+c)\\-2-c-4(1+c)\\4-3\end{pmatrix} = \begin{pmatrix}-5-4c\\-6-5c\\1\end{pmatrix}$

In this solution, the vector product has be found using the formula $\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix} \times \begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix} = \begin{pmatrix}a_2b_3-a_3b_2\\a_3b_1-a_1b_3\\a_1b_2-a_2b_1\end{pmatrix}$

This formula can be found in the Edexcel formulae booklet which is provided for the examination.

b $\begin{pmatrix}-5-4c\\-6-5c\\1\end{pmatrix} = \begin{pmatrix}3\\d\\1\end{pmatrix}$

Equating the x components

$-5-4c = 3 \Rightarrow 4c = -8 \Rightarrow c = -2$

Equating the y components

$d = -6-5c = -6-5\times(-2) = -6+10$

$= 4$, as required.

c When $c = -2$

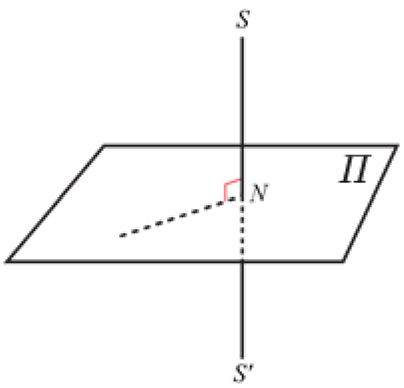
$\overrightarrow{RP} \times \overrightarrow{RQ} = \begin{pmatrix}-5-4c\\-6-5c\\1\end{pmatrix} = \begin{pmatrix}-5+8\\-6+10\\1\end{pmatrix} = \begin{pmatrix}3\\4\\1\end{pmatrix}$

An equation of Π is

$\mathbf{r} \cdot \begin{pmatrix}3\\4\\1\end{pmatrix} = \begin{pmatrix}-1\\3\\-2\end{pmatrix} \cdot \begin{pmatrix}3\\4\\1\end{pmatrix} = -3+12-2$

$\mathbf{r} \cdot \begin{pmatrix}3\\4\\1\end{pmatrix} = 7$

13 d



In this diagram, the point N is the intersection of SS' and the plane. As S' is the reflection of S in Π, SS' is perpendicular to Π and N is the midpoint of SS'. Hence the translation (or displacement) from S to N is the same as the translation (or displacement) from N to S'. The method used in this solution is to find the position vector of N and, then, to find the translation which maps S to N. This translation can then be used to find the position vector of S' from the position vector of N.

A vector equation of SS' is

$$\mathbf{r} = \begin{pmatrix} 1 \\ 5 \\ 10 \end{pmatrix} + t\begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix}$$

Parametric equations for SS' are

$$x = 1 + 3t,\ y = 5 + 4t,\ z = 10 + t \qquad \textbf{(1)}$$

A Cartesian equation of Π is

$$3x + 4y + z = 7 \qquad \textbf{(2)}$$

To find the position vector of N, the point of intersection of SS' and Π, substitute **(1)** into **(2)**

$$3(1+3t) + 4(5+4t) + 10 + t = 7$$
$$3 + 9t + 20 + 16t + 10 + t = 7$$
$$26t = -26 \Rightarrow t = -1$$

The position vector of N is $\begin{pmatrix} 1+3t \\ 5+4t \\ 10+t \end{pmatrix} = \begin{pmatrix} 1-3 \\ 5-4 \\ 10-1 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 9 \end{pmatrix}$

The translation which maps S to N is represented by the vector

$$\overrightarrow{SN} = \begin{pmatrix} -2 \\ 1 \\ 9 \end{pmatrix} - \begin{pmatrix} 1 \\ 5 \\ 10 \end{pmatrix} = \begin{pmatrix} -3 \\ -4 \\ -1 \end{pmatrix}$$

The translation which maps S to N will also map N to S'.

The position vector of S' is given by

$$\begin{pmatrix} -2 \\ 1 \\ 9 \end{pmatrix} + \begin{pmatrix} -3 \\ -4 \\ -1 \end{pmatrix} = \begin{pmatrix} -5 \\ -3 \\ 8 \end{pmatrix}$$

The position vector of S' is the position vector of N added to the vector representing the translation.

14 a $\mathbf{b}-\mathbf{a}=3\mathbf{i}+3\mathbf{j}-4\mathbf{k}-(\mathbf{i}+3\mathbf{j}-\mathbf{k})=2\mathbf{i}-3\mathbf{k}$

$\mathbf{c}-\mathbf{a}=5\mathbf{i}-2\mathbf{j}-2\mathbf{k}-(\mathbf{i}+3\mathbf{j}-\mathbf{k})=4\mathbf{i}-5\mathbf{j}-\mathbf{k}$

$$(\mathbf{b}-\mathbf{a})\times(\mathbf{c}-\mathbf{a})=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 2 & 0 & -3\\ 4 & -5 & -1\end{vmatrix}$$

$$=\begin{vmatrix}0 & -3\\ -5 & -1\end{vmatrix}\mathbf{i}-\begin{vmatrix}2 & -3\\ 4 & -1\end{vmatrix}\mathbf{j}+\begin{vmatrix}2 & 0\\ 4 & -5\end{vmatrix}\mathbf{k}$$

$$=-15\mathbf{i}-10\mathbf{j}-10\mathbf{k}$$

The vector $(\mathbf{b}-\mathbf{a})\times(\mathbf{c}-\mathbf{a})$ is perpendicular to both AB and AC and, so, is perpendicular to the plane Π_1
You can use this vector, or any parallel vector, as the $\mathbf{n}$ in the equation $\mathbf{r}.\mathbf{n}=p$ in part **b**. Here each coefficient has been divided by –5
This eases later working and avoids negative signs.

b A vector perpendicular to Π_1 is $3\mathbf{i}+2\mathbf{j}+2\mathbf{k}$
A vector equation of Π_1 is

$$\mathbf{r}\cdot(3\mathbf{i}+2\mathbf{j}+2\mathbf{k})=(\mathbf{i}+3\mathbf{j}-\mathbf{k})\cdot(3\mathbf{i}+2\mathbf{j}+2\mathbf{k})$$
$$=3+6-2=7$$

c The line l is parallel to the vector

$$(\mathbf{i}+\mathbf{k})\times(3\mathbf{i}+2\mathbf{j}+2\mathbf{k})=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 1 & 0 & 1\\ 3 & 2 & 2\end{vmatrix}$$
$$=-2\mathbf{i}+\mathbf{j}+2\mathbf{k}$$

To find one point on both Π_1 and Π_2

For Π_1 $x+z=3$

Let $z=0$, then $x=3$

The form $(\mathbf{r}-\mathbf{p})\times\mathbf{q}=0$ is that of a line passing through a point with position vector $\mathbf{p}$, parallel to the vector $\mathbf{q}$. So you need to find one point on the line; that is any point which is on both Π_1 and Π_2
As there are infinitely many such points, there are many possible answers to this question. The choice of $z=0$ here is an arbitrary one.

Substituting $z=0$, $x=3$ into a Cartesian equation of Π_2

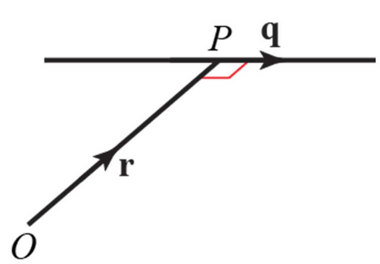
$3x+2y+z=7$

$9+2y+0=7\Rightarrow y=-1$

One point on Π_1 and Π_2 and, hence on l is (3, –1, 0)

Hence, a vector equation of l is $\left(\mathbf{r}-(3\mathbf{i}-\mathbf{j})\right)\times(-2\mathbf{i}+\mathbf{j}+2\mathbf{k})=0$

d A vector equation of l is



$\mathbf{r}=(3\mathbf{i}-\mathbf{j})+t(-2\mathbf{i}+\mathbf{j}+2\mathbf{k})$

$\quad=(3-2t)\mathbf{i}+(-1+t)\mathbf{j}+2t\mathbf{k}$

At P, $\mathbf{r}$ is perpendicular to l

$$\left((3-2t)\mathbf{i}+(-1+t)\mathbf{j}+2t\mathbf{k}\right)\cdot(-2\mathbf{i}+\mathbf{j}+2\mathbf{k})=0$$

$$-6+4t-1+t+4t=0\Rightarrow 9t=7\Rightarrow t=\frac{7}{9}$$

At the point P which is nearest to the origin O, the position vector of P, $\mathbf{r}$, is perpendicular to the direction of the line, $\mathbf{q}$.
Forming the scalar product $\mathbf{r}.\mathbf{q}$ and equating to zero gives you an equation in t.

The coordinates of P are

$$(3-2t,\ -1+t,\ 2t)=\left(\frac{13}{9},\ -\frac{2}{9},\ \frac{14}{9}\right)$$

15 a $\mathbf{a}\times\mathbf{b} = (2\mathbf{i}-\mathbf{k})\times(4\mathbf{i}+3\mathbf{j}+\mathbf{k})$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 0 & -1 \\ 4 & 3 & 1 \end{vmatrix} = 3\mathbf{i}-6\mathbf{j}+6\mathbf{k}$$

b Substituting (0, 0, 0) into $x - 2y + 2z$

$0-2\times0+2\times0=0$

So the plane with equation $x - 2y + 2z = 0$ contains O.

Similarly as

$2-2\times0+2\times(-1)=2-2=0$

and $4-2\times3+2\times1=4-6+2=0$,

the plane with equation $x - 2y + 2z = 0$

contains $A(2, 0, -1)$ and $B(4, 3, 1)$

'Verify' means check that the equation is satisfied by the data of this particular question. To do this you can just show that the coordinates of O, A and B satisfy $x - 2y + 2z = 0$
You do not have to show any general methods.

c For B to lie on the plane with equation

$$\mathbf{r}.\mathbf{n} = d$$

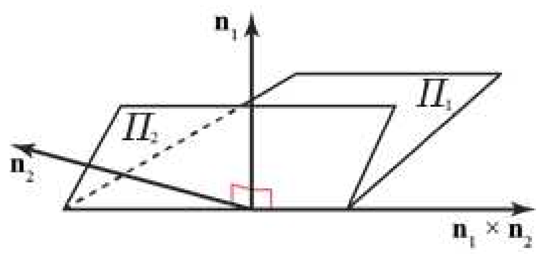
$$(4\mathbf{i}+3\mathbf{j}+\mathbf{k})\cdot(3\mathbf{i}+\mathbf{j}-\mathbf{k}) = d$$

$$d = 4\times3+3\times1+1\times(-1) = 12+3-1 = 14$$

d The line of intersection L lies in the direction given by

$$(\mathbf{i}-2\mathbf{j}+2\mathbf{k})\times(3\mathbf{i}+\mathbf{j}-\mathbf{k}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -2 & 2 \\ 3 & 1 & -1 \end{vmatrix}$$

$$= 0\mathbf{i}+7\mathbf{j}+7\mathbf{k}$$



The vector $\mathbf{n}_1 = \mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$ is perpendicular to Π_1 and the vector $\mathbf{n}_2 = 3\mathbf{i} + \mathbf{j} - \mathbf{k}$ is perpendicular to Π_2
This diagram illustrates the line of intersection of the planes is parallel to $\mathbf{n}_1\times\mathbf{n}_2$
This gives you the direction of L. To find the equation of L, you also need one point on L. In this case, the information given in the question shows you that you already have such a point, B.

A vector parallel to $7\mathbf{j} + 7\mathbf{k}$ is $\mathbf{j} + \mathbf{k}$ and this is parallel to the line L.

The point B, which has position vector $4\mathbf{i} + 3\mathbf{j} + \mathbf{k}$, lies on both Π_1 and Π_2 and, hence, on L.

A vector equations of L is

$\mathbf{r} = 4\mathbf{i} + 3\mathbf{j} + \mathbf{k} + t(\mathbf{j} + \mathbf{k})$

e Rearranging the answer to part **d**
$\mathbf{r} = 4\mathbf{i} + (3 + t)\mathbf{j} + (1 + t)\mathbf{k}$

At the point X on L where OX is perpendicular to L

$$\mathbf{r}.(\mathbf{j}+\mathbf{k}) = 0$$

$$\left(4\mathbf{i}+(3+t)\mathbf{j}+(1+t)\mathbf{k}\right)\cdot(\mathbf{j}+\mathbf{k}) = 3+t+1+t = 0$$

$$2t = -4 \Rightarrow t = -2$$

The positive vector of X is

$4\mathbf{i}+(3-2)\mathbf{j}+(1-2)\mathbf{k} = 4\mathbf{i}+\mathbf{j}-\mathbf{k}$

16 a Let $\mathbf{a} = \mathbf{j} + 2\mathbf{k}$, $\mathbf{b} = 2\mathbf{i} + 3\mathbf{j} + \mathbf{k}$, and $\mathbf{c} = \mathbf{i} + \mathbf{j} + 3\mathbf{k}$

$\mathbf{b} - \mathbf{a} = 2\mathbf{i} + 3\mathbf{j} + \mathbf{k} - (\mathbf{j} + 2\mathbf{k}) = 2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$

$\mathbf{c} - \mathbf{a} = \mathbf{i} + \mathbf{j} + 3\mathbf{k} - (\mathbf{j} + 2\mathbf{k}) = \mathbf{i} + \mathbf{k}$

A vector which is perpendicular to Π is

$$(\mathbf{b} - \mathbf{a}) \times (\mathbf{c} - \mathbf{a}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 2 & -1 \\ 1 & 0 & 1 \end{vmatrix}$$

$$= 2\mathbf{i} - 3\mathbf{j} - 2\mathbf{k}$$

The vector product $(\mathbf{b} - \mathbf{c}) \times (\mathbf{c} - \mathbf{a})$ is, by definition, perpendicular to both $\mathbf{b} - \mathbf{a}$ and $\mathbf{c} - \mathbf{a}$ and, so, it is perpendicular to both AB and AC. It will also be perpendicular to the plane containing AB and AC.

b $$\Delta ABC = \frac{1}{2}\left|(\mathbf{b} - \mathbf{a}) \times (\mathbf{c} - \mathbf{a})\right|$$

$$= \frac{1}{2}\left|2\mathbf{i} - 3\mathbf{j} - 2\mathbf{k}\right|$$

$$= \frac{1}{2}\sqrt{(2^2 + (-3)^2 + (-2)^2)}$$

$$= \frac{\sqrt{17}}{2}$$

The vector product can be interpreted as a vector with magnitude twice the area of the triangle which has the vectors as two of its sides.

c A vector equation of Π is

$$\mathbf{r} \cdot (2\mathbf{i} - 3\mathbf{j} - 2\mathbf{k}) = (\mathbf{j} + 2\mathbf{k}) \cdot (2\mathbf{i} - 3\mathbf{j} - 2\mathbf{k}) = 0 - 3 - 4$$

$$= -7$$

d A Cartesian equation of Π is $2x - 3y - 2z = -7$

The vector equation $\mathbf{r} \cdot \begin{pmatrix} a \\ b \\ c \end{pmatrix} = p$ and the Cartesian equation $ax + by + cz = p$ are equivalents.

e The distance from a point (α, β, γ) to a plane

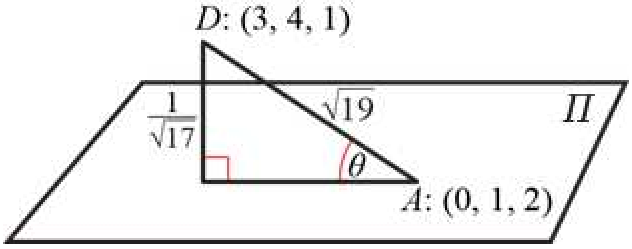
$n_1 x + n_2 y + n_2 z + d = 0$ is $\left|\dfrac{n_1\alpha + n_2\beta + n_3\gamma + d}{\sqrt{(n_1^2 + n_2^3 + n_3^2)}}\right|$

This formula is given in the Edexcel formulae booklet. If you use a formula from the booklet, it is sensible to quote it in your solution. The distance of a point from a plane is defined to be the shortest distance from the point to the plane; that is the perpendicular distance from the point to the plane.

Hence the distance from (0, 0, 0) to $2x - 3y - 2z = -7$

is $\left|\dfrac{7}{\sqrt{(2^2 + (-3)^2 + (-2)^2)}}\right| = \dfrac{7}{\sqrt{17}}$

16 f



Let the angle between AD and Π be θ

$$AD^2 = (3-0)^2 + (4-1)^2 + (1-2)^2 = 9+9+1 = 19$$

$$AD = \sqrt{19}$$

$$\sin\theta = \frac{\left(\frac{1}{\sqrt{17}}\right)}{\sqrt{19}} = 0.055641\ldots$$

$$\theta = 3.2^\circ \text{ (1 d.p.)}$$

17 a $\overrightarrow{AB} = \begin{pmatrix}4\\2\\1\end{pmatrix} - \begin{pmatrix}-1\\-1\\1\end{pmatrix} = \begin{pmatrix}5\\3\\0\end{pmatrix}$

$$\overrightarrow{AC} = \begin{pmatrix}2\\1\\0\end{pmatrix} - \begin{pmatrix}-1\\-1\\1\end{pmatrix} = \begin{pmatrix}3\\2\\-1\end{pmatrix}$$

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{pmatrix}5\\3\\0\end{pmatrix} \times \begin{pmatrix}3\\2\\-1\end{pmatrix} = \begin{pmatrix}-3\\5\\1\end{pmatrix}$$

The vector product $\overrightarrow{AB} \times \overrightarrow{AC}$ is, by definition, perpendicular to both AB and AC. So it will also be perpendicular to the plane containing AB and AC.

An equation of the line, l say, which passes through D and is perpendicular to Π is $\mathbf{r} = \begin{pmatrix}1\\2\\3\end{pmatrix} + t\begin{pmatrix}-3\\5\\1\end{pmatrix}$

b $\overrightarrow{AD} = \begin{pmatrix}1\\2\\3\end{pmatrix} - \begin{pmatrix}-1\\-1\\1\end{pmatrix} = \begin{pmatrix}2\\3\\2\end{pmatrix}$

$$\overrightarrow{AD} \cdot (\overrightarrow{AB} \times \overrightarrow{AC}) = \begin{pmatrix}2\\3\\2\end{pmatrix} \cdot \begin{pmatrix}-3\\5\\1\end{pmatrix} = -6+15+2 = 11$$

The volume of the tetrahedron, V say, is given by

$$V = \frac{1}{6}\left|\overrightarrow{AD} \cdot (\overrightarrow{AB} \times \overrightarrow{AC})\right| = \frac{1}{6}|11| = \frac{11}{6}$$

17 c An equation for Π is

$$\mathbf{r}\cdot\begin{pmatrix}-3\\5\\1\end{pmatrix}=\begin{pmatrix}-1\\-1\\1\end{pmatrix}\cdot\begin{pmatrix}-3\\5\\1\end{pmatrix}=3-5+1$$

$$\mathbf{r}\cdot\begin{pmatrix}-3\\5\\1\end{pmatrix}=-1$$

The vector equation $\mathbf{r}\cdot\begin{pmatrix}a\\b\\c\end{pmatrix}=p$ and the Cartesian equation $ax+by+cz=p$ are equivalents and one can always be replaced by the other.

d A Cartesian equation for Π is

$$-3x+5y+z=-1$$

Parametric equation corresponding to the equation of l found in part **a** are

$x=1-3t,\ y=2+5t,\ z=3+t$

Substituting these parametric equations into the Cartesian equation for Π

$$-3(1-3t)+5(2+5t)+3+t=-1$$

$$-3+9t+10+25t+3+t=-1$$

$$35t=-11\Rightarrow t=-\frac{11}{35}$$

The coordinates of E are given by

$$(1-3t,\ 2+5t,\ 3+t)$$

$$=\left(1+3\times\frac{11}{35},\ 2-5\times\frac{11}{35},\ 3-\frac{11}{35}\right)$$

$$=\left(\frac{68}{35},\frac{15}{35},\frac{94}{35}\right)$$

Use your calculator to help you work out these awkward fractions. Of course, $\frac{15}{35}=\frac{5}{7}$ and this is acceptable as part of the answer. However, the subsequent working is easier if all the coordinates have the same denominator.

e $$DE^2=\left(1-\frac{68}{35}\right)^2+\left(2-\frac{15}{35}\right)^2+\left(3-\frac{94}{35}\right)^2$$

$$=\left(\frac{33}{35}\right)^2+\left(\frac{55}{35}\right)^2+\left(\frac{11}{35}\right)^2$$

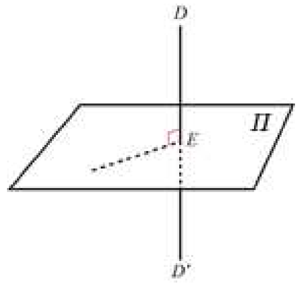
$$=\frac{33^2+55^2+11^2}{35^2}=\frac{4235}{1225}=\frac{121}{35}$$

The distance d between points with coordinates (x_1, x_2, x_3) and (y_1, y_2, y_3) is given by $d^2=(x_1-y_1)^2+(x_2-y_2)^2+(x_3-y_3)^2$

Hence $DE=\sqrt{\left(\frac{121}{35}\right)}=\frac{11}{\sqrt{35}}=\frac{11\sqrt{35}}{35}$, as required.

17 f The translation mapping D to E is represented by the vector

$$\overrightarrow{DE} = \begin{pmatrix} \frac{68}{35} \\ \frac{15}{35} \\ \frac{94}{35} \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} \frac{33}{35} \\ -\frac{55}{35} \\ -\frac{11}{35} \end{pmatrix}$$



As D' is the reflection of D in Π, E is the midpoint of DD' and the translation which maps D to E also maps E to D'.
So you can find the position vector of D' by adding $\begin{pmatrix} \frac{33}{35} \\ -\frac{55}{35} \\ -\frac{11}{35} \end{pmatrix}$ to the position vector of E.

The position vector of D' is given by

$$\overrightarrow{OD'} = \overrightarrow{OE} + \overrightarrow{DE} = \begin{pmatrix} \frac{68}{35} \\ \frac{15}{35} \\ \frac{94}{35} \end{pmatrix} + \begin{pmatrix} \frac{33}{35} \\ -\frac{55}{35} \\ -\frac{11}{35} \end{pmatrix} = \begin{pmatrix} \frac{101}{35} \\ -\frac{40}{35} \\ \frac{83}{35} \end{pmatrix}$$

The coordinates of D' are $\left(\frac{101}{35}, -\frac{40}{35}, \frac{83}{35}\right)$

18 a Equating the x components

To show that two lines intersect, you find the values of the two parameters, here s and t, which make two of the components equal and then you show that these values make the third components equal.

$-1-2s=-t$ **(1)**

Equating the y components

$2+s=-1+t$ **(2)**

(1) + **(2)** $1-s=-1 \Rightarrow s=2$

Substitute $s=2$ into **(2)** $4=-1+t \Rightarrow t=5$

Checking the z components

For l_1: $-4+3s=-4+6=2$

For l_2: $7-t=7-5=2$

These are the same, so l_1 and l_2 intersect.

The lines l_1 and l_2 are parallel to

$-2\mathbf{i}+\mathbf{j}+3\mathbf{k}$ and $-\mathbf{i}+\mathbf{j}-\mathbf{k}$ respectively.

$(-2\mathbf{i}+\mathbf{j}+3\mathbf{k})\cdot(-\mathbf{i}+\mathbf{j}-\mathbf{k}) = 2+1-3=0$

As the scalar product $\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta$, where θ is the angle between the vectors, if, for non-zero vectors, the scalar produce is zero then $\cos\theta = 0$ and $\theta = 90^\circ$

Hence l_1 is perpendicular to l_2

b Substituting $s=2$ into the equation for l_1, the common point has position vector

$-\mathbf{i}+2\mathbf{j}-4\mathbf{k}+2(-2\mathbf{i}+\mathbf{j}+3\mathbf{k}) = -5\mathbf{i}+4\mathbf{j}+2\mathbf{k}$

Using $\mathbf{r}=\mathbf{a}+\mu(\mathbf{b}-\mathbf{a})$, an equation of l_3 is

$\mathbf{r}=\mathbf{a}+u(\mathbf{b}-\mathbf{a})$ is the appropriate form of the equation of a straight line going through two points with position vectors $\mathbf{a}$ and $\mathbf{b}$

Here $\mathbf{a}=-5\mathbf{i}+4\mathbf{j}+2\mathbf{k}$
$\mathbf{b}=4\mathbf{i}+\lambda\mathbf{j}-3\mathbf{k}$

$$\begin{aligned} \mathbf{r} &= -5\mathbf{i}+4\mathbf{j}+2\mathbf{k}+\mu(4\mathbf{i}+\lambda\mathbf{j}-3\mathbf{k}-(-5\mathbf{i}+4\mathbf{j}+2\mathbf{k})) \\ &= -5\mathbf{i}+4\mathbf{j}+2\mathbf{k}+\mu(9\mathbf{i}+(\lambda-4)\mathbf{j}-5\mathbf{k}) \end{aligned}$$

18 c A vector **n** perpendicular to the plane, Π say, containing l_1 and l_2 is

$$\mathbf{n} = (-\mathbf{i}+\mathbf{j}-\mathbf{k})\times(-2\mathbf{i}+\mathbf{j}+3\mathbf{k})$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 1 & -1 \\ -2 & 1 & 3 \end{vmatrix} = 4\mathbf{i}+5\mathbf{j}+\mathbf{k}$$

Let the angle between l_3 and Π be θ

$$|\mathbf{n}|^2 = 4^2+5^2+1^2 = 42$$

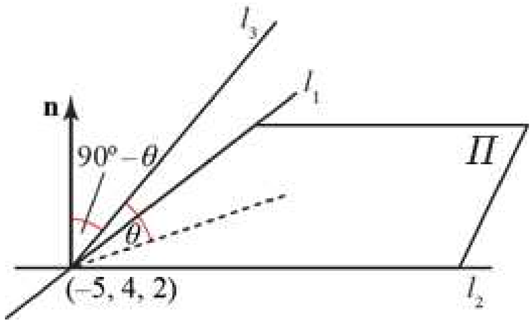
$$|9\mathbf{i}+(\lambda-4)\mathbf{j}-5\mathbf{k}| = 9^2+(\lambda-4)^2+(-5)^2$$

$$= 81+\lambda^2-8\lambda+16+25 = \lambda^2-8\lambda+122$$

$$\mathbf{n}\cdot(9\mathbf{i}+(\lambda-4)\mathbf{j}-5\mathbf{k}) = |\mathbf{n}||(9\mathbf{i}+(\lambda-4)\mathbf{j}-5\mathbf{k})|\cos(90^\circ-\theta)$$

$$(4\mathbf{i}+5\mathbf{j}+\mathbf{k})\cdot(9\mathbf{i}+(\lambda-4)\mathbf{j}-5\mathbf{k})$$

$$= \sqrt{42}\times\sqrt{(\lambda^2-8\lambda+122)}\sin\theta$$



The cosine of the angle between **n** and l_3 can be found using the scalar product of **n** and a vector parallel to l_3
This cosine is $\sin\theta$

$$\sin\theta = \frac{4\times9+5(\lambda-4)+1\times(-5)}{\sqrt{42}\sqrt{(\lambda^2-8\lambda+122)}}$$

$$= \frac{5\lambda+11}{\sqrt{42}\sqrt{(\lambda^2-8\lambda+122)}}$$

d If l_1, l_2 and l_3 are coplanar then

$\theta = 0$ and $\sin\theta = 0$

Looking at the diagram in part **b** above, if l_3 lies in the plan Π, then $\theta = 0$

Hence $5\lambda+11 = 0 \Rightarrow \lambda = \dfrac{-11}{5}$

19 a The Cartesian equations of the planes are

$P_1: 2x - y + 2z = 9$ **(1)**

$P_2: 4x + 3y - z = 8$ **(2)**

Points on the line of intersection of the two planes can be found by solving simultaneous the Cartesian equations of the two planes. As there are 2 equations in 3 unknowns, there are infinitely many solutions. A free choice can be made for one variable, here x is given the parameter t, and the other variables cab then be found in terms of t.

(1) + 2 × **(2)**

$10x + 5y = 25$

$2x + y = 5$

Let $x = t$, then $y = 5 - 2x = 5 - 2t$

From **(2)**

$z = 4x + 3y - 8$

$= 4t + 3(5 - 2t) - 8 = 7 - 2t$

The general point on the line of intersection of the planes has coordinates $(t, 5 - 2t, 7 - 2t)$

This is the equivalent of the parametric equations of the common line $x = t,\ y = 5 - 2t,\ z = 7 - 2t$

The equivalent vector equation of this line is $\mathbf{r} = 5\mathbf{j} + 7\mathbf{k} + t(\mathbf{i} - 2\mathbf{j} - 2\mathbf{k})$

The distance, y say, from O to this general point is given by

$y^2 = t^2 + (5 - 2t)^2 + (7 - 2t)^2$

$= t^2 + 25 - 20t + 4t^2 + 49 - 28t + 4t^2$

$= 9t^2 - 48t + 74$ **(3)**

Differentiating both sides of **(3)** with respect to t

$$2y\frac{dy}{dt} = 18t - 48$$

A calculus method of finding the minimum distance is shown here. You could instead use the property that, at the shortest distance, the position vector of the point is perpendicular to the common line. This method is illustrated in Question **13**.

At a minimum distance $\frac{dy}{dt} = 0$

$$18t - 48 = 0 \Rightarrow t = \frac{48}{18} = \frac{8}{3}$$

Substituting into **(3)**

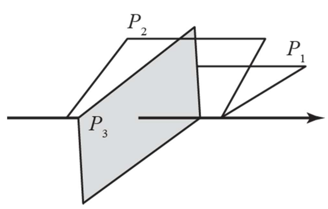
$$y^2 = 9 \times \left(\frac{8}{3}\right)^2 - 48 \times \frac{8}{3} + 74$$

$= 64 - 128 + 74 = 10$

The shortest distance from O to the line of intersection of the planes is $\sqrt{10}$

b The line of intersection of P_1 and P_2 has vector equation $\mathbf{r} = 5\mathbf{j} + 7\mathbf{k} + t(\mathbf{i} - 2\mathbf{j} - 2\mathbf{k})$

The common on line of P_1 and P_2 is a normal to the plane P_3 which is perpendicular to P_1 and P_2



Hence the vector $\mathbf{i} - 2\mathbf{j} - 2\mathbf{k}$ is perpendicular to Π_3

An equation of P_3 is

$\mathbf{r} \cdot (\mathbf{i} - 2\mathbf{j} - 2\mathbf{k}) = (2\mathbf{j} + \mathbf{k}) \cdot (\mathbf{i} - 2\mathbf{j} - 2\mathbf{k})$

$= -4 - 2 = -6$

19 c Substituting $(t, 5-2t, 7-2t)$ into $x-2y-2z=-6$

$t-2(5-2t)-2(7-2t)=-6$

$t-10+4t-14+4t=-6 \Rightarrow 9t=18 \Rightarrow t=2$

The position vector of the common point is

$t\mathbf{i}+(5-2t)\mathbf{j}+(7-2t)\mathbf{k}=2\mathbf{i}+\mathbf{j}+3\mathbf{k}$

The point that lies on the three planes is given by substituting the general point on the line of intersection of P_1 and P_2 into the Cartesian equation of P_3

20 Equating the equation of the hyperbola with that of the straight line we find that the y-coordinates of A and B are given by the solutions of $y(4y-16)=64$

This equation can be rewritten as

$4y^2-16y=64$
$y^2-4y-16=0$

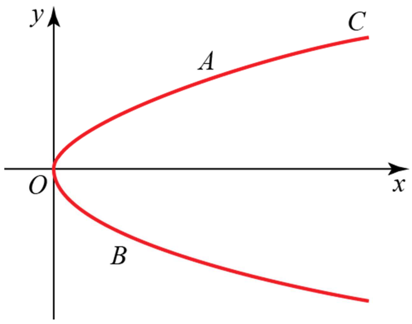
which has solutions $2\pm2\sqrt{5}$

Using $x=\frac{64}{y}$ and rationalising we find that the x-coordinate of the midpoint is

$$\frac{\frac{64}{2+2\sqrt{5}}+\frac{64}{2-2\sqrt{5}}}{2}=-\frac{256}{32}=-8$$

The y-coordinate is then obtained from the equation of the straight line at the point $x=-8$, so that the coordinates of M are $(-8, 2)$

21 a



You have to recognise that $x=3t^2$, $y=6t$ is a parabola and draw it passing through the origin with the correct orientation.

b For the intersections, substitute $x=3t^2$, $y=6t$ into $y=x-72$

$$\begin{aligned}
6t &= 3t^2-72\\
3t^2-6t-72 &= 0\\
(\div 3)\quad t^2-2t-24 &= 0\\
(t-6)(t+4) &= 0\\
t &= 6, -4
\end{aligned}$$

$(3t^2, 6t)$ is a general point on the parabola. The points A and B must be of this form and, if they also lie on the line with equation $y=x-72$, the points on the parabola must also satisfy the equation of the line.

The question does not tell you which point is A and which point is B but, as you are only asked to find the distance between them, it does not matter which is which and you can make your own choice.

For A, say, $t=6$

$x=3t^2=108, y=6t=36$

For B, say, $t=-4$

$x=3t^2=3\times(-4)^2=48,\ y=6t=-24$

$$\begin{aligned}
\text{Using } d^2 &= (x_1-x_2)^2+(y_1-y_2)^2\\
AB^2 &= (108-48)^2+(36-(-24))^2\\
&= 60^2+60^2=2\times60^2\\
AB &= \sqrt{(2\times60^2)}=60\sqrt{2}
\end{aligned}$$

You are asked to give your answer as a surd in its simplest form. 84.85 is not acceptable as it is not a surd and $\sqrt{7200}$ is not the simplest form. A surd in its simplest form contains the square root of the smallest possible single number.

22 Since P and Q lie on the parabola, their coordinates are (1, 2) and (9, 6) respectively. Therefore, the line through these points has equation $y=\frac{x}{2}+\frac{3}{2}$, and the midpoint of PQ is R=(5, 4)

Then we can easily find that the equation of the perpendicular bisector to PQ is

$y-4=-2(x-5)$
$y-4=-2x+10$
$y=-2x+14$

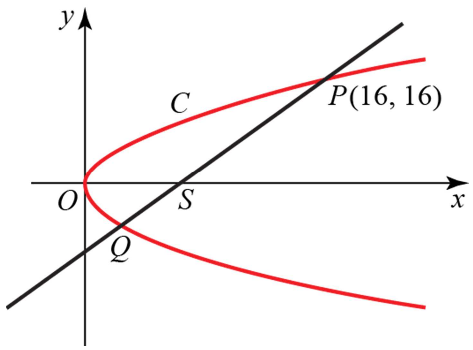
Now if we intersect this with the equation of C we find that the x-coordinates of M and N are the solutions of $(-2x+14)^2=4x$

We solve this:

$4x^2-56x+196=4x$
$4x^2-60x+196=0$
$x^2-15x+49=0$
$$x_{1,2}=\frac{15\pm\sqrt{29}}{2}$$

In particular, they can be written in the form $\frac{15}{2}\pm\frac{1}{2}\sqrt{29}$, so $\lambda=\frac{15}{2},\mu=\frac{1}{2}$

23



You should mark on your diagram any points given in the question. Here mark S, P and Q. Diagrams often help you check your working. Here, for example, it is obvious from the diagram that Q must have a negative y-coordinate. If you get $y = 4$ (a mistake it is easy to make), you would know you were wrong and look for an error in your working.

a $S(4, 0)$

The focus of the parabola with equation $y^2=4ax$ has coordinates $(a, 0)$.
Here $a = 4$.
The question asks you to write down the answer, so you do not have to show working.

b Using $\frac{y-y_1}{y_2-y_1}=\frac{x-x_1}{x_2-x_1}$ with $(x_1, y_1) = (4, 0)$ and $(x_2, y_2)=(16, 16)$, an equation of SP is

$$\frac{y-0}{16-0}=\frac{x-4}{16-4}$$
$$\frac{y}{16}=\frac{x-4}{12}$$
$$\cancel{12}^{3}y=\cancel{16}^{4}(x-4)$$
$$3y=4x-16$$
$$4x-3y-16=0$$

Methods for finding the equation of a straight line are given in Chapter 5 of Edexcel AS and A Level Modular Mathematics AS and A-Level, Core Mathematics 1
You can use any correct method for finding the line.

23 c From **(b)**

$$x = \frac{3y+16}{4}$$

Substitute for x in $y^2 = 16x$

$$y^2 = \cancel{16}^4\left(\frac{3y+16}{\cancel{4}}\right) = 12y+64$$

$y^2 - 12y - 64 = 0.$

$(y-16)(y+4) = 0$

$y = 16$ corresponds to the point P.

For Q, $y = -4$

$$x = \frac{3\times -4+16}{4} = \frac{4}{4} = 1$$

The coordinates of Q are $(1, -4)$.

To find Q you solve the simultaneous equations $4x - 3y - 16 = 0$ and $y^2 = 16x$

The method of using substitution, when one equation is linear and the other is quadratic, is given in Chapter 3 of Edexcel AS and A Level Modular Mathematics AS and A-Level, Core Mathematics 1

24 First of all, we need to find the coordinates of the focus S.
Obviously, the y-coordinate is 0; as for the x-coordinate, if we consider the equation of C to be $x = \frac{y^2}{20}$ then we know that the coordinate must be $\frac{1}{\frac{4}{20}} = 5$

Now, this allows us to find the equation of the straight line l: it is $y = \frac{4}{3}(x-5)$

This intersects C in two points, whose y-coordinates satisfy $y = \frac{4}{3}\left(\frac{y^2}{20} - 5\right)$

We solve this:

$$y = \frac{4}{3}\left(\frac{y^2}{20} - 5\right)$$
$$3y = \frac{y^2}{5} - 20$$
$$15y = y^2 - 100$$
$$y^2 - 15y - 100 = 0$$

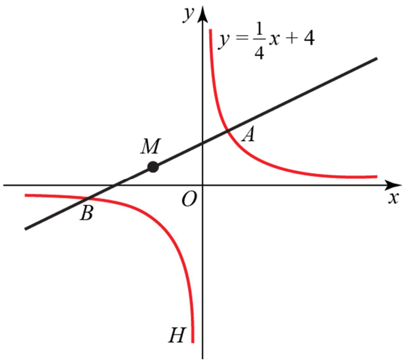
This has only the one positive solution $\frac{15+\sqrt{625}}{2} = 20$

Then the point P has coordinates $(20, 20)$
The area of the region R is given by the area enclosed by C between O and P minus the area of the triangle PQS, where Q is the projection of P on the x-axis. As in the first Cartesian quadrant the parabola is the graph of the function $y = 2\sqrt{5x}$, the area can be written as follows:

$$2\sqrt{5}\int_0^{20}\sqrt{x}\,dx - \frac{15\cdot 20}{2} =$$
$$= 2\sqrt{5}\left(\tfrac{2}{3}20\sqrt{20}\right) - 150 =$$
$$= \tfrac{4}{3}\sqrt{5}(40\sqrt{5}) - 150 =$$
$$= \tfrac{800-450}{3} = \tfrac{350}{3}$$

25 The Cartesian equation of the hyperbola is $xy = 16$
Therefore, the x-coordinates of P and Q are the solutions of the equation $x(2x+4)=16$
We solve this:
$2x^2+4x=16$
$x^2+2x-8=0$
$x_{1,2}=-1\pm3$
Then $P = (2, 8)$ and $Q=(-4,-4)$

26



Substitute $x=8t,\ y=\frac{16}{t}$ into $y=\frac{1}{4}x+4$

$$\frac{16}{t} = \frac{1}{4}\times 8t+4$$
$$\frac{16}{t} = 2t+4$$

Multiplying by t and rearranging
$2t^2+4t-16 = 0$
$(\div 2)\quad t^2+2t-8 = (t+4)(t-2)=0$
$t = 2, -4$

For A, say, $t=2 \Rightarrow x=8t=8\times2=16$

and $y=\frac{16}{t}=\frac{16}{2}=8$

The coordinates of A are (16, 8)

For B, say, $t=-4\Rightarrow x=8t=8\times-4=-32$

and $y=\frac{16}{t}=\frac{16}{-4}=-4$

The coordinates of B are (–32, –4)

The x-coordinate of the midpoint of AB is given by

$$x_M=\frac{16-32}{2}=-8$$

The y-coordinate of the midpoint of AB is given by

$$y_M=\frac{8-4}{2}=2$$

The coordinates of M are (–8, 2)

$\left(8t, \frac{16}{t}\right)$ is a general point on the rectangular hyperbola. The points A and B must be of this form and, if they also lie on the line with equation $y=\frac{1}{4}x+4$, the points on the parabola must also satisfy the equation of the line.

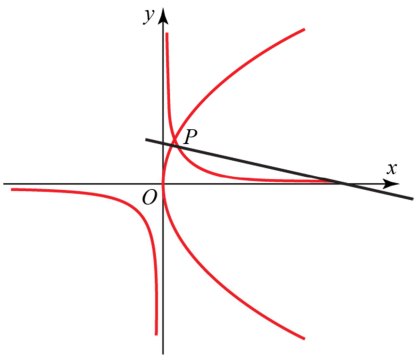
The question does not tell you which point is A and which point is B but, as the midpoint is not affected by the choice, it does not matter which is which and you can make your own choice.

The coordinates of the midpoint M of $A(x_1, y_1)$ and $B(x_2, y_2)$ are given by $(x_M, y_M)=\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$

27 a The y-coordinates of P and Q are the solutions of the equation $(12-2y)y=10$
We solve this:
$(12-2y)y=10$
$12y-2y^2=10$
$y^2-6y+5=0$
$(y-5)(y-1)=0$
We substitute y=5 and y=1 into the equation of the hyperbola to find the coordinates of P and Q:
P=(2, 5) and Q=(10, 1)

b The line intersects the coordinate axis in the points (0, 6) and (12, 0)
The area of the triangle these points form with the origin is 36
The area enclosed by the hyperbola between the points P and Q is
$\int_2^{10} \frac{10}{x}\,\mathrm{d}x =$
$=10\int_2^{10} \frac{1}{x}\,\mathrm{d}x =$
$=10(\ln 10-\ln 2)=$
$=10\ln 5$
Therefore, the area of the triangle minus the area enclosed by the hyperbola is $36 - 10\ln 5$
However, we must also remove the parts that go from the intersection with the axes to P and Q: the trapezoid that is determined by P, the point (0, 6) and their projections on the x-axis has area 11, while the triangle determined by Q, its projection and (0, 12) has area 1
Then the area of the shaded region is $36-10\ln 5-12=24-10\ln 5$
Hence $a = 24$, $b=-10$ and $c=5$

28 a



$(24t^2, 48t)$ must satisfy the equation $xy = 144$

$$24t^2 \times 48t = 144$$

$$t^3 = \frac{144}{24\times 48} = \frac{1}{8} \Rightarrow t = \frac{1}{2}$$

For P, $x = 24t^2 = 24\times\left(\frac{1}{2}\right)^2 = 6$

$$y = 48t = 48\times\frac{1}{2} = 24$$

The coordinates of P are (6, 24)

The point with coordinates $(at^2, 2at)$ always lies on the parabola with equation $y^2 = 4ax$, in this case $a = 24$, so P is on the parabola for all t.
There will however only be one value of t for which P also lies on the rectangular hyperbola and you find it by substituting $(24t^2, 48t)$ into $xy =$ 144

28 b $y^2 = 96x \Rightarrow y = (96)^{\frac{1}{2}}x^{\frac{1}{2}} = 4\sqrt{6}x^{\frac{1}{2}}$

$96 = 16\times 6 = 4^2\times 6$. So $\sqrt{96} = \sqrt{(4^2\times 6)} = 4\sqrt{6}$

$\frac{dy}{dx} = \frac{1}{2}\times 4\sqrt{6}x^{-\frac{1}{2}} = \frac{2\sqrt{6}}{x^{\frac{1}{2}}}$

At $x = 6$, $\frac{dy}{dx} = \frac{2\sqrt{6}}{\sqrt{6}} = 2$

Using $y - y_1 = m(x - x_1)$, an equation of the tangent to the parabola at P is

$y - 24 = 2(x - 6) = 2x - 12$

$y = 2x + 12$

c $y = \frac{144}{x} = 144x^{-1}$

$\frac{dy}{dx} = -144x^{-2} = -\frac{144}{x^2}$

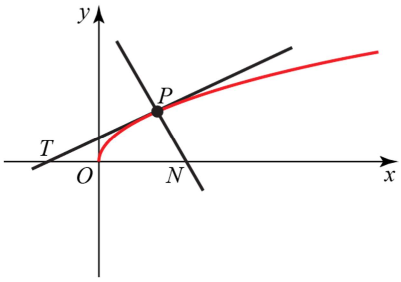
At $x = 6$, $\frac{dy}{dx} = -\frac{144}{6^2} = -4$

Using $y - y_1 = m(x - x_1)$, an equation of the tangent to the hyperbola at P is

$y - 24 = -4(x - 6) = -4x + 24$

$y = -4x + 48$

29



The $t > 0$ in the question implies that you only need to consider the part of the parabola where $y > 0$, that is the part above the x-axis.

To find an equation of the tangent PT

$$y^2 = 4ax \Rightarrow y = 2a^{\frac{1}{2}}x^{\frac{1}{2}} (y > 0)$$

$$y = 2a^{\frac{1}{2}}x^{\frac{1}{2}}$$

$$\frac{dy}{dx} = \frac{1}{2} \times 2a^{\frac{1}{2}}x^{-\frac{1}{2}} = \frac{a^{\frac{1}{2}}}{x^{\frac{1}{2}}}$$

Using $\frac{d}{dx}(x^n) = nx^{n-1}$, $\frac{d}{dx}\left(x^{\frac{1}{2}}\right) = \frac{1}{2}x^{\frac{1}{2}-1} = \frac{1}{2}x^{-\frac{1}{2}}$

At $x = at^2$, $x^{\frac{1}{2}} = a^{\frac{1}{2}}t$

$$\frac{dy}{dx} = \frac{a^{\frac{1}{2}}}{a^{\frac{1}{2}}t} = \frac{1}{t}$$

Using $y - y_1 = m(x - x_1)$ with $(x_1, y_1) = (at^2, 2at)$, an equation of the tangent to the parabola at P is

$$y - 2at = \frac{1}{t}(x - at^2)$$

$$ty - 2at^2 = x - at^2$$

$$ty = x + at^2 \quad \textbf{(1)}$$

To find x-coordinate of T, substitute $y = 0$ into **(1)**

The tangent crosses the x-axis where $y = 0$

$$0 = x + at^2 \Rightarrow x = -at^2$$

Using $d^2 = (x_1 - x_2)^2 + (y_1 + y_2)^2$ with

$(x_1, y_1) = (at^2, 2at)$ and $(x_2, y_2) = (-at^2, 0)$

$$PT^2 = (at^2 - (-at^2))^2 + (2at - 0)^2$$

$$= (2at^2)^2 + 4a^2t^2 = 4a^2t^2 + 4a^2t^2$$

$$= 4a^2t^2(t^2 + 1) \quad \textbf{(2)}$$

To find an equation of the normal PN

Using $mm' = -1$,

$$\frac{1}{t} \times m' = -1 \Rightarrow m' = -t$$

The normal is perpendicular to the tangent. From working earlier in the question, you know that the gradient of the tangent is $\frac{1}{t}$

29 cont.

Using $y - y_1 = m'(x - x_1)$ with $(x_1, y_1) = (at^2, 2at)$,
an equation of the normal to the parabola at P is

$$\begin{aligned} y - 2at &= -t(x - at^2) \\ &= -tx + at^3 \\ y + tx &= 2at + at^3 \quad \textbf{(3)} \end{aligned}$$

The normal crosses the x-axis where $y = 0$

To find the x-coordinate of N, substitute $y = 0$ into **(3)**

$$tx = 2at + at^3 \Rightarrow x = 2a + at^2$$

Using $d^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2$ with

$(x_1, y_1) = (at^2, 2at)$ and $(x_2, y_2) = (2a + at^2, 0)$

$$\begin{aligned} PN^2 &= (at^2 - (2a + at^2))^2 + (2at - 0)^2 \\ &= (2a)^2 + (2at)^2 = 4a^2 + 4a^2t^2 \\ &= 4a^2(1 + t^2) \quad \textbf{(4)} \end{aligned}$$

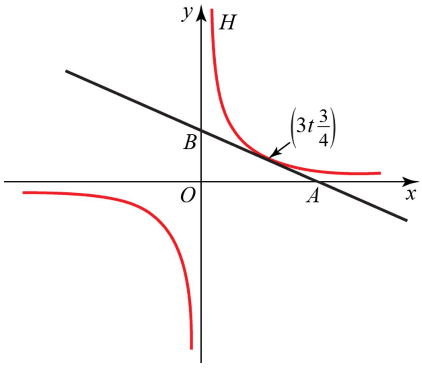
From **(2)** and **(4)**

$$\frac{PT^2}{PN^2} = \frac{4a^2t^2(t^2+1)}{4a^2(t^2+1)} = t^2$$

Hence

$$\frac{PT}{PN} = t, \text{ as required.}$$

30 a



$y = \frac{9}{x} = 9x^{-1}$

$\frac{dy}{dx} = -9x^{-2} = -\frac{9}{x^2}$

When $x = 3t$, $\frac{dy}{dx} = -\frac{9}{(3t)^2} = -\frac{1}{t^2}$

Using $y - y_1 = m(x - x_1)$ with $(x_1, y_1) = \left(3t, \frac{3}{t}\right)$, the tangent to H

$$y - \frac{3}{t} = -\frac{1}{t^2}(x - 3t)$$

$(\times t^2)$ $\quad t^2 y - 3t = -x + 3t$

$x + t^2 y = 6t$, as required.

Using $\frac{d}{dx}(x^n) = nx^{n-1}$,
$\frac{d}{dx}(9x^{-1}) = -1 \times 9x^{-1-1} = -9x^{-2} = -\frac{9}{x^2}$

You can use $(x_1, y_1) = \left(3t, \frac{3}{t}\right)$ in the formula $y - y_1 = m(x - x_1)$ in exactly the same way as you use coordinates with numerical values like, say, (6, 4)

b For A, substitute, $y = 0$ into $x + t^2 y = 6t$

$x = 6t \Rightarrow OA = 6t$

For, B, substitute, $x = 0$ into $x + t^2 y = 6t$

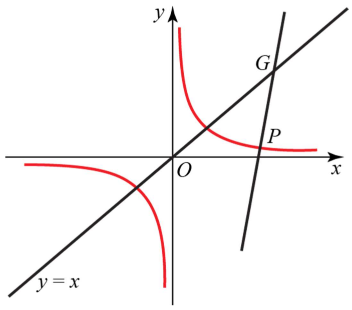
$t^2 y = 6t \Rightarrow y = \frac{6}{t} \Rightarrow OB = \frac{6}{t}$

Area $\Delta OAB = \frac{1}{2} OA \times OB = \frac{1}{2} \times 6t \times \frac{6}{t} = 18$

This area, 18, is a constant independent of t.

This result means that no matter which point you take on this rectangular hyperbola the area of the triangle OAB is always the same, 18

31 a



$$y = \frac{c^2}{x} = c^2x^{-1}$$

$$\frac{dy}{dx} = -c^2x^{-2} = \frac{c^2}{x^2}$$

At $P, x = ct$

$$\frac{dy}{dx} = \frac{c^2}{c^2t^2} = -\frac{1}{t^2}$$

For the gradient of the normal, using $mm' = -1$,

$$\left(-\frac{1}{t^2}\right)m' = -1 \Rightarrow m' = t^2$$

Using $y - y_1 = m(x - x_1)$ with $(x_1, y_1) = \left(ct, \frac{c}{t}\right)$, an equation of the normal to the hyperbola at P is

$$y - \frac{c}{t} = t^2(x - ct)$$

$$y - \frac{c}{t} = t^2x - ct^3$$

$$(\times t) \quad yt - c = t^3x - ct^4$$

$$t^3x - ty - ct^4 + c = 0$$

$$t^3x - ty - c(t^4 - 1) = 0, \text{ as required}$$

The normal to H at P is perpendicular to the tangent at P. To work out perpendicular gradients you will need the formula $mm' = -1$
So you have to find the gradient of tangent before you can find the gradient of the normal. You find the gradient of the tangent using differentiation.

When you are asked to show that an equation is true, you must use algebra to transform your equation to the equation exactly as it is printed in the question.

b For G, substitute $y = x$ into the result in part (a)

$$t^3x - tx - c(t^4 - 1) = 0$$

$$(t^3 - t)x = c(t^4 - 1)$$

$$x = \frac{c(t^4 - 1)}{t^3 - t} = \frac{c\,(t^2 - 1)\,(t^2 + 1)}{t\,(t^2 - 1)} = \frac{ct^2 + c}{t} = ct + \frac{c}{t}$$

You could not "cancel" the $(t^2 - 1)$ terms if $t = \pm 1$, as then $(t^2 - 1)$ would be 0, but these cases are explicitly ruled out in the question.

The coordinates of G are $\left(ct + \frac{c}{t},\ ct + \frac{c}{t}\right)$

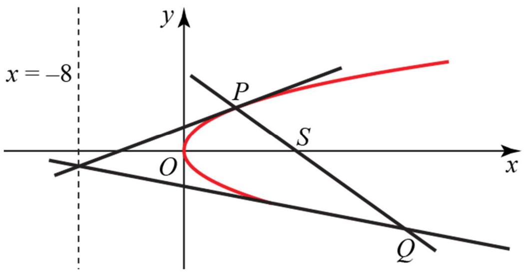
$$PG^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2$$

$$= \left(ct + \frac{c}{t} - ct\right)^2 + \left(ct + \frac{c}{t} - \frac{c}{t}\right)^2$$

$$= \frac{c^2}{t^2} + c^2t^2 = c^2\left(t^2 + \frac{1}{t^2}\right), \text{ as required.}$$

Using $d^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2$ with $(x_1, y_1) = \left(ct + \frac{c}{t}, ct + \frac{c}{t}\right)$ and $(x_2, y_2) = \left(ct, \frac{c}{t}\right)$

32



a (8, 0)

b $x = -8$

c Using $\frac{y-y_1}{y_2-y_1} = \frac{x-x_1}{x_2-x_1}$ with

$(x_1, y_1) = (2, 8)$ and $(x_2, y_2) = (32, -32)$,

an equation of PQ is

$$\frac{y-8}{-32-8} = \frac{x-2}{32-2}$$

$$\frac{y-8}{-40} = \frac{x-2}{30}$$

$$3y - 24 = -4x + 8$$

$$3y + 4x = 32$$

Substitute $y = 0$

$$0 + 4x = 32 \Rightarrow x = 8$$

The coordinates of S (8, 0) satisfy the equation of PQ.

Hence S lies on the joining P and Q.

If $y^2 = 4ax$, the focus has coordinates $(a, 0)$ and the directrix has equation $x = -a$
Comparison of $y^2 = 4a$ *with* $y^2 = 32x$,
shows that, in this case, $a = 8$

Methods for finding the equation of a straight line are given in Chapter 5 of Edexcel AS and A level Modular Mathematics AS and A-Level, Core Mathematics 1
You can use any correct method for finding the line.

$\sqrt{32} = \sqrt{(16 \times 2)} = \sqrt{16} \times \sqrt{2} = 4\sqrt{2}$

On the upper half of the parabola, in the first quadrant, the y-coordinates of P are positive.

32 d $y^2 = 32x \Rightarrow y = \pm 4\sqrt{2}x^{\frac{1}{2}}$

P is on the upper half of the parabola where $y = \pm 4\sqrt{2}x^{\frac{1}{2}}$

$$\frac{dy}{dx} = \frac{1}{2}4\sqrt{2}x^{-\frac{1}{2}} = \frac{2\sqrt{2}}{x^{\frac{1}{2}}}$$

At $x = 2$, $\frac{dy}{dx} = \frac{2\sqrt{2}}{\sqrt{2}} = 2$

Using $y - y_1 = m(x - x_1)$, the tangent to C at P is

$$y - 8 = 2(x - 2) = 2x - 4$$
$$y = 2x + 4 \ldots\ldots\textbf{(1)}$$

Q is on the lower half of the parabola where $y = -4\sqrt{2}x^{\frac{1}{2}}$

On the lower half of the parabola, in the fourth quadrant, the y-coordinates of P are negative.

$$\frac{dy}{dx} = -\frac{1}{2}4\sqrt{2}x^{-\frac{1}{2}} = -\frac{2\sqrt{2}}{x^{\frac{1}{2}}}$$

At $x = 32$, $\frac{dy}{dx} = -\frac{2\sqrt{2}}{\sqrt{32}} = -\frac{2\sqrt{2}}{4\sqrt{2}} = -\frac{1}{2}$

Using $y - y_1 = m(x - x_1)$, the tangent to C at Q is

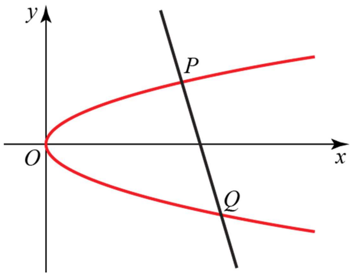
$$y + 32 = -\frac{1}{2}(x - 32) = -\frac{1}{2}x + 16$$
$$y = -\frac{1}{2}x - 16 \ldots\ldots\textbf{(2)}$$

To find the x-coordinate of the intersection of the tangents, from **(1)** and **(2)**

$$2x + 4 = -\frac{1}{2}x - 16$$
$$\frac{5}{2}x = -20 \Rightarrow x = -20 \times \frac{2}{5} = -8$$

The equation of the directrix is $x = -8$ and, hence, the intersection of the tangents lies on the directrix.

33



a $y^2 = 4ax \Rightarrow y = 2a^{\frac{1}{2}}x^{\frac{1}{2}}$

$$y = 2a^{\frac{1}{2}}x^{\frac{1}{2}}$$

$$\frac{dy}{dx} = \frac{1}{2} \times 2a^{\frac{1}{2}}x^{-\frac{1}{2}} = \frac{a^{\frac{1}{2}}}{x^{\frac{1}{2}}}$$

At $x = at^2$, $x^{\frac{1}{2}} = a^{\frac{1}{2}}t$

$$\frac{dy}{dx} = \frac{a^{\frac{1}{2}}}{a^{\frac{1}{2}}t} = \frac{1}{t}$$

Using $mm' = -1$

$$\frac{1}{t} \times m' = -1 \Rightarrow m' = -t$$

Using $y - y_1 = m'(x - x_1)$ with $(x_1, y_1) = (at^2, 2at)$, an equation of the normal to the parabola at P is

$$y - 2at = -t(x - at^2)$$
$$= -tx + at^3$$
$$y + tx = 2at + at^3, \text{as required.}$$

The normal is perpendicular to the tangent, so you must first find the gradient of the tangent. Then you use $mm' = -1$ to find the gradient of the normal.

When you are asked to show that an equation is true, you must use algebra to transform your equation to the equation exactly as it is printed in the question.
You substitute $x = aq^2$ and $y = 2aq$ into the answer to part **(a)**.

b Let the coordinates of Q be $(aq^2, 2aq)$

The point Q lies on the normal at P, so

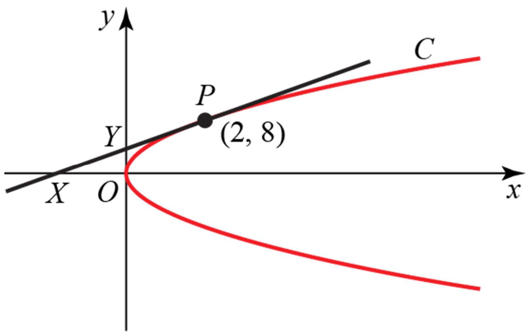
$$2aq + taq^2 = 2at + at^3$$
$$2aq - 2at + atq^2 - at^3 = 0$$
$$2a(q - t) + at(q^2 - t^2) = 0$$
$$2a(q - t) + at(q - t)(q + t) = 0$$
$$a(q - t)(2 + t(q + t)) = 0$$
$$2 + tq + t^2 = 0$$
$$q = -\frac{t^2 + 2}{t}$$

There are two possibilities here: $q - t = 0$ and $2 + t(q + t) = 0$
As P and Q are different points, $q \neq t$, so you need only consider the second possibility.
You use this to find q in terms of t.

Replace the q in $(aq^2, 2aq)$ by $-\frac{t^2 + 2}{t}$
You need not attempt to simplify this further.

The coordinates of Q are $\left(a\left(\frac{t^2 + 2}{t}\right)^2, -2a\left(\frac{t^2 + 2}{t}\right)\right)$

34



a Substitute (2,8) into $y^2 = 4ax$

$$64 = 4a \times 2 = 8a \Rightarrow a = \frac{64}{8} = 8$$

b $y = 2a^{\frac{1}{2}}x^{\frac{1}{2}}$

$$\frac{dy}{dx} = \frac{1}{2} \times 2a^{\frac{1}{2}}x^{-\frac{1}{2}} = \frac{a^{\frac{1}{2}}}{x^{\frac{1}{2}}}$$

Using $\frac{d}{dx}(x^n) = nx^{n-1}, \frac{d}{dx}\left(x^{\frac{1}{2}}\right) = \frac{1}{2}x^{\frac{1}{2}-1} = \frac{1}{2}x^{-\frac{1}{2}}$

When $a = 8$ and $x = 2$

$$\frac{dy}{dx} = \frac{\sqrt{8}}{\sqrt{2}} = \frac{2\sqrt{2}}{\sqrt{2}} = 2$$

Using $y - y_1 = m(x - x_1)$ the tangent to C at P is

$$y - 8 = 2(x - 2) = 2x - 4$$
$$y = 2x + 4$$

The method for obtaining the equation of a straight line when you know its gradient and one point which it passes through is given in Chapter 5 of Edexcel AS and A Level Modular Mathematics AS and A-Level, Core Mathematics 1

c At $X, y = 0 \Rightarrow 0 = 2x + 4 \Rightarrow x = -2$

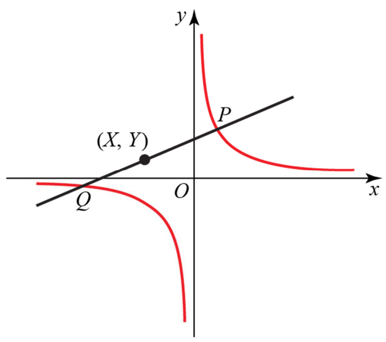
SO $OX = 2$

At $Y, x = 0 \Rightarrow y = 2 \times 0 + 4 \Rightarrow y = 4$

SO $OY = 4$

$$\text{Area } \Delta OXY = \frac{1}{2}OX \times OY = \frac{1}{2} \times 2 \times 4 = 4$$

35 a



a $y = \frac{c^2}{x} = c^2x^{-1}$

$\frac{dy}{dx} = -c^2x^{-2} = -\frac{c^2}{x^2}$

At P, $x = ct$

$\frac{dy}{dx} = -\frac{c^2}{c^2t^2} = \frac{1}{t^2}$

For the gradient of the normal, using $mm' = -1$,

$\left(-\frac{1}{t^2}\right)m' = -1 \Rightarrow m' = t^2$

Using $y - y_1 = m(x - x_1)$ with $(x_1, y_1) = \left(ct, \frac{c}{t}\right)$, an

equation of the normal to the hyperbola at P is

$y - \frac{c}{t} = t^2(x - ct)$

$= t^2x - ct^3$

$y = t^2x + \frac{c}{t} - ct^3$, as required...**(1)**

The normal to H at P is perpendicular to the tangent at P. To work out perpendicular gradients you will need the formula $mm' = -1$
So you have to find the gradient of the tangent before you can find the gradient of the normal. You find the gradient of the tangent by differentiating.

When you are asked to show that an equation is true, you must use algebra to transform your equation to the equation exactly as it is printed in the question. In this case, the form of the printed equation suggests a method for the next part of the question.

35 b $xy = c^2 \Rightarrow y = \frac{c^2}{x}$ … …**(2)**

For Q, from **(1)** and **(2)**

$$t^2 x + \frac{c}{t} - ct^3 = \frac{c^2}{x}$$

$\times t$ and collect terms as a quadratic in x

$$t^2 x^2 + (c - ct^4)x - c^2 t = 0$$

$$(x - ct)(t^3 x + c) = 0$$

$x = ct$ corresponds to P

For Q, $x = -\frac{c}{t^3}$

Substitute the x-coordinate into **(2)**

$$y = \frac{c^2}{x} = \frac{c^2}{-\frac{c}{t^3}} = -ct^3$$

The coordinates of Q are $\left(-\frac{c}{t^3}, -ct^3\right)$

> Writing the equation of the rectangular hyperbola, in the form $y = ...$, enables you to eliminate y quickly between **(1)** and **(2)**.

> The x-coordinate of P ct, must be a solution of the equation. So $(x - ct)$ must be a factor of the quadratic and, so, you can write down the second factor using $x \times t^3 x = t^3 x^2$ and $-ct \times c = -c^2 t$

c $X = \frac{ct + \left(-\frac{c}{t^3}\right)}{2} = \frac{ct^4 - c}{2t^3} = \frac{c(t^4 - 1)}{2t^3}$

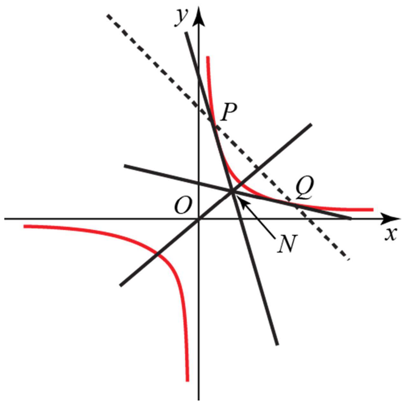
$$Y = \frac{\frac{c}{t} + \left(-ct^3\right)}{2} = \frac{ct - ct^4}{2t} = \frac{c(1 - t^4)}{2t}$$

$$\frac{X}{Y} = \frac{\frac{c\left(\frac{t^4 - 1}{2t^3}\right)}{2t^3}}{\frac{c(1 - t^4)}{2t}} = \frac{c(t^4 - 1)}{2t^3} \times \frac{2t}{c(1 - t^4)}$$

$= -\frac{1}{t^2}$, as required.

> Multiplying all terms on the top and bottom of the fraction by t^3
>
> The coordinates of the midpoint of $A(x_1, y_1)$ and $B(x_2, y_2)$ are given by
>
> $(X, Y) = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$
>
> Multiplying all terms on the top and bottom of the fraction by t.

36 a



$y = \frac{c^2}{x} = c^2x^{-1}$

$\frac{dy}{dx} = -c^2x^{-2} = -\frac{c^2}{x^2}$

At $x = cp$

$\frac{dy}{dx} = -\frac{c^2}{c^2p^2} = -\frac{1}{p^2}$

Using $y - y_1 = m(x - x_1)$ with $x_1, y_1 = \left(cp, \frac{c}{p}\right)$, an equation of the tangent to the hyperbola is

$y - \frac{c}{p} = -\frac{1}{p^2}(x - cp)$

$y - \frac{c}{p} = -\frac{x}{p^2} + \frac{c}{p}$

$y = -\frac{x}{p^2} + \frac{2c}{p}$

$(\times p^2)\quad p^2y = -x + 2cp$, as required…**(1)**

b The tangent at Q is

$q^2y = -x + 2cq$ … …**(2)**

The equation of the tangent at Q is the same as the equation of the tangent at P with the ps replaced by qs. You do not have to work out the equation twice.

To find the y-coordinate of N subtract **(2)** from **(1)**

$(p^2 - q^2)y = 2c(p - q)$

$y = \frac{2c(p-q)}{p^2-q^2} = \frac{2c\,\cancel{(p-q)}}{\cancel{(p-q)}(p+q)} = \frac{2c}{p+q}$, as required.

To find y, you eliminate x from equations **(1)** and **(2)**. These equations are a pair of simultaneous linear equations and the method of solving them is essentially the same as you learnt for GCSE.

36 c To find the x-coordinate of N substitute the result of part (b) into **(1)**

$$\frac{2cp^2}{p+q} = -x + 2cp$$
$$x = 2cp - \frac{2cp^2}{p+q} = \frac{2cp(p+q) - 2cp^2}{p+q} = \frac{2cpq}{p+q}$$

The gradient of PQ, m say, is given by

$$m = \frac{\frac{c}{p} - \frac{c}{p}}{cp - cq} = \frac{\frac{\not{c}\,(q-p)^{-1}}{pq}}{\not{c}\,(p-q)} = -\frac{1}{pq}$$

The gradient m is found using $m = \frac{y_2 - y_1}{x_2 - x_1}$ with $(x_1, y_1) = \left(cp, \frac{c}{p}\right)$ and $(x_2, y_2) = \left(cq, \frac{c}{q}\right)$

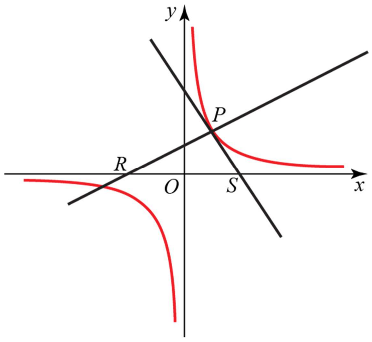
The gradient of ON, m' say, is given by

$$m' = \frac{\frac{2c}{p+q}}{\frac{2cpq}{p+q}} = \frac{1}{pq}$$

Given that ON is perpendicular to PQ

$$mm' = -1$$
$$-\frac{1}{pq} \times \frac{1}{pq} = -1 \Rightarrow p^2q^2 = 1$$

37 a



$y = \frac{c^2}{x} = c^2x^{-1}$

$\frac{dy}{dx} = -c^2x^{-2} = -\frac{c^2}{x^2}$

At $x = cp$

$\frac{dy}{dx} = -\frac{c^2}{c^2p^2} = -\frac{1}{p^2}$

Using $y - y_1 = m(x - x_1)$ with $(x_1,\ y_1) = \left(cp,\ \frac{c}{p}\right)$,

an equation of the tangent to the hyperbola is

$y - \frac{c}{p} = -\frac{1}{p^2}(x - cp)$

$y - \frac{c}{p} = -\frac{x}{p^2} + \frac{c}{p}$

$y + \frac{x}{p^2} = \frac{2c}{p}$

$(\times p^2) \quad p^2y + x = 2cp$, as required... **(1)**

b $(2cp, 0)$

The tangent crosses the x-axis at $y = 0$

You can put $y = 0$ into **(1)** in your head and just write down the coordinates of S. No working is needed.

c Using $d^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2$ with

$(x_1, y_1) = \left(cp, \frac{c}{p}\right)$ and $(x_2, y_2) = (2cp, 0)$

$PS^2 = (cp - 2cp)^2 + \left(\frac{c}{p} - 0\right)^2 = c^2p^2 + \frac{c^2}{p^2}$

$= c^2\left(p^2 + \frac{1}{p^2}\right) = c^2\left(\frac{p^4 + 1}{p^2}\right)$

$PS = \frac{c}{p}(1 + p^4)^{\frac{1}{2}}$

There are many possible forms for this answer. Any equivalent form would gain full marks.

37 d To find the equation of the normal at P

The working in part (a) shows the gradient of the tangent is $-\frac{1}{p^2}$

Let the gradient of the normal be m'

Using $mm' = -1$,

$$-\frac{1}{p^2} \times m' = -1 \Rightarrow m' = p^2$$

Using $y - y_1 = m'(x - x_1)$ with $(x_1, y_1) = \left(cp, \frac{c}{p}\right)$ an equation of the normal to the hyperbola at P is

$$y - \frac{c}{p} = p^2(x - cp)$$
$$= p^2x - cp^3$$
$$p^2x = y - \frac{c}{p} + cp^3$$

To find an expression for the area of the triangle you can obtain the length of the side RS and use that as the base of the triangle in the formula for the area of the triangle. First you need to obtain an equation of the normal and use it to find the coordinates of R.

To find the x-coordinate of R, substitute $y = 0$

$$p^2x = -\frac{c}{p} + cp^3 \Rightarrow x = cp - \frac{c}{p^3}$$

$$RS = 2cp - \left(cp - \frac{c}{p^3}\right) = cp + \frac{c}{p^3} = c\left(\frac{p^4 + 1}{p^3}\right)$$

$$\text{Area } \Delta RPS = \frac{1}{2} RS \times \text{height}$$

If RS is taken as the base of the triangle, the height of the triangle is the y-coordinate of P.

$$41c^2 = \frac{1}{2} \times c\left(\frac{p^4 + 1}{p^3}\right) \times \frac{c}{p}$$
$$= \frac{c^2}{2p^4}(p^4 + 1)$$

$$82p^4 = p^4 + 1 \Rightarrow p^4 = \frac{1}{81} \Rightarrow p = \frac{1}{3}$$

As $p > 0$ is given in the question, you need not consider the alternative solution $p = -\frac{1}{3}$

The coordinates of P are $\left(cp, \frac{c}{p}\right) = \left(\frac{c}{3}, 3c\right)$

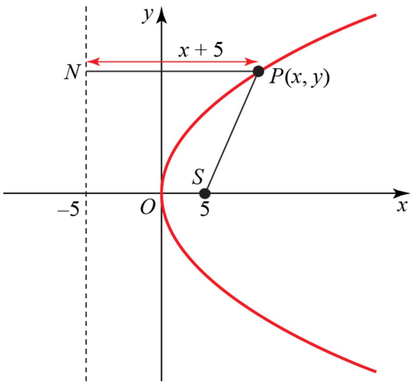
38 The coordinates of P are $\left(ct, \frac{c}{t}\right)$; therefore, the coordinates of the midpoint between P and the origin are $\left(\frac{ct}{2}, \frac{c}{2t}\right)$

The product $\frac{ct}{2} \cdot \frac{c}{2t}$ yields $\frac{c^2}{4}$: in other words, the locus of the midpoint is described by the equation

$$xy = \frac{c^2}{4}$$

This is the equation of a hyperbola.

39



A diagram helps you see and understand what is going on. You should label important points. If letters are not given in the question, you can make your own up. Putting them on a diagram makes your method clear to the examiner.

By the definition of a parabola

$SP = PN$

$SP^2 = PN^2$

$S = (5, 0), P = (x, y)$

$$SP^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2$$
$$= (x-5)^2 + y^2$$

Using the formula $d^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2$, it is often easier to work with the distances squared rather than with distances.

$PN = x + 5$

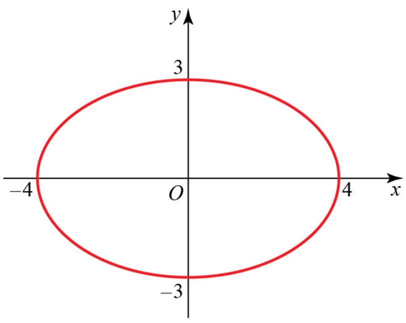
Along PN, the distance from P to the y-axis is x, and it is a further distance 5 from the y-axis to N.

$$SP^2 = PN^2$$
$$(x-5)^2 + y^2 = (x+5)^2$$
$$\cancel{x^2} - 10x + \cancel{25} + y^2 = \cancel{x^2} + 10x + \cancel{25}$$
$$y^2 = 20x$$

Multiply out the brackets using $(a+b)^2 = a^2 + 2ab + b^2$
Then "cancel" the equal terms on both sides of the equation.

Comparing with $y^2 = 4ax$, this is the required form with $a = 5$

40 a



When you draw a sketch, you should show the important features of the curve. When drawing an ellipse, you should show that it is a simple closed curve and indicate the coordinates of the points where the curve intersects the axes.

b $b^2 = a^2(1 - e^2)$

$9 = 16(1 - e^2) = 16 - 16e^2$

$$e^2 = \frac{16-9}{16} = \frac{7}{16}$$

$$e = \frac{\sqrt{7}}{4}$$

The formula you need for calculating the eccentricity and the coordinates of the foci are given in the Edexcel formula booklet you are allowed to use in the examination. You should be familiar with the formulae in that booklet. You should quote any formulae you use in your solution.

c The coordinates of the foci are given by

$$(\pm ae, 0) = \left(\pm 4 \times \frac{\sqrt{7}}{4}, 0\right) = \left(\pm\sqrt{7}, 0\right)$$

41 a $b^2 = a^2(e^2 - 1)$

$4 = 16(e^2 - 1) = 16e^2 - 16$

$e^2 = \frac{16+4}{16} = \frac{20}{16} = \frac{5}{4}$

$e = \frac{\sqrt{5}}{2}$

> The formula for calculating the eccentricity is $b^2 = a^2(e^2 - 1)$
> It is important not to confuse this with the formula for calculating the eccentricity of an ellipse $b^2 = a^2(1 - e^2)$

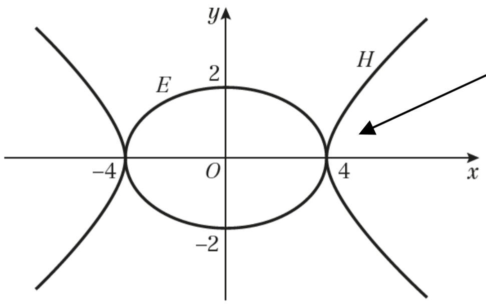
b The coordinates of the foci are given by

$$(\pm ae, 0) = \left(\pm 4 \times \frac{\sqrt{5}}{2}, 0\right) = \left(\pm 2\sqrt{5}, 0\right)$$

> The formulae for the foci of an ellipse and a hyperbola are the same $(\pm ae, 0)$

Therefore the distance between the foci is $2\sqrt{5} + 2\sqrt{5} = 4\sqrt{5}$

c



> In this sketch, you should show where the curves cross the axes. Label which curve is H and which is E. These two curves touch each other on the x-axis.

42 a $b^2 = a^2(1 - e^2)$

$4 = 9(1 - e^2) = 9 - 9e^2$

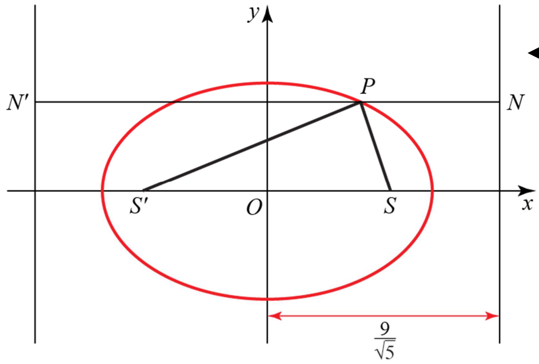
$e^2 = \frac{9-4}{9} = \frac{5}{9} \Rightarrow e = \frac{\sqrt{5}}{3}$

> As the coordinates of the foci of an ellipse are $(\pm ae, 0)$, you first need to find the eccentricity of the ellipse using $b^2 = a^2(1 - e^2)$ with $a = 3$ and $b = 2$

The coordinates of the foci are given by

$$(\pm ae, 0) = \left(\pm 3 \times \frac{\sqrt{5}}{3}, 0\right) = \left(\pm\sqrt{5}, 0\right)$$

42 b



In this question, you are not asked to draw a diagram but with questions on coordinate geometry it is usually a good idea to sketch a diagram so you can see what is going on.

The equations of the directrices are $x = \pm\frac{a}{e}$

$$x = \pm\frac{3}{\frac{\sqrt{5}}{3}} = \pm\frac{9}{\sqrt{5}}$$

Let the line through P parallel to the x-axis intersect the directrices at N and N', as shown in the diagram

$$N'N = 2\times\frac{9}{\sqrt{5}} = \frac{18}{\sqrt{5}}$$

If you introduce points, like N and N' here, you should define them in your solution and mark them on your diagram. This helps the examiner follow your solution.

The focus directrix property of the ellipse gives that

$$SP = ePN \quad \text{and} \quad S'P = ePN'$$

$$SP + S'P = ePN' + ePN'$$

$$= e(PN + PN') = eN'N$$

$$= \frac{\sqrt{5}}{3}\times\frac{18}{\sqrt{5}} = 6, \text{ as required.}$$

43 a $3x^2 + 4y^2 = 12$

$$\frac{x^2}{4} + \frac{y^2}{3} = 1$$

$$b^2 = a^2\left(1-e^2\right)$$

$$3 = 4\left(1-e^2\right) = 4-4e^2$$

$$e^2 = \frac{4-3}{4} = \frac{1}{4} \Rightarrow e = \frac{1}{2}$$

You divide this equation by 12

Comparing the result with $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $a^2 = 4$ and $b^2 = 3$ and you use $b^2 = a^2(1 - e^2)$ to calculate e.

43 b $3x^2 + 4y^2 = 12$

Differentiate implicitly with respect to x

$6x + 8y\frac{dy}{dx} = 0$

$\frac{dy}{dx} = -\frac{6x}{8y} = -\frac{3x}{4y}$

Differentiating implicitly using the chain rule,
$\frac{d}{dx}(4y^2) = \frac{dy}{dx}\frac{d}{dy}(4y^2) = \frac{dy}{dx} \times 8y$

At $\left(1, \frac{3}{2}\right)$

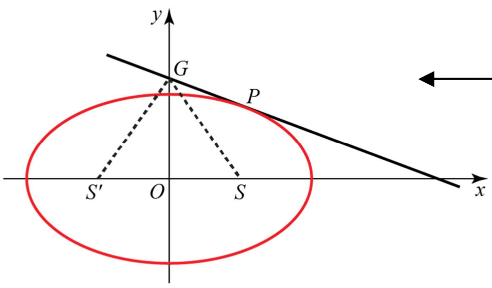
$\frac{dy}{dx} = \frac{-3 \times 1}{4 \times \frac{3}{2}} = -\frac{1}{2}$

Using $y - y_1 = m(x - x_1)$, an equation of the tangent is

$y - \frac{3}{2} = -\frac{1}{2}(x - 1) = -\frac{1}{2}x + \frac{1}{2}$

$y = -\frac{1}{2}x + 2$

c



Sketching a diagram makes it clear that the area of the triangle is to be found using the standard expression $\frac{1}{2}$base × height with the base $S'S$ and the height OG.

The coordinates of S are

$(ae, 0) = \left(2 \times \frac{1}{2}, 0\right) = (1, 0)$

By symmetry, the coordinates of S' are $(-1, 0)$
The y-coordinate of G is given by

$y = 0 + 2 = 2$

You find the y-coordinate of G by substituting $x = 0$ into the answer to part **a**.

$\Delta SS'G = \frac{1}{2}\text{base} \times \text{height}$

$= \frac{1}{2}S'S \times OG$

$= \frac{1}{2}2 \times 2 = 2$

44 a S_2 has coordinates $\left(\frac{a}{2}\sqrt{3}, 0\right)$

Hence

$$e = \frac{\sqrt{3}}{2}$$

Comparing $\left(\frac{a}{2}\sqrt{3}, 0\right)$ with the formula for the focus $(ae, 0)$, $e = \frac{\sqrt{3}}{2}$

$$b^2 = a^2\left(1 - e^2\right)$$

$$= a^2\left(1 - \frac{3}{4}\right) = \frac{a^2}{4} \quad *$$

An equation of the ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

You are given that a is the semi-major axis, so a can be left in the equation. The data in the question does not include b, so b must be replaced.

Using $*$

$$\frac{x^2}{a^2} + \frac{y^2}{\frac{a^2}{4}} = 1$$

The required equation is

$$\frac{x^2}{a^2} + \frac{4y^2}{a^2} = 1$$

$$x^2 + 4y^2 = a^2$$

b Equations of the directrices are

$$x = \pm\frac{a}{e} = \pm\frac{a}{\frac{\sqrt{3}}{2}} = \pm\frac{2a}{\sqrt{3}}$$

c From $*$ above, $b = \frac{a}{2}$

44 d For Q

$$\left(a\cos\phi, \frac{1}{2}a\sin\varphi\right) = \left(a\cos\frac{\pi}{4}, \frac{1}{2}a\sin\frac{\pi}{4}\right)$$
$$= \left(\frac{a}{\sqrt{2}}, \frac{a}{2\sqrt{2}}\right) = \left(\frac{a\sqrt{2}}{2}, \frac{a\sqrt{2}}{4}\right)$$

For P

$$\left(a\cos\phi, \frac{1}{2}a\sin\varphi\right) = \left(a\cos\frac{\pi}{2}, \frac{1}{2}a\sin\frac{\pi}{2}\right)$$
$$= \left(0, \frac{a}{2}\right)$$

For PQ

$$\frac{y - \frac{a}{2}}{\frac{a\sqrt{2}}{4} - \frac{a}{2}} = \frac{x - 0}{\frac{a\sqrt{2}}{2} - 0}$$

Using the formula from module C1 for a line, $\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}$

$$\frac{4y - 2a}{\sqrt{2} - 2} = \frac{2x}{\sqrt{2}}$$

The a cancels throughout the denominators of this equation. On the left-hand side $\frac{4\left(y - \frac{a}{2}\right)}{4\left(\frac{\sqrt{2}}{4} - \frac{1}{2}\right)} = \frac{4y - 2a}{\sqrt{2} - 2}$

$$4\sqrt{2}y - 2\sqrt{2}a = (2\sqrt{2} - 4)x$$
$$\left(4 - 2\sqrt{2}\right)x + 4\sqrt{2}y - 2\sqrt{2}a = 0$$

Dividing throughout by $2\sqrt{2}$

$\left(\sqrt{2} - 1\right)x + 2y - a = 0,$ as required.

45 a $b^2 = a^2(1 - e^2)$

$4 = 9(1 - e^2) = 9 - 9e^2$

$$e^2 = \frac{9 - 4}{9} = \frac{5}{9} \Rightarrow e = \frac{\sqrt{5}}{3}$$

The formulae you need for calculating the eccentricity, the coordinates of the foci, and the equations of the directrices are given in the Edexcel formula booklet you are allowed to use in the examination. However, it wastes time checking your textbook every time you need to use these formulae and it is worthwhile remembering them. **Remember** to quote any formulae you use in your solution.

b The coordinates of the foci are

$$(\pm ae, 0) = \left(\pm 3 \times \frac{\sqrt{5}}{3}, 0\right) = (\pm\sqrt{5}, 0)$$

The equation of the directrices are

$$x = \pm\frac{a}{e} = \pm\frac{3}{\frac{\sqrt{5}}{3}} = \pm\frac{9}{\sqrt{5}}$$

45 c $x = 3\cos\theta,\ y = 2\sin\theta$

$$\frac{dx}{d\theta} = -3\sin\theta, \frac{dy}{d\theta} = 2\cos\theta$$

$$\frac{dy}{dx} = \frac{dy}{d\theta} \times \frac{d\theta}{dx} = \frac{2\cos\theta}{3\sin\theta}$$

$$y - y_1 = m(x - x_1)$$

$$y - 2\sin\theta = \frac{2\cos\theta}{3\sin\theta}(x - 3\cos\theta)$$

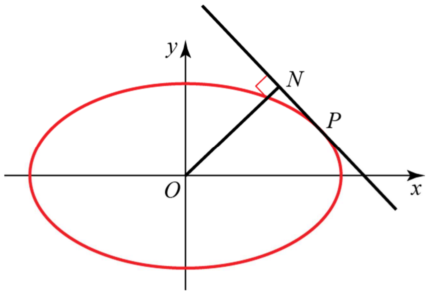
$$3y\sin\theta - 6\sin^2\theta = -2x\cos\theta + 6\cos^2\theta$$

$$2x\cos\theta + 3y\sin\theta = 6(\cos^2\theta + \sin^2\theta) = 6$$

Divide this line throughout by 6

$$\frac{x\cos\theta}{3} + \frac{y\sin\theta}{2} = 1, \text{ as required.}$$

45 d



Let the foot of the perpendicular from O to the tangent at P be N.

Using $mm' = -1$ the gradient of ON is given by

$$m' = -\frac{1}{\frac{dy}{dx}} = \frac{3\sin\theta}{2\cos\theta}$$

An equation of ON is $y = \frac{3\sin\theta}{3\sin\theta}$ *

Eliminating y between equation * and the answer to part c

$$\frac{x\cos\theta}{3} + \frac{\sin\theta}{2}\left(\frac{3\sin\theta}{2\cos\theta}x\right) = 1$$

$$x\left(\frac{4\cos^2\theta + 9\sin^2\theta}{12\cos\theta}x\right) = 1$$

$$x = \frac{12\cos\theta}{4\cos^2\theta + 9\sin^2\theta}$$

$x = \frac{12\cos\theta}{4\cos^2\theta + 9\sin^2\theta}$ and $y = \frac{18\sin\theta}{4\cos^2\theta + 9\sin^2\theta}$ are parametric equations of the locus. Eliminating θ between them to obtain a Cartesian equation is not easy and you will need to use the printed answer to help you.

Substituting this expression for x into equation *

$$y = \frac{3\sin\theta}{2\cos\theta} \times \frac{12\cos\theta}{4\cos^2\theta + 9\sin^2\theta} = \frac{18\sin\theta}{4\cos^2\theta + 9\sin^2\theta}$$

$$x^2 + y^2 = \left(\frac{12\cos\theta}{4\cos^2\theta + 9\sin^2\theta}\right)^2 + \left(\frac{18\sin\theta}{4\cos^2\theta + 9\sin^2\theta}\right)^2$$

$$= \frac{144\cos^2\theta + 324\sin^2\theta}{(4\cos^2\theta + 9\sin^2\theta)^2} = \frac{36(4\cos^2\theta + 9\sin^2\theta)}{(4\cos^2\theta + 9\sin^2\theta)^2}$$

$$= \frac{36}{4\cos^2\theta + 9\sin^2\theta}$$

$$9x^2 + 4y^2 = \frac{9 \times 144\cos^2\theta + 4 \times 324\sin^2\theta}{(4\cos^2\theta + 9\sin^2\theta)^2}$$

$$= \frac{1296\cos^2\theta + 1296\sin^2\theta}{(4\cos^2\theta + 9\sin^2\theta)^2} = \frac{1296}{(4\cos^2\theta + 9\sin^2\theta)^2}$$

$$= \left(\frac{36}{4\cos^2\theta + 9\sin^2\theta}\right)^2 = (x^2 + y^2)^2$$

The locus of N is $(x^2 + y^2)^2 = 9x^2 + 4y^2$, as required.

46 a $x = a\cos\theta,\ y = b\sin\theta$

$$\frac{dx}{d\theta} = -a\sin\theta,\ \frac{dy}{d\theta} = b\cos\theta$$

$$\frac{dy}{dx} = \frac{dy}{d\theta} \times \frac{d\theta}{dx} = -\frac{b\cos\theta}{a\sin\theta}$$

Using $mm' = -1$, the gradient of the normal is given by

$$m' = \frac{a\sin\theta}{b\cos\theta}$$

$$y - y_1 = m'(x - x_1)$$

$$y - b\sin\theta = \frac{a\sin\theta}{b\sin\theta}(x - a\cos\theta)$$

$$by\cos\theta - b^2\sin\theta\cos\theta = ax\sin\theta - a^2\sin\theta\cos\theta$$

$$ax\sin\theta - by\cos\theta = (a^2 - b^2)\sin\theta\cos\theta$$

Divide this equation throughout by sin θ cos θ

$$\frac{ax}{\cos\theta} - \frac{by}{\sin\theta} = a^2 - b^2$$

$ax\sec\theta - by\operatorname{cosec}\theta = a^2 - b^2$, as required

b Substituting $y = 0$ in the result to part **a**

$$ax\sec\theta = a^2 - b^2$$

You find the x-coordinate of G by substituting $y = 0$ into the equation of the normal at P and solving the resulting equation for x.

$$x = \frac{a^2 - b^2}{a}\cos\theta$$

$P: (a\cos\theta,\ b\sin\theta)$, $G: \left(\frac{a^2 - b^2}{a}\cos\theta,\ 0\right)$

The coordinates $(x_M,\ y_M)$ of M the midpoint of PG are given by

$$\left(\frac{x_1 + x_2}{2},\ \frac{y_1 + y_2}{2}\right).$$

$$x_M = \frac{a\cos\theta + \frac{a^2 - b^2}{a}\cos\theta}{2}$$

$$= \frac{\cos\theta}{2}\left(\frac{a^2 + a^2 - b^2}{a}\right) = \left(\frac{2a^2 - b^2}{2a}\right)\cos\theta$$

$$y_M = \frac{b\sin\theta + 0}{2} = \frac{b\sin\theta}{2}$$

Hence, the coordinates of M are

$\left[\left(\frac{2a^2 - b^2}{2a}\right)\cos\theta,\ \left(\frac{b}{2}\right)\sin\theta\right]$, as required.

46 c For M

$$x=\left(\frac{2a^2-b^2}{2a}\right)\cos\theta,\ y=\left(\frac{b}{2}\right)\sin\theta$$

$$\cos\theta\frac{x}{\left(\frac{2a^2-b^2}{2a}\right)},\ \sin\theta=\frac{y}{\left(\frac{b}{2}\right)}$$

$$\cos^2\theta+\sin^2\theta=1$$

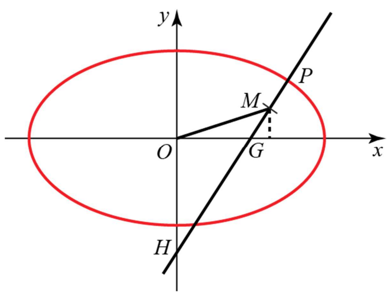
$$\frac{x^2}{\left(\frac{2a^2-b^2}{2a}\right)^2}+\frac{y^2}{\left(\frac{b}{2}\right)^2}=1$$

Any curve with an equation of the form $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is an ellipse. If you are asked to show that a locus is an ellipse, it is sufficient to show that it has a Cartesian equation of this form.

This is an ellipse. A Cartesian equation of this ellipse is

$$\frac{4a^2x^2}{\left(2a^2-b^2\right)^2}+\frac{4y^2}{b^2}=1$$

d



Substituting $x = 0$ into the equation of the normal

$$-by\operatorname{cosec}\theta=a^2-b^2\Rightarrow y=-\frac{a^2-b^2}{b}\sin\theta$$

Hence $OH=\frac{a^2-b^2}{b}\sin\theta$

$$\frac{\text{area }\Delta OMG}{\text{area }\Delta OGH}=\frac{y-\text{coordinate of }M}{OH}$$

$$=\frac{\left(\frac{b}{2}\right)\sin\theta}{\frac{a^2-b^2}{b}\sin\theta}$$

$$=\frac{b^2}{2(a^2-b^2)}\text{, as required.}$$

The triangles OMG and OGH can be looked at as having the same base OG.
As the area of a triangle is $\frac{1}{2}\times\text{base}\times\text{height}$, triangles with the same base will have areas proportional to their heights. The height of the triangle OGM is shown by a dotted line in the diagram and is given by the y-coordinate of M.

47 In the first Cartesian quadrant, the ellipse is the graph of the function $y = 4\sqrt{1-\frac{x^2}{8^2}}$

The area it encloses for x greater than 4 is $4\int_4^8 \sqrt{1-\frac{x^2}{8^2}}\,\mathrm{d}x$; we solve the integral by putting $x = 8\sin u$, getting:

$$4\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 8\cos^2 u \,\mathrm{d}u =$$
$$= 16\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 1+\cos 2u \,\mathrm{d}u =$$
$$= \frac{16\pi}{3} + 16\left[-\frac{\sqrt{3}}{4}\right] =$$
$$= \frac{16\pi}{3} - 4\sqrt{3}$$

In order to get the total area we need the area of the triangle PQO, where O is the origin and Q is the projection of P on the x-axis. The area of this triangle is easily $4\sqrt{3}$: therefore, the area of the shaded region is $\frac{16}{3}\pi$ and $a = \frac{16}{3}$

48 a Substituting $y = mx + c$ into $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$$\frac{x^2}{a^2} + \frac{(mc+c)^2}{b^2} = 1$$
$$b^2x^2 + a^2(mx+c)^2 = a^2b^2$$
$$b^2x^2 + a^2m^2x^2 + 2a^2mxc + a^2c^2 = a^2b^2$$
$$(a^2m^2 + b^2)x^2 + 2a^2mcx + a^2(c^2 - b^2) = 0$$

Multiply this equation throughout by a^2b^2
Then multiply out the brackets and collect the terms together as a quadratic in x.

As the line is a tangent this equation has repeated roots

'$b^2 - 4ac = 0$'

$$4a^4m^2c^2 - 4(a^2m^2 + b^2)a^2(c^2 - b^2) = 0$$
$$a^2m^2c^2 - (a^2m^2 + b^2)(c^2 - b^2) = 0$$
$$\cancel{a^2m^2c^2} - \cancel{a^2m^2c^2} + a^2m^2b^2 - b^2c^2 + b^4 = 0$$

Divide this equation throughout by b^2 and then rearrange to make c^2 the subject of the formula.

$c^2 = a^2m^2 + b^2$, as required.

48 b $(3,4) \in y = mx + c$

Hence $4 = 3m + c \Rightarrow c = 4 - 3m$ **(1)**

The tangents have equations of the form $y = mx + c$ and $x = 3$, $y = 4$ must satisfy this relation.

For this ellipse, $a = 4$ and $b = 5$ and the result in part **a** becomes

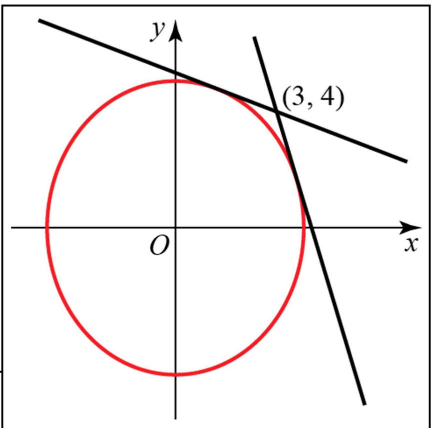
$c^2 = 16m^2 + 25$ **(2)**

Substituting **(1)** into **(2)**

$$(4-3m)^2 = 16m^2 + 25$$
$$16 - 24m + 9m^2 = 16m^2 + 25$$
$$7m^2 + 24m + 9 = (m+3)(7m+3) = 0$$
$$m = -3, -\frac{3}{7}$$

If $m = -3$, $c = 4 - 3m = 4 + 9 = 13$

If $m = -\frac{3}{7}$, $c = 4 - 3m = 4 + \frac{9}{7} = \frac{37}{7}$



There are two tangents to the ellipse which pass through (3, 4)
Both have negative gradients.

The equations of the tangents are

$$y = -3x + 13 \text{ and } y = -\frac{3}{7}x + \frac{37}{7}$$

49 a Substituting $y = mx + c$ into $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$$\frac{x^2}{a^2} + \frac{(mx+c)^2}{b^2} = 1$$
$$b^2x^2 + a^2(mx+c)^2 = a^2b^2$$
$$b^2x^2 + a^2m^2x^2 + 2a^2mxc + a^2c^2 = a^2b^2$$
$$(b^2 + a^2m^2)x^2 + 2a^2mcx + a^2(c^2 - b^2) = 0, \text{ as required.}$$

Multiply this equation throughout by a^2b^2
Then multiply out the brackets and collect the terms together as a quadratic in x.

b As the line is a tangent the result of part **a** has repeated roots

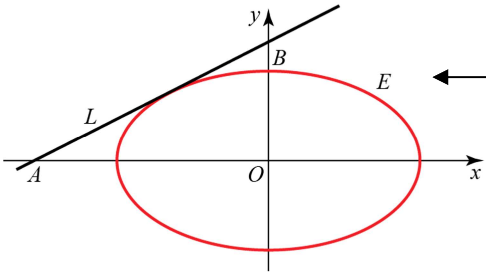
'$b^2 - 4ac = 0$'

$$4a^4m^2c^2 - 4(b^2 + a^2m^2)a^2(c^2 - b^2) = 0$$
$$a^2m^2c^2 - (b^2 + a^2m^2)(c^2 - b^2) = 0$$
$$a^2m^2c^2 - b^2c^2 + b^4 - a^2m^2c^2 + a^2m^2b^2 = 0$$
$$c^2 = a^2m^2 + b^2, \text{ as required.}$$

Divide this equation throughout by $4a^2$

Divide this equation throughout by b^2 and then rearrange to make c^2 the subject of the formula.

49 c



As $c^2 = a^2m^2 + b^2$, $y = mx + c$ could have the forms

$y = mx \pm \sqrt{(b^2 + a^2m^2)}$

However, the question specifies that the tangent crosses the positive y-axis. As the line has a positive y intercept, you can reject the negative possibility.

An equation of L is $y = mx + \sqrt{b^2 + a^2m^2}$

For A $y = 0$

$0 = mx + \sqrt{b^2 + a^2m^2} \Rightarrow x = -\dfrac{\sqrt{b^2 + a^2m^2}}{m}$

Hence $OA = \dfrac{\sqrt{b^2 + a^2m^2}}{m}$

For B $x = 0$

$y = \sqrt{b^2 + a^2m^2}$

Hence $OB = \sqrt{b^2 + a^2m^2}$

The area of triangle OAB, T say, is given by

$$T = \frac{1}{2} OA \times OB = \frac{1}{2} \frac{\sqrt{b^2 + a^2m^2}}{m} \sqrt{b^2 + a^2m^2}$$
$$= \frac{b^2 + a^2m^2}{2m}$$

49 d $T = \frac{b^2 + a^2m^2}{2m} = \frac{1}{2}b^2m^{-1} + \frac{1}{2}a^2m$

For a minimum

$\frac{dT}{dm} = -\frac{1}{2}b^2m^{-2} + \frac{1}{2}a^2 = 0$

$\frac{b^2}{m^2} = a^2 \Rightarrow m^2 = \frac{b^2}{a^2}$

As L has a positive gradient

$m = \frac{b}{a}$

The diagram shows that the tangent has a positive gradient and so the possible value $-\frac{b}{a}$ can be ignored.

$\frac{d^2T}{dm^2} = b^2m^{-3} = \frac{b^2}{m^3}$

At $m = \frac{b}{a}, \frac{d^2T}{dm^2} = \frac{b^2}{m^3} = \frac{a^3}{b} > 0$ and so this gives a minimum value of

$T = \frac{b^2 + a^2\left(\frac{b}{a}\right)^2}{2\left(\frac{b}{a}\right)} = \frac{2b^2}{2\left(\frac{b}{a}\right)} = ab$, as required.

e At $m = \frac{b}{a}, c^2 = a^2m^2 + b^2 = a^2\left(\frac{b}{a}\right)^2 + b^2 = 2b^2$

Substituting $m = \frac{b}{a}$ and $c = \sqrt{2}b$ into the result in part **a**.

$\left(b^2 + a^2 \times \frac{b^2}{a^2}\right)x^2 + 2a^2 \times \frac{b}{a} \times \sqrt{2}bx + a^2(2b^2 - b^2) = 0$

$2b^2x^2 + 2ab^2\sqrt{2}x + a^2b^2 = 0$

Divide this equation throughout by b^2

$2x^2 + 2a\sqrt{2}x + a^2 = 0$

$\left(\sqrt{2}x + a\right)^2 = 0$

As the line is a tangent, this quadratic must factorise to a complete square. If you cannot see the factors, you can use the quadratic formula.

$x = -\frac{a}{\sqrt{2}}$

50 a To find an equation of the tangent at P.

$x = \cosh t, y = \sinh t$

$$\frac{dx}{dt} = \sinh t, \quad \frac{dy}{dt} = \cosh t$$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = \frac{\cosh t}{\sinh t}$$

Using $y - y_1 = m(x - x_1)$

$$y - \sinh t = \frac{\cosh t}{\sinh t}(x - \cosh t)$$

$$y \sinh t - \sinh^2 t = x \cosh t - \cosh^2 t$$

$$y \sinh t = x \cosh t - (\cosh^2 t - \sinh^2 t)$$

$$= x \cosh t - 1$$

Using the identity $\cosh^2 t - \sinh^2 t = 1$

$x \cosh t - y \sinh t = 1$ **(1)**

To find the equation of the normal at P.

Using $mm' = -1$, the gradient of the normal is given by

$$m' = -\frac{\sinh t}{\cosh t}$$

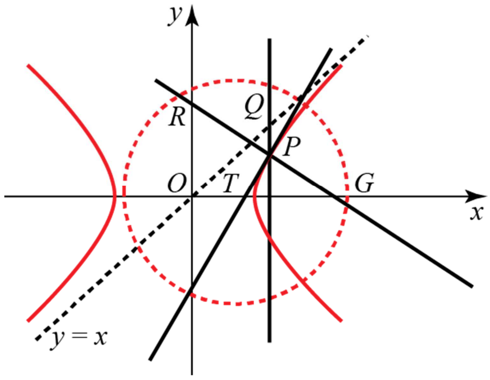
$$y - y_1 = m'(x - x_1)$$

$$y - \sinh t = -\frac{\sinh t}{\cosh t}(x - \cosh t)$$

$$y \cosh t - \sinh t \cosh t = -x \sinh t + \sinh t \cosh t$$

$x \sinh t + y \cosh t = 2 \sinh t \cosh t$ **(2)**

50 b



Substitute $y = 0$ into **(2)**

$x\sinh t = 2\sinh t\cosh t$

$x = 2\cosh t$

The coordinates of G are (2 cosh t, 0)

The x-coordinate of Q is cosh t

The asymptote in the first quadrant has equation $y = x$

Hence the coordinates of Q are (cosh t, cosh t)

The gradient of GQ is given by $\frac{y_1 - y_2}{x_1 - x_2} = \frac{0 - \cosh t}{2\cosh t - \cosh t} = -1$

As the gradient of $y = x$ is 1 and $1 \times -1 = -1$, GQ is perpendicular to the asymptote.

To find the coordinates of G, you substitute $y = 0$ into the equation of the normal found in part **a**.

To asymptotes to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ are $y = \pm\frac{b}{a}x$

These formulae are given in the Edexcel formulae booklet. With the hyperbola $a = b = 1$ and the asymptotes are $y = \pm x$

The asymptote in the first quadrant has equation $y = x$

c Substitute $y = 0$ into **(1)**

$x\cosh t = 1 \Rightarrow x = \frac{1}{\cosh t}$

The coordinates of T are $\left(\frac{1}{\cosh t}, 0\right)$

Substitute $x = 0$ into **(2)**

$y\cosh t = 2\sinh t\cosh t \Rightarrow y = 2\sinh t$

The coordinates of R are $(0, 2\sinh t)$

$TG = 2\cosh t - \frac{1}{\cosh t}$

$TR^2 = OR^2 + OT^2 = (2\sinh t)^2 + \left(\frac{1}{\cosh t}\right)^2$

$= 4\sinh^2 t + \frac{1}{\cosh^2 t} = 4(\cosh^2 t - 1) + \frac{1}{\cosh^2 t}$

$= 4\cosh^2 t - 4 + \frac{1}{\cosh^2 t}$

$= \left(2\cosh t - \frac{1}{\cosh t}\right)^2 = TG^2$

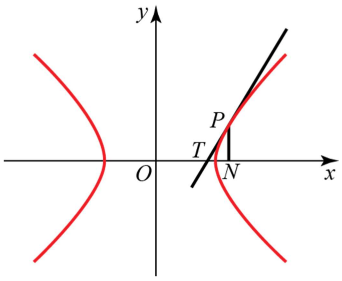
Hence $TR = TG$ and R lies on the circle with centre at T and radius TG.

To find the coordinates of T, you substitute $y = 0$ into the equation of the tangent found in part **a**.

To find the coordinates of R, you substitute $x = 0$ into the equation of the normal found in part **a**.

If a circle can be drawn through R with centre T and radius TG then TR must also be a radius of the circle. So you can solve the problem by showing that TR and TG have the same length.

51



To find the coordinates of T, it is easiest to carry out your calculation in terms of a parameter. As the question specifies no particular parametric form, you can choose your own. The hyperbolic form has been used here but ($a \sec t$, $b \tan t$) would work as well and there are other possible alternatives.

Let the point P have coordinates $(a\cosh t, b\sinh t)$

To find an equation of the tangent PT,

$$\frac{dx}{dt} = a\sinh t, \frac{dy}{dt} = b\cosh t$$

$$\frac{dy}{dx} = \frac{dy}{dt}\times\frac{dt}{dx} = \frac{dy}{dt}\div\frac{dx}{dt} = \frac{b\cosh t}{a\sinh t}$$

Using $y - y_1 = m(x - x_1)$

$$y - b\sinh t = \frac{b\cosh t}{a\sinh t}(x - a\cosh t)$$

To find the x-coordinate of T, you substitute $y = 0$ into an equation of the tangent at P, so first you must obtain an equation for the tangent.

$$ay\sinh t - ab\sinh^2 t = bx\cosh t - ab\cosh^2 t$$

$$ay\sinh t = bx\cosh t - ab(\cosh^2 t - \sinh^2 t)$$

$$= bx\cosh t - ab$$

Using the identity $\cosh^2 t - \sinh^2 t = 1$

For T, $y = 0$

$$bx\cosh t = ab \Rightarrow x = \frac{a}{\cosh t}$$

The coordinates of N are ($a \cosh t$, 0)

$$OT \cdot ON = \frac{a}{\cosh t}\times a\cosh t = a^2, \text{ as required.}$$

52 a $\frac{dx}{dt} = a\sec t\tan t, \frac{dy}{dt} = b\sec^2 t$

$$\frac{dy}{dx} = \frac{b\sec^2 t}{a\sec t\tan t} = \frac{b\sec t}{a\tan t} = \frac{b}{a\sin t}$$

To find the gradient of the tangent, you use a version of the chain rule $\frac{dy}{dx} = \frac{dy}{dt}\times\frac{dt}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$

Using $mm' = -1$, the gradient of the normal

is given by $m' = -\frac{a\sin t}{b}$

An equation of the normal is

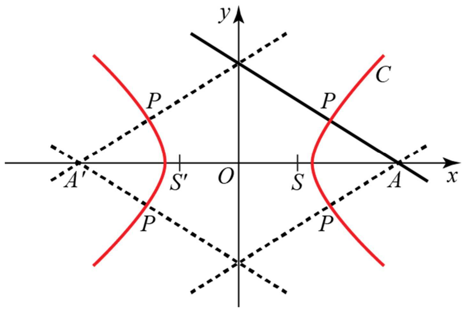
$$y - y_1 = m'(x - x_1)$$

$$y - b\tan t = -\frac{a\sin t}{b}(x - a\sec t)$$

$$by - b^2\tan t = -ax\sin t + a^2\tan t$$

$$ax\sin t + by = (a^2 + b^2)\tan t, \text{ as required.}$$

52 b



A diagram is essential here. Without it, you would be unlikely to see that there are four possible point where $OA = 3OS$

There are two to the right of the y-axis, corresponding to the focus S with coordinates $(ae, 0)$, and two to the left of the y-axis, corresponding to the focus, here marked S', with coordinates $(-ae, 0)$

The x-coordinate of A is given by

$ax\sin t + 0 = (a^2 + b^2)\tan t$

$$x = \frac{a^2 + b^2}{a} \times \frac{\tan t}{\sin t} = \frac{a^2 + b^2}{a\cos t}$$

Hence $OA = \dfrac{a^2 + b^2}{a\cos t}$

Using $b^2 = a^2(e^2 - 1)$ with $e = \dfrac{3}{2}$

$$b^2 = a^2\left(\frac{9}{4} - 1\right) = \frac{5a^2}{4}$$

and $OA = \dfrac{a^2 + b^2}{a\cos t} = \dfrac{a^2 + \frac{5a^2}{4}}{a\cos t} = \dfrac{9a}{4\cos t}$

You need to eliminate b from the length OA to obtain a solvable equation in t from the condition $OA = 3AS$

As $e = \dfrac{3}{2}$,

$$OS = ae = \frac{3a}{2}$$

$$OA = 3OS$$

$$\frac{9a}{4\cos t} = \frac{9a}{2} \Rightarrow \cos t = \frac{1}{2}$$

$$t = \frac{\pi}{3}, \frac{5\pi}{3}$$

These values give two points P, $(2a, \sqrt{3}b)$ and $(2a, -\sqrt{3}b)$

These are the solutions in the first and fourth quadrants.

From the diagram, by symmetry, there are also solutions in the second and third quadrants giving

$$t = \frac{2\pi}{3}, \frac{4\pi}{3}$$

These correspond to the two points $(-2a, \sqrt{3}b)$ and $(-2a, -\sqrt{3}b)$ where $\cos t = -\dfrac{1}{2}$

The possible values of t are

$$t = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$$

53 a $\frac{x^2}{a^2}-\frac{y^2}{a^2}=1$

$b^2=a^2(e^2-1)$

For this hyperbola $b^2=a^2$

$a^2=a^2(e^2-1)\Rightarrow 1=e^2-1\Rightarrow e^2=2$

$e=\sqrt{2}$, as required.

> $x^2-y^2=a^2\Rightarrow \frac{x^2}{a^2}-\frac{y^2}{a^2}=1$
>
> This is an hyperbola in which $a=b$

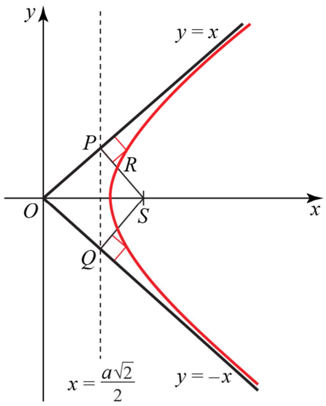
b The coordinates of S are

$(ae,0)=(a\sqrt{2},0)$

An equation of L is

$x=\frac{a}{e}=\frac{a}{\sqrt{2}}=\frac{a\sqrt{2}}{2}$

c



SP is perpendicular to $y=x$, so its gradients is -1

An equation of SP is

$y=-1(x-a\sqrt{2})=-x+a\sqrt{2}$

$y+x=a\sqrt{2}$

SP meets $y=x$ where

$x+x=a\sqrt{2}\Rightarrow x=\frac{a\sqrt{2}}{2}$

Hence P is on the directrix L.

> The asymptotes to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ are $y=\pm\frac{b}{a}x$
>
> These formulae are given in the Edexcel formulae booklet. With this hyperbola $a=b$ and the asymptotes are $y=\pm x$
>
> This question is about the intersection of line with the asymptotes. The lines $y=x$ and $y=-x$ are perpendicular to each other and a hyperbola with perpendicular asymptotes is called a rectangular hyperbola. In Module FP 1, you studied another rectangular hyperbola, $xy=c^2$

SQ is perpendicular to $y=-x$, so its gradient is 1

An equation of SQ is

$y=1(x-a\sqrt{2})=x-a\sqrt{2}$

$y=x-a\sqrt{2}$

SQ meets $y=-x$ where $-x=x-a\sqrt{2}\Rightarrow x=\frac{a\sqrt{2}}{2}$

Hence Q is on the directrix L.

Both P and Q lie on the directrix L.

The coordinates of P are $\left(\frac{a\sqrt{2}}{2},\frac{a\sqrt{2}}{2}\right)$

The coordinates of Q are $\left(\frac{a\sqrt{2}}{2},-\frac{a\sqrt{2}}{2}\right)$

53 d $SP: y + x = a\sqrt{2}$ **(1)**

Hyperbola $x^2 - y^2 = a^2$ **(2)**

Form **(1)** $y = a\sqrt{2} - x$ **(3)**

Substitute **(3)** into **(2)**

$$x^2 - (a\sqrt{2} - x)^2 = a^2$$

$$x^2 - 2a^2 + 2\sqrt{2}ax - x^2 = a^2$$

$$2\sqrt{2}ax = 3a^2 \Rightarrow x = \frac{3a}{2\sqrt{2}} = \frac{3\sqrt{2}}{4}a$$

To find the coordinates of R, you solve equations **(1)** and **(2)** simultaneously.

Substituting for x in **(3)**

$$y = a\sqrt{2} - \frac{3\sqrt{2}}{4}a = \frac{\sqrt{2}}{4}a$$

The coordinates of R are $\left(\frac{3\sqrt{2}}{4}a, \frac{\sqrt{2}}{4}a\right)$

To find the tangent to the hyperbola at R

$$x^2 - y^2 = a^2$$

$$2x - 2y\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{x}{y}$$

Differentiating the equation of the hyperbola implicitly with respect to x.

At R

$$\frac{dy}{dx} = \frac{x}{y} = \frac{\frac{3\sqrt{2}}{4}a}{\frac{\sqrt{2}}{4}a} = 3$$

$$y - y_1 = m(x - x_1)$$

$$y - \frac{\sqrt{2}}{4}a = 3\left(x - \frac{3\sqrt{2}}{4}a\right) = 3x - \frac{9\sqrt{2}}{4}a$$

$$y = 3x - 2\sqrt{2}a$$

This is the equation of the tangent to the hyperbola at R. To establish that R passes through Q, you substitute the x-coordinate of Q into this equation and show that this gives the y-coordinate of Q.

At $x = \frac{a\sqrt{2}}{2}$, $y = 3\left(\frac{a\sqrt{2}}{2}\right) - 2\sqrt{2}a = -\frac{a\sqrt{2}}{2}$

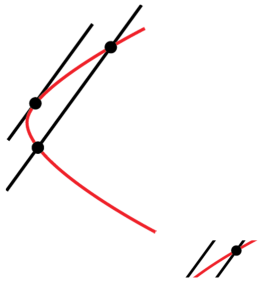
This is the y-coordinate of Q.

Hence the tangent at R passes through Q.

54 Let the equation of the tangent be $y = mx + c$

Eliminating y between $y = mx + c$ and $x^2 - 4y^2 = 4$

$$x^2 - 4(mx + c)^2 = 4$$
$$x^2 - 4m^2x^2 - 8mcx - 4c^2 = 4$$
$$(4m^2 - 1)x^2 + 8mcx + 4(c^2 + 1) = 0 \quad *$$



If the line was a chord, it would cut the curve in two distinct points and this equation would have a positive discriminant. As the line is a tangent it touches the curve at just one point and this equation has a repeated root. The discriminant is zero.

As the line is a tangent, equation * has repeated roots

'$b^2 - 4ac = 0$'

$$64m^2c^2 - 16(4m^2 - 1)(c^2 + 1) = 0$$
$$64m^2c^2 - 64m^2c^2 - 64m^2 + 16c^2 + 16 = 0$$
$$16c^2 = 64m^2 - 16$$
$$c^2 = 4m^2 - 1 \Rightarrow c = \pm\sqrt{(4m^2 - 1)}$$

The equation of the tangent is

$y = mx \pm \sqrt{(4m^2 - 1)}$, where $|m| > \frac{1}{2}$, as required.

If $|m| < \frac{1}{2}$, then $\sqrt{(4m^2 - 1)}$ would be the square root of a negative number and there would be no real answer. The cases $m = \pm\frac{1}{2}$ are interesting. For these values the equations are $y = \pm\frac{1}{2}x$

These are the asymptotes of the hyperbola and do not touch it at any point with finite coordinates. Asymptotes can be thought of as tangents to the curve 'at infinity'.

55 a $x = a\cos t,\ \ y = b\sin t$

$$\frac{dx}{dt} = -a\sin t, \frac{dy}{dt} = b\cos t$$
$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = -\frac{b\cos t}{a\sin t}$$

For the tangent

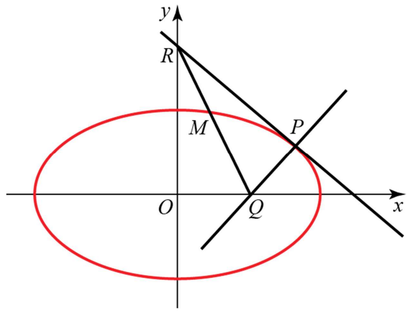
$$y - y_1 = m(x - x_1)$$
$$y - b\sin t = -\frac{b\cos t}{a\sin t}(x - a\cos t)$$
$$ay\sin t - ab\sin^2 t = -bx\cos t + ab\cos^2 t$$
$$ay\sin t + bx\cos t = ab(\sin^2 t + \cos^2 t)$$
$$ay\sin t + bx\cos t = ab$$

As the question asks for no particular form for the equation of the tangent this is an acceptable form for the answer. However the calculation in part **c** will be easier if you simplify the equation at this stage using $\sin^2 t + \cos^2 t = 1$

b As $\frac{dy}{dx} = -\frac{b\cos t}{a\sin t}$, using $mm' = -1$, the gradient of the normal is given by

$$m' = \frac{a\sin t}{b\cos t}$$
$$y - y_1 = m'(x - x_1)$$
$$y - b\sin t = \frac{a\sin t}{b\cos t}(x - a\cos t)$$
$$by\cos t - b^2\sin t\cos t = ax\sin t - a^2\sin t\cos t$$
$$ax\sin t - by\cos t = (a^2 - b^2)\sin t\cos t$$

55 c



The condition $0 < t < \frac{\pi}{2}$ implies that P is in the first quadrant.

Substituting $y = 0$ into the answer to part **b**

$ax\sin t = (a^2 - b^2)\sin t\cos t \Rightarrow x = \frac{a^2 - b^2}{a}\cos t$

You find the x-coordinate of Q by substituting $y = 0$ into the equation you found for the normal in part **b** and solving for x.

The coordinates of Q are $\left(\frac{a^2 - b^2}{a}\cos t, 0\right)$

Substituting $x = 0$ into the answer to part **a**

$ay\sin t = ab \Rightarrow y = \frac{b}{\sin t}$

You find the y-coordinate of R by substituting $x = 0$ into the equation you found for the tangent in part **a** and solving for y.

The coordinates of R are $\left(0, \frac{b}{\sin t}\right)$

The coordinates of M are given by

$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) = \left(\frac{a^2 - b^2}{2a}\cos t, \frac{b}{2\sin t}\right)$

d If the coordinates of M are (x, y) then $x = \frac{a^2 - b^2}{2a}\cos t \Rightarrow \cos t = \frac{2ax}{a^2 - b^2}$ and

$y = \frac{b}{2\sin t} \Rightarrow \sin t = \frac{b}{2y}$

As $\cos^2 t + \sin^2 t = 1$, the locus of

M is $\left(\frac{2ax}{a^2 - b^2}\right)^2 + \left(\frac{b}{2y}\right)^2 = 1$, as required.

$x = \frac{a^2 - b^2}{2a}\cos t$ and $y = \frac{b}{2\sin t}$ are the parametric equations of the locus of M. To find the Cartesian equation, you must eliminate t. The form of the answer given in the question gives you a hint that you can use the identity $\cos^2 t + \sin^2 t = 1$ to do this.

56 a To find the equation of the tangent at $(a\sec\theta, b\tan\theta)$

$$x = a\sec\theta, \quad y = b\tan\theta$$

$$\frac{dx}{d\theta} = a\sec\theta\tan\theta, \frac{dy}{dt} = b\sec^2\theta$$

$$\frac{dy}{dx} = \frac{b\sec^2\theta}{a\sec\theta\tan\theta} = \frac{b\sec\theta}{a\tan\theta} = \frac{b}{a\sin\theta}$$

$$y - y_1 = m(x - x_1)$$

$$y - b\tan\theta = \frac{b}{a\sin\theta}(x - a\sec\theta)$$

As the question asks for no particular form for the equation of the tangent this is an acceptable form for the answer. However, the calculation in part **b** will be easier if you simplify the equation at this stage.

$$ay\sin\theta - \frac{ab\sin^2\theta}{\cos\theta} = bx - ab\sec\theta$$

$$bx - ay\sin\theta = ab\left(\frac{1-\sin^2\theta}{\cos\theta}\right) = ab\frac{\cos^2\theta}{\cos\theta}$$

$$bx - ay\sin\theta = ab\cos\theta \quad \textbf{(1)}$$

To find the equation of the normal at $(a\sec\theta, b\tan\theta)$

Using $mm' = -1$, the gradient of the normal is given by

$$m' = -\frac{a\sin\theta}{b}$$

$$y - y_1 = m'(x - x_1)$$

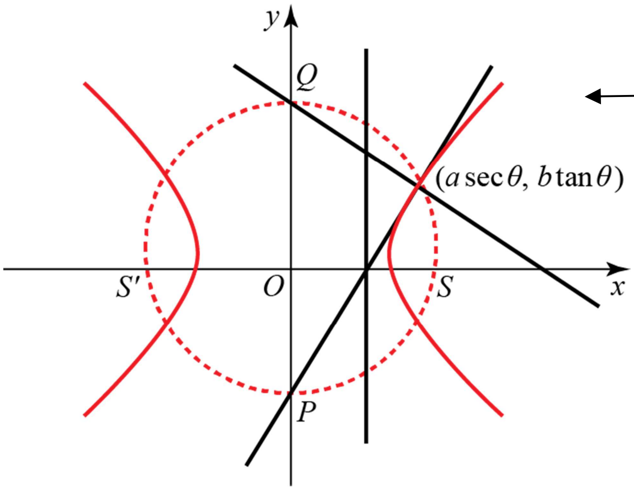
$$y - b\tan\theta = -\frac{a\sin\theta}{b}(x - a\sec\theta)$$

When you multiply the brackets out, $\sin\theta\sec\theta = \frac{\sin\theta}{\cos\theta} = \tan\theta$

$$by - b^2\tan\theta = -ax\sin\theta + a^2\tan\theta$$

$$ax\sin\theta + by = (a^2 + b^2)\tan\theta \quad \textbf{(2)}$$

56 b



This problem will be solved using the property that the angle in a semi-circle is a right angle and you need to show that *PS* and *QS* are perpendicular. All five of the points, *P*, *Q*, $(a\sec\theta, b\tan\theta)$ and the two foci lie on the same circle.

Substitute $x = 0$ into **(1)**
$-ay\sin\theta = ab\cos\theta \Rightarrow y = -b\cot\theta$

To find the coordinates of *P*, you substitute $x = 0$ into the equation of the tangent found in part **a**.

The coordinates of the *P* are $(0, -b\cot\theta)$

Substitute $x = 0$ into **(2)**

$$by = (a^2 + b^2)\tan\theta \Rightarrow y = \frac{a^2 + b^2}{b}\tan\theta$$

To find the coordinates of *Q*, you substitute $x = 0$ into the equation of the normal found in part **a**.

The coordinates of *Q* are $\left(0, \dfrac{a^2 + b^2}{b}\tan\theta\right)$

The focus *S* has coordinates $(ae, 0)$

The gradient of *PS* is given by $m = \dfrac{y_1 - y_2}{x_1 - x_2} = \dfrac{-b\cot\theta - 0}{0 - ae} = \dfrac{b}{ae}\cot\theta$

The gradient of *QS* is given by

$$m' = \frac{y_1 - y_2}{x_1 - x_2} = \frac{\frac{a^2 + b^2}{b}\tan\theta - 0}{0 - ae} = \frac{-(a^2 + b^2)}{abe}\tan\theta$$

$$mm' = \frac{b}{ae}\cot\theta \times -\frac{a^2 + b^2}{abe}\tan\theta = -\frac{a^2 + b^2}{a^2e^2}$$

The formula for the eccentricity is

$$b^2 = a^2(e^2 - 1)$$
$$b^2 = a^2e^2 - a^2 \Rightarrow a^2e^2 = a^2 + b^2$$

Hence $mm' = -\dfrac{a^2 + b^2}{a^2e^2} = -\dfrac{a^2 + b^2}{a^2 + b^2} = -1$

So *PS* is perpendicular to *QS* and $\angle PSQ = 90°$
By the converse of the theorem that the angle in a semi-circle is a right angle, the circle described on *PQ* as diameter passes through the focus *S*.
By symmetry, the circle also passes through the focus S'.

There is no need to repeat the calculations for PS' and QS'. It is evident from the diagram that the whole diagram is symmetrical about the *y*-axis, so, if the circle passes through *S*, it passes through S'. It is quite acceptable to appeal to symmetry to complete your proof.

57 The inequality can be solved as follows:

$$\frac{2}{x-2} < \frac{1}{x+1}$$
$$\frac{2}{x-2} - \frac{1}{x+1} < 0$$
$$\frac{2x+2-x+2}{(x-2)(x+1)} < 0$$
$$\frac{x+4}{(x-2)(x+1)} < 0$$

The inequality is satisfied when numerator and denominator do not have the same sign. The numerator is positive for $x > -4$, while the denominator is positive for $x > 2$ or $x < -1$

Therefore, the inequality holds for $x < -4$ or for $-1 < x < 2$

58
$$\frac{x^2}{x-2} > 2x$$
$$\frac{x^2}{x-2} - 2x > 0$$
$$\frac{x^2 - 2x(x-2)}{x-2} > 0$$
$$\frac{4x - x^2}{x-2} > 0$$
$$\frac{x(4-x)}{x-2} > 0$$

You collect the terms together on one side of the inequality, write the expression as a single fraction and factorise the result as far as possible.

Considering $f(x) = \frac{x(4-x)}{x-2}$,

the critical values are $x = 0$, 2 and 4

You find the critical values by solving the numerator equal to zero and the denominator equal to zero. In this case the numerator = 0, gives $x = 0$, 4 and the denominator gives $x = 2$

	$x < 0$	$0 < x < 2$	$2 < x < 4$	$4 < x$
Sign of f(x)	+	–	+	–

For example if $4 < x$, then $\frac{x(4-x)}{x-2} = \frac{\text{positive} \times \text{negative}}{\text{positive}}$, which is negative.

The solution of $\frac{x^2}{x-2} > 2x$ is

$$\{x : x < 0\} \cup \{x : 2 < x < 4\}$$

59 $\frac{x^2-12}{x} > 1$

Multiply both sides by x^2

$\frac{x^2-12}{x} \times x^2 > x^2$

$x(x^2-12)-x^2 > 0$

$x^3-12x-x^2 > 0$

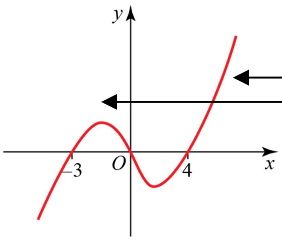
$x(x^2-x-12) > 0$

$x(x-4)(x+3) > 0$

x cannot be zero as $\frac{x^2-12}{x}$ would be undefined, so x^2 is positive and you can multiply both sides of an inequality by a positive number or expression without changing the inequality. You could **not** multiply both sides of the inequality by x as x could be positive or negative.

Sketching $y = x(x-4)(x+3)$

The graph of $y = x(x-4)(x+3)$ crosses the y axis at $x = -3$, 0 and 4



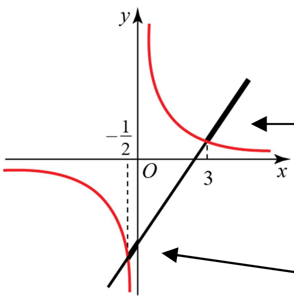
You can see from the sketch that the graph is above the x-axis for $-3 < x < 0$ and $x > 4$

You can then just write down this answer.

The solution of $\frac{x^2-12}{x} > 1$ is $\{x : -3 < x < 0\} \cup \{x : x > 4\}$

If you preferred, you could solve this question using the method illustrated in the solutions to questions 2 and 3 above.

60



Both $y = 2x-5$ and $y = \frac{3}{x}$ are straightforward graphs to sketch and so this is a suitable question for a graphical method. The question, however, specifies no method and so you can use any method which gives an exact answer.

After sketching the two graphs, $2x-5 > \frac{3}{x}$ is the set of values of x for which the line is above the curve. These parts of the line have been drawn thickly on the sketch.

$2x-5 = \frac{3}{x}$

$x(2x-5) = 3$

$2x^2-5x-3 = 0$

$(2x+1)(x-3) = 0$

$x = -\frac{1}{2}, 3$

You need to find the x-coordinates of the points where the line and curve meet to find two end points of the intervals. The other end point ($x = 0$) can be seen by inspecting the sketch.

The solution to $2x-5 > \frac{3}{x}$ is $\left\{x : -\frac{1}{2} < x < 0\right\} \cup \{x : x > 3\}$

which in set notation can be written as

$\left\{x : -\frac{1}{2} < x < 0\right\} \cup \{x : x > 3\}$

61 $$\frac{x+k}{x+4k} > \frac{k}{x}$$

$$\frac{x+k}{x+4k} - \frac{k}{x} > 0$$

$$\frac{(x+k)x - k(x+4k)}{(x+4k)x} > 0$$

$$\frac{x^2 - 4k^2}{(x+4k)x} > 0$$

$$\frac{(x+2k)(x-2k)}{(x+4k)x} > 0$$

Considering $f(x) = \frac{(x+2k)(x-2k)}{(x+4k)x}$,

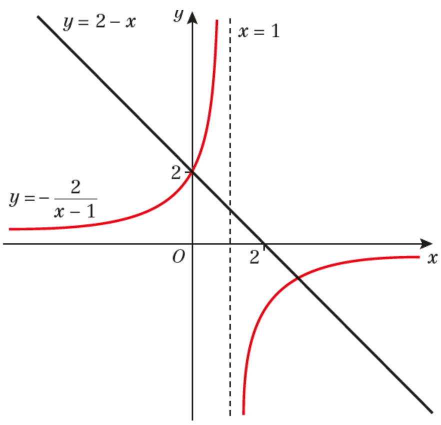
the critical values are $x = -4k, -2k, 0$ and $2k$

For example, when k is positive, in the interval $0 < x < 2k$, $\frac{(x+2k)(x-2k)}{(x+4k)x} = \frac{\text{positive} \times \text{negative}}{\text{positive} \times \text{positive}}$, which is negative.

	$x < -4k$	$-4k < x < -2k$	$-2k < x < 0$	$0 < x < 2k$	$2k < x$
Sign of f(x)	+	–	+	–	+

The solution of $\frac{x+k}{x+4k} > \frac{k}{x}$ is $\{x : x < -4k\} \cup \{x : -2k < x < 0\} \cup \{x : x < 0\} > 2k$

62 a

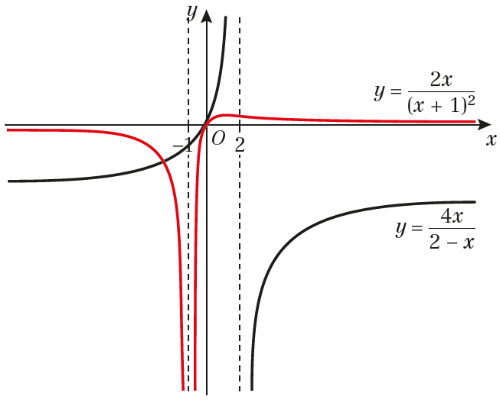


b The points of intersection are those whose x-coordinate satisfies the equation $2 - x = -\frac{2}{x-1}$

We solve this:
$(2-x)(x-1) = -2$
$(x-2)(x-1) = 2$
$x^2 - 3x + 2 = 2$
$x^2 - 3x = 0$
Which is solved by $x = 0$ and $x = 3$
Both solutions are acceptable. Therefore, the points of intersection are (0, 2) and $(3, -1)$

c It is clear from the graph that the solution to the inequality is $x < 0$ or $1 < x < 3$

63 a



b The points of intersection are those whose x-coordinate satisfies the equation $\frac{4x}{2-x} = \frac{2x}{(x+1)^2}$

We solve this:

$4x(x+1)^2 = 2x(2-x)$
$4x(x^2+2x+1) = 4x-2x^2$
$4x^3+8x^2+4x = 4x-2x^2$
$4x^3+10x^2 = 0$
$x^2(2x+5) = 0$

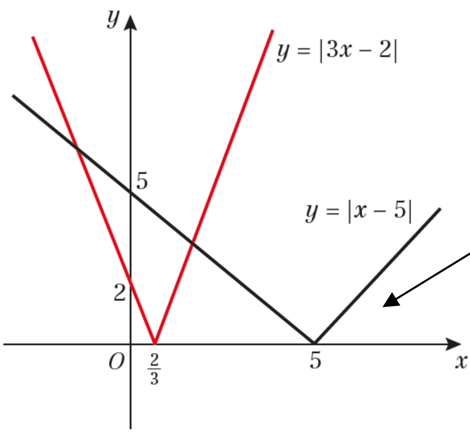
Which is solved by $x = 0$ and $x = -\frac{5}{2}$

These are both acceptable solutions. Therefore, the points of intersection are $(0,0)$ and $\left(-\frac{5}{2}, -\frac{20}{9}\right)$

c It is clear from the graph that the set of the solutions to the inequality is

$\left\{x : x \le -\frac{5}{2}\right\} \cup \{x : x = 0\} \cup \{x : x > 2\}$

64 a



You should mark the coordinates of the points where the graphs meet the axes.

Inequalities which contain both an expression in x with a modulus sign and an expression in x without a modulus sign, are usually best answered by drawing a sketch. In this case, you have been instructed to draw the sketch first. The continuous line is the graph of $y = |x+2|$

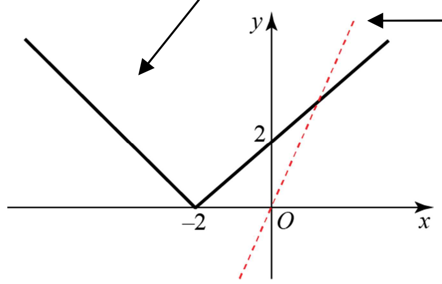
You should mark the coordinates of the points where the graph cuts the axis.

b The points of intersection are those whose x-coordinate satisfies the equation $|x-5| = |3x-2|$

For $x > 5$ or $x < \frac{2}{3}$, this is equivalent to $x - 5 = 3x - 2$, which is solved by $x = -\frac{3}{2}$, which is acceptable. For $\frac{2}{3} < x < 5$, it is equivalent to $5 - x = 3x - 2$, which is solved by $x = \frac{7}{4}$, which is also acceptable. Therefore, there are two points of intersection: $\left(-\frac{3}{2}, \frac{13}{2}\right)$ and $\left(\frac{7}{4}, \frac{13}{4}\right)$

c It is clear from the graph that the set of the solutions to the inequality is $\{x : x < -\frac{3}{2}\} \cup \{x : x > \frac{7}{4}\}$

65 a



Inequalities which contain both an expression in x with a modulus sign and an expression in x without a modulus sign, are usually best answered by drawing a sketch. In this case, you have been instructed to draw the sketch first. The continuous line is the graph of $y = |x+2|$

You should mark the coordinates of the points where the graph cuts the axis.

You should now add the graph $y = 2x$ to your sketch. This has been done with a dotted line. You find the solution to the inequality by identifying the values of x where the dotted line is above the continuous line.

b The intersection occurs when $x > -2$

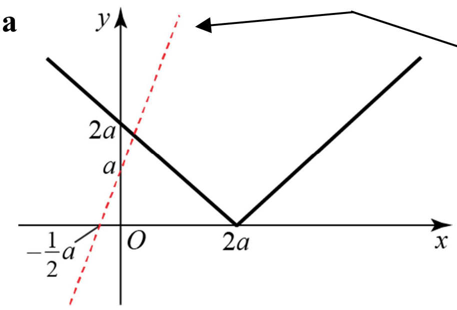
When $-2, |x+2| = x+2$

$$2x = x+2$$
$$x = 2$$

The solution of $2x > |x+2|$ is $x > 2$

When f(x) is positive, $|f(x)| = f(x)$

66 a



The dotted line is added to the sketch in part **a** to help you to solve part **b**. The dotted line is the graph of $y = 2x + a$ and the solution to the Inequality in part **b** is found by identifying where the continuous line, which corresponds to $|x - 2a|$, is above the dotted line, which corresponds to $2x + a$

b The intersection occurs when $x < 2a$

When $x < 2a, |x-2a| = 2a - x$

$$2a - x = 2x + a$$
$$-3x = -a \Rightarrow x = \frac{1}{3}a$$

The solution of $|x-2a| > 2x + a$ is $x < \frac{1}{3}a$

If f(x) is negative, then $|f(x)| = -f(x)$

67 We have two cases, depending on the sign of $\frac{x}{x-3}$

If $x>3$ or $x<0$, then it is positive and the inequality becomes $\frac{x}{x-3}<8-x$, which can be solved as follows:

$$\frac{x}{x-3}+x-8<0$$
$$\frac{x+x^2-3x-8x+24}{x-3}<0$$
$$\frac{x^2-10x+24}{x-3}<0$$
$$\frac{(x-6)(x-4)}{x-3}<0$$

This holds when numerator and denominator do not have the same sign; the numerator is positive for $x>6$ or $x<4$, while the denominator is positive for $x>3$

Therefore, this inequality is solved for $x<3$ or for $4<x<6$; we were looking for solutions with $x>3$ or $x<0$, therefore the set satisfying the first case is $\{x:x<0\}\cup\{x:4<x<6\}$

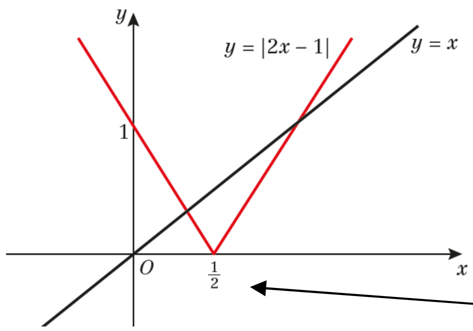
In the second case, we have $0<x<3$

Here, the inequality becomes $\frac{x}{3-x}<8-x$, which leads to $\frac{x^2-12x+24}{x-3}<0$

This is solved by $x<6-2\sqrt{3}$ or $x>3$

Clearly only the first one of these solutions is acceptable. Therefore, the set of the solutions is $\{x:4<x<6\}\cup\{x:x<6-2\sqrt{3}\}$

68 a



At $x=0, y=|-1|=1$

At $x=\frac{1}{2}, y=0$

b There are two points of intersection.

At the right hand point of intersection,

$x>\frac{1}{2}\Rightarrow|2x-1|=2x-1$

$2x-1=x\Rightarrow x=1$

At the left hand point of intersection,

$x<\frac{1}{2}\Rightarrow|2x-1|=1-2x$

$1-2x=x\Rightarrow x=\frac{1}{3}$

If $f(x)>0$, then $|f(x)|=f(x)$

If $f(x)<0$, then $|f(x)|=-f(x)$

The points of intersection of the two graphs are

$\left(\frac{1}{3},\frac{1}{3}\right)$ and (1, 1)

You need to give both the x-coordinates and the y-coordinates.

68 c The solution of $|2x-1| > x$ is $\left\{x : x < \frac{1}{3}\right\} \cup \{x : x > 1\}$

You identify the regions on the graph where the **V** shape representing $y = |2x-1|$ is above the line representing $y = x$

69

$$|x-3| > 2|x+1|$$
$$(x-3)^2 > 4(x+1)^2$$
$$x^2 - 6x + 9 > 4x^2 + 8x + 4$$
$$0 > 3x^2 + 14x - 5$$
$$(x+5)(3x-1) < 0$$

As both $|x-3|$ and $2|x+1|$ are positive you can square both sides of the inequality without changing the direction of the inequality sign. If a and b are both positive, it is true that $a > b \Rightarrow a^2 > b^2$

You cannot make this step if either or both of a and b are negative.

Considering $\mathrm{f}(x) = (x+5)(3x-1)$,

the critical values are $x = -5$ and $\frac{1}{3}$

Alternatively you can draw a sketch of $y = (x + 5)(3x - 1)$ and identify the region where the curve is below the y-axis.

	$x < -5$	$-5 < x < \frac{1}{3}$	$\frac{1}{3} < x$
Sign of $\mathrm{f}(x)$	+	–	+

The solution of $|x-3| > 2|x+1|$ is

$$\left\{x : -5 < x < \frac{1}{3}\right\}$$

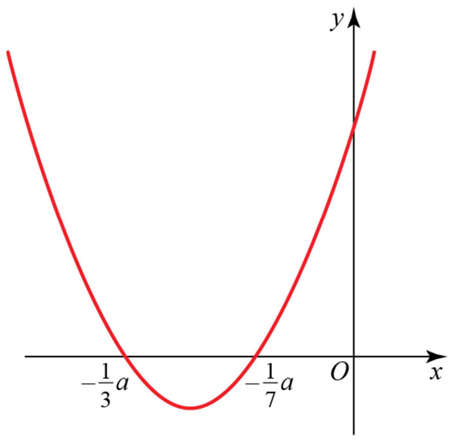
70

$$|5x+a| \leqslant |2x|$$
$$(5x+a)^2 \leqslant (2x)^2$$
$$25x^2 + 10ax + a^2 \leqslant 4x^2$$
$$21x^2 + 10ax + a^2 \leqslant 0$$
$$(3x+a)(7x+a) \leqslant 0$$

As a is positive, both $|5x+a|$ and $|2x|$ are positive and you can square both sides of the inequality.

Sketching $y = (3x+a)(7x+a)$

The graph is a parabola intersecting the x-axis at $x = -\frac{1}{3}a$ and $x = -\frac{1}{7}a$

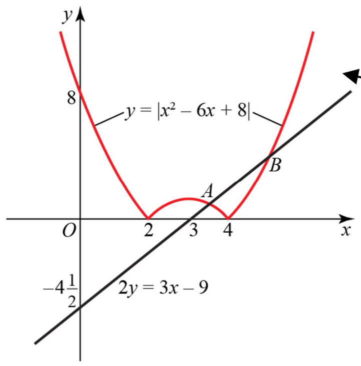


A common error here is not to realise that, for a positive a, $-\frac{1}{3}a$ is a smaller number than $-\frac{1}{7}a$

It is very easy to get the inequality the wrong way round.

The solution of $|5x+a| \leqslant |2x|$ is $-\frac{1}{3}a \leqslant x \leqslant -\frac{1}{7}a$

71 a



The curve meets the x-axis at (2, 0) and (4, 0)
The line meets the x-axis at (3, 0)

> As $x^2 - 6x + 8 = (x-2)(x-4)$ the curve meets the x-axis at $x = 2$ and $x = 4$
> The sketching of the graphs of modulus functions is in Chapter 5 of book C3

b To find the coordinates of A. The x-coordinate of A is in the interval $2 < x < 4$

In this interval $x^2 - 6x + 8$ is negative and, hence,

$$|x^2 - 6x + 8| = -x^2 + 6x - 8$$

> If $f(x) < 0$, then $|f(x)| = -f(x)$

$$-x^2 + 6x - 8 = \frac{3x-9}{2}$$
$$-2x^2 + 12x - 16 = 3x - 9$$
$$2x^2 - 9x + 7 = 0$$
$$(2x-7)(x-1) = 0$$
$$x = \frac{7}{2}, \cancel{1}$$

> As the x-coordinate of A is in the interval $2 < x < 4$, the solutions $x = 1$ must be rejected.

$$y = \frac{3 \times \frac{7}{2} - 9}{2} = \frac{3}{4}$$

The coordinates of A are $\left(\frac{7}{2}, \frac{3}{4}\right)$,

To find the coordinates of B. The x-coordinate of B is in the interval $x > 4$

In this interval $x^2 - 6x + 8$ is positive and, hence,

$$|x^2 - 6x + 8| = x^2 - 6x - 8$$

> If $f(x) > 0$, then $|f(x)| = f(x)$

$$x^2 + 6x + 8 = \frac{3x-9}{2}$$
$$2x^2 - 12x + 16 = 3x - 9$$
$$2x^2 - 15x + 25 = 0$$
$$(x-5)(2x-5) = 0$$
$$x = 5, \cancel{2\frac{1}{2}}$$

> As the x-coordinate of B is in the interval $x > 4$, the solution $x = 2\frac{1}{2}$ must be rejected.

$$y = \frac{3 \times 5 - 9}{2} = 3$$

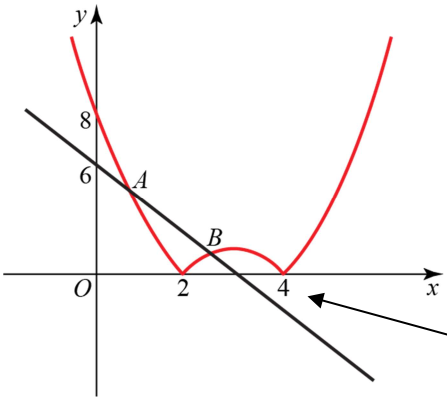
The coordinates of B are (5, 3)

The solution of $2|x^2 - 6x + 8| > 3x - 9$ is

> You solve the inequality by inspecting the graphs. You look for the values of x where the curve is above the line.

$x < 3\frac{1}{2}, x > 5$

72 a



You should mark the coordinates of the points where the graphs meet the axes.

b Let the points where the graphs intersect be A and B

For A, $(x – 2)(x – 4)$ is positive

$$(x-2)(x-4)=6-2x$$
$$x^2-6x+8=6-2x$$
$$x^2-4x=-2$$
$$x^2-4x+4=2$$
$$(x-2)^2=2$$
$$x=2-\sqrt{2}$$

The quadratic equations have been solved by completing the square. You could use the formula for solving a quadratic but the conditions of the question require exact solutions and you should not use decimals.

The quadratic equation has another solution $2+\sqrt{2}$ but the diagram shows that the x-coordinate of A is less than 2, so this solution is rejected.

For B, $(x – 2)(x – 4)$ is negative

$$-(x-2)(x-4)=6-2x$$
$$-x^2+6x-8=6-2x$$
$$x^2-8x=-14$$
$$x^2-8x+16=2$$
$$(x-4)^2=2$$
$$x=4-\sqrt{2}$$

The quadratic equation has another solution $2+\sqrt{2}$ but the diagram shows that the x-coordinate of B is less than 4, so this solution is rejected.

The values of x for which $|(x-2)(x-4)|=6-2x$ are $2-\sqrt{2}$ and $4-\sqrt{2}$

c The solution of $|(x-2)(x-4)|<6-2x$ is $2-\sqrt{2}<x<4-\sqrt{2}$

You look for the values of x where the curve is below the line.

73 a For $x > -2$, $x + 2$ is positive and the equation is

$$\frac{x^2-1}{x+2} = 3(1-x)$$

$$x^2 - 1 = 3(1-x)(x+2) = -3x^2 - 3x + 6$$

$$4x^2 + 3x - 7 = (4x+7)(x-1) = 0$$

$$x = -\frac{7}{4}, 1$$

As both of these answers are greater than –2 both are valid.

For $x < -2$, $x + 2$ is negative and the equation is

$$\frac{x^2-1}{-(x+2)} = 3(1-x)$$

$$x^2 - 1 = -3(1-x)(x+2) = 3x^2 + 3x - 6$$

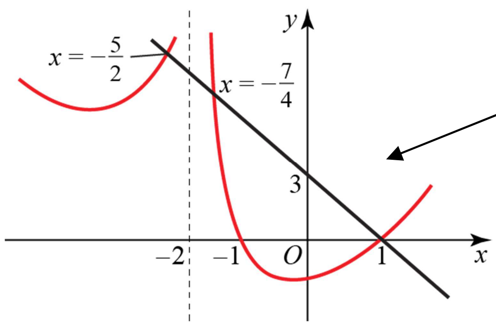
$$2x^2 + 3x - 5 = (2x+5)(x-1) = 0$$

$$x = -\frac{5}{2}, \cancel{1}$$

As 1 is not less than –2 the answer 1 should be 'rejected' here. However, the earlier working has already shown 1 to be a correct solution.

The solutions are $-\frac{5}{2}, -\frac{7}{4}$ and 1

b



To complete the question, you add the graph of y = 3(1 – x) to the graph which has already been drawn for you. You know the x-coordinates of the points of intersection from part **a**.

The solution of $\frac{x^2-1}{|x+2|} < 3(1-x)$ is

$$\left\{x : x < -\frac{5}{2}\right\} \cup \left\{x : -\frac{7}{4} < x < 1\right\}$$

You look for the values of x on the graph where the curve is below the line.

Challenge

1 A general point of the hyperbola is $\begin{pmatrix} ct \\ \frac{c}{t} \end{pmatrix}$. The matrix of a 135° anticlockwise rotation is $\begin{pmatrix} -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{pmatrix}$

Therefore, the general point of the rotated hyperbola is $\begin{pmatrix} -\frac{ct\sqrt{2}}{2} - \frac{c\sqrt{2}}{2t} \\ \frac{ct\sqrt{2}}{2} - \frac{c\sqrt{2}}{2t} \end{pmatrix}$

If we compute $x^2 - y^2$, we get:

$$\left(-\frac{ct\sqrt{2}}{2} - \frac{c\sqrt{2}}{2t}\right)^2 - \left(\frac{ct\sqrt{2}}{2} - \frac{c\sqrt{2}}{2t}\right)^2 =$$

$$= \frac{c^2t^2}{2} + \frac{c^2}{2t^2} + c^2 - \left(\frac{c^2t^2}{2} + \frac{c^2}{2t^2} - c^2\right) =$$

$$= 2c^2$$

Therefore, the equation of the rotated hyperbola is $x^2 - y^2 = 2c^2 = k^2$, with $k = \sqrt{2}c$

Challenge

2 The inequality can be solved as follows:

$$\frac{1}{1-\sin x}-\frac{1}{\sin x}<0$$

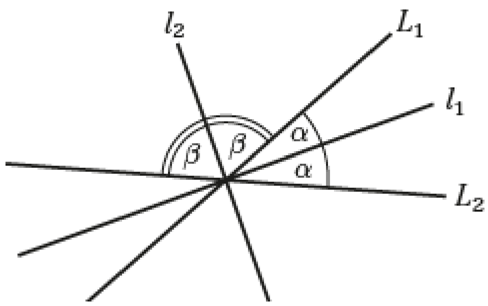
$$\frac{\sin x-1+\sin x}{(1-\sin x)(\sin x)}<0$$

$$\frac{2\sin x-1}{(1-\sin x)(\sin x)}<0$$

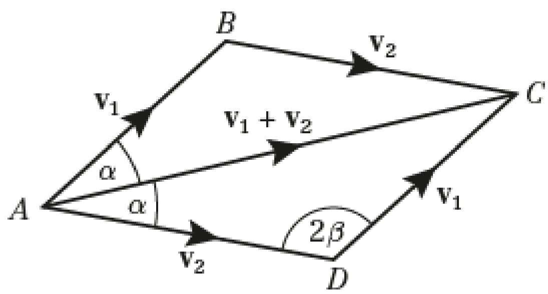
This is satisfied when numerator and denominator do not have the same sign. The numerator is positive for $\frac{\pi}{6}<x<\frac{5}{6}\pi$, while the denominator is positive for $0<x<\frac{\pi}{2}$ or $\frac{\pi}{2}<x<\pi$

Then the inequality is satisfied by the set $\left\{x:0<x<\frac{\pi}{6}\right\}\cup\left\{x:\frac{5}{6}\pi<x<\pi\right\}$

3 a Bisectors in red:



b L_1 has direction vector $\mathbf{v}_1=\begin{pmatrix}l_1\\m_1\\n_1\end{pmatrix}$ and L_2 has direction vector $\mathbf{v}_2=\begin{pmatrix}l_2\\m_2\\n_2\end{pmatrix}$. Then $\mathbf{v}_1$ and $\mathbf{v}_2$ form the rhombus with diagonal $\mathbf{v}_1+\mathbf{v}_2$.



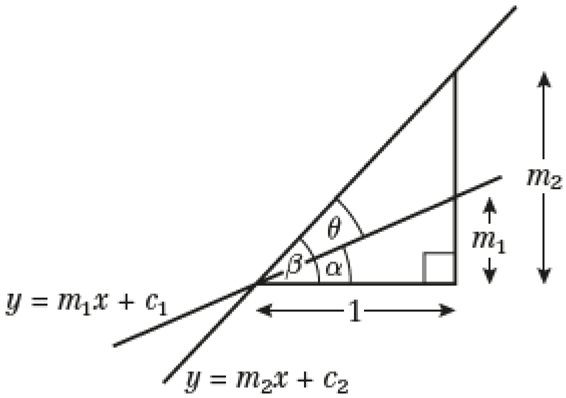
$\mathbf{v}_1+\mathbf{v}_2=\begin{pmatrix}l_1+l_2\\m_1+m_2\\n_1+n_2\end{pmatrix}$ bisects angle BAD so is parallel to l_1. Hence l_1 has direction ratios $l_1+l_2 : m_1+m_2 : n_1+n_2$. The other diagonal of the rhombus is given by

$\mathbf{v}_1+\mathbf{v}_2=\begin{pmatrix}l_1+l_2\\m_1+m_2\\n_1+n_2\end{pmatrix}$ and bisects angle ADC so it is parallel to l_2. Hence l_2 has direction ratios $l_1-l_2 : m_1-m_2 : n_1-n_2$ respectively.

In general, $|\mathbf{v}_1+\mathbf{v}_2|$ and $|\mathbf{v}_1-\mathbf{v}_2|$ are not equal to 1, so these values are not direction cosines.

Challenge

4 a



Using the identity for $\tan(A \pm B)$:

$\tan\theta = \tan(\beta - \alpha) = \dfrac{\tan\beta - \tan\alpha}{1+\tan\beta\tan\alpha} = \dfrac{m_2 - m_1}{1+m_1m_2}$ as required.

b Assuming the point P has coordinates ($a\cos t$, $b\sin t$), the gradient of the normal at P is $\dfrac{a\cos t}{b\sin t}$

Knowing that the coordinates of the foci are $(\pm ae, 0)$, we can easily find that the gradients of PS and PS' are $\dfrac{b\sin t}{a\cos t \pm ae}$

So, using the result from part **a**, the tangent of the angle between PS and the normal is:

$$\frac{\dfrac{a\sin t}{b\cos t} - \dfrac{b\sin t}{a\cos t - ae}}{1+\left(\dfrac{a\sin t}{b\cos t}\right)\left(\dfrac{b\sin t}{a\cos t - ae}\right)} =$$

$$= \frac{\dfrac{a^2\cos t\sin t - a^2 e\sin t - b^2\cos t\sin t}{ab\cos^2 t - abe\cos t}}{1+\dfrac{ab\sin^2 t}{ab\cos^2 t - abe\cos t}} =$$

$$= \frac{(a^2-b^2)\cos t\sin t - a^2 e\sin t}{ab\cos^2 t - abe\cos t + ab\sin^2 t} =$$

$$= \frac{a^2e^2\cos t\sin t - a^2 e\sin t}{ab(1-e\cos t)} =$$

$$= \frac{-a^2 e\sin t(1-e\cos t)}{ab(1-e\cos t)} = -\frac{ae\sin t}{b}$$

Similarly, we find that this is also the value of the tangent of the angle between the normal and PS'
Therefore, since the tangent is injective between 0 and 2π (where it is defined), we can conclude that the two angles are the same.