Trigonometric identities and equations 10E

1 a $\sin 4\theta = 0$ $0° \le \theta \le 360°$

Let $X = 4\theta$ so $0° \le X \le 1440°$

Solve $\sin X = 0$ in the interval
$0° \le X \le 1440°$

From the graph of $y = \sin X$, $\sin X = 0$ where

$X = 0°,\ 180°,\ 360°,\ 540°,\ 720°,\ 900°,$
     $1080°,\ 1260°,\ 1440°$

$\theta = \dfrac{X}{4}$

  $= 0°, 45°, 90°, 135°, 180°, 225°, 270°,$

    $315°, 360°$

b $\cos 3\theta = -1$ $0° \le \theta \le 360°$

Let $X = 3\theta$ so $0° \le X \le 1080°$

Solve $\cos X = 0$ in the interval
$0° \le X \le 1080°$

From the graph of $y = \cos X$, $\cos X = -1$ where

$X = 180°,\ 540°,\ 900°,$

$\theta = \dfrac{X}{3}$

  $= 60°,\ 180°,\ 300°$

c $\tan 2\theta = 1$ $0° \le \theta \le 360°$

Let $X = 2\theta$

Solve $\tan X = 1$ in the interval
$0° \le X \le 720°$.

A solution is $X = \tan^{-1}(1) = 45°$

As $\tan X$ is +ve, X is in the first and third quadrants.

So $X = 45°,\ 225°,\ 405°,\ 585°$

$\theta = \dfrac{X}{2}$

  $= 22\tfrac{1}{2}°,\ 112\tfrac{1}{2}°,\ 202\tfrac{1}{2}°,\ 292\tfrac{1}{2}°$

d $\cos 2\theta = \dfrac{1}{2}$ $0 \le \theta \le 360°$

Let $X = 2\theta$

Solve $\cos X = \dfrac{1}{2}$ in the interval
$0 \le X \le 720°$.

A solution is $X = \cos^{-1}\left(\dfrac{1}{2}\right) = 60°$

As $\cos X$ is +ve, X is in the first and fourth quadrants.

So $X = 60°,\ 300°,\ 420°,\ 660°$

d $\theta = \dfrac{X}{2}$

  $= 30°, 150°, 210°, 330°$

e $\tan \dfrac{1}{2}\theta = -\dfrac{1}{\sqrt{3}}$ $0 \le \theta \le 360°$

Let $X = \dfrac{1}{2}\theta$
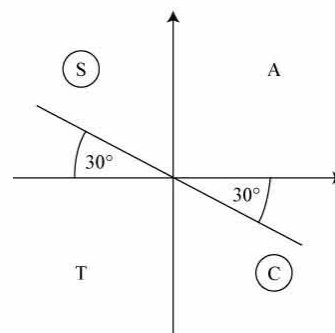
Solve $\tan X = -\dfrac{1}{\sqrt{3}}$ in the interval

$0° \le X \le 180°$.

A solution is $X = \tan^{-1}\left(-\dfrac{1}{\sqrt{3}}\right) = -30°$

(This is not in the interval.)

As $\tan X$ is −ve, X is in the second and fourth quadrants.

Read off solution in the interval
$0° \le X \le 180°$.
   $X = 150°$

So $\theta = 2X$

   $= 300°$

f $\sin(-\theta) = \dfrac{1}{\sqrt{2}}$ $0° \le \theta \le 360°$

Let $X = -\theta$

Solve $\sin X = \dfrac{1}{\sqrt{2}}$ in the interval

$0° \le X \le 360°$.

A solution is $X = \sin^{-1}\left(\dfrac{1}{\sqrt{2}}\right) = 45°$

As $\sin X$ is +ve, X is in the first and second quadrants.
   $X = -315°,\ -225°$

So $\theta = -X$

   $= 225°,\ 315°$

2 a $\tan(45° - \theta) = -1$ $0° \le \theta \le 360°$
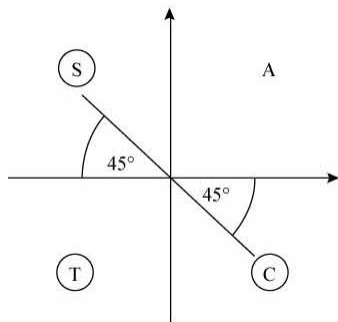
Let $X = 45° - \theta$ so $0° \ge -\theta \ge -360°$

Solve $\tan X = -1$ in the interval

$45° \ge X \ge -315°$

A solution is $X = \tan^{-1}(-1) = -45°$

As $\tan X$ is −ve, X is in the second and fourth quadrants.

$X = -225°, -45°$

So $\theta = 45° - X = 90°, 270°$

b $2\sin(\theta - 20°) = 1$

So $\sin(\theta - 20°) = \dfrac{1}{2}$ $0° \le \theta \le 360°$

Let $X = \theta - 20°$

Solve $\sin X = \dfrac{1}{2}$ in the interval

$-20° \le X \le 340°$

A solution is $X = \sin^{-1}\left(\dfrac{1}{2}\right) = 30°$

As $\sin X$ is +ve, solutions are in the first and second quadrants.

   $X = 30°, 150°$

So $\theta = X + 20°$

   $= 50°, 170°$

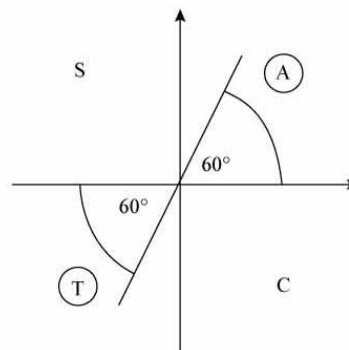**c** Solve $\tan X = \sqrt{3}$ where $X = (\theta + 75°)$.

The interval for X is $75° \le X \le 435°$

One solution is $\tan^{-1}\left(\sqrt{3}\right) = 60°$

(This is not in the interval)

As $\tan X$ is +ve, solutions are in the first and third quadrants.

c

$X = 240°, 420°$

So $\theta = X - 75°$

   $= 165°, 345°$

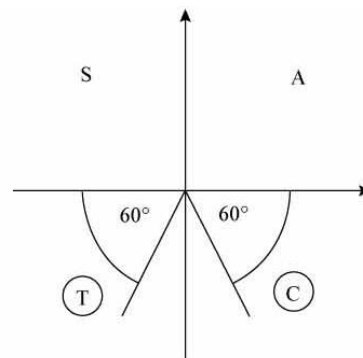**d** Solve $\sin X = \dfrac{\sqrt{3}}{2}$ where $X = (\theta - 10°)$.

The interval for X is $-10° < X \le 350°$

First solution is $\sin^{-1}\left(-\dfrac{\sqrt{3}}{2}\right) = -60°$

(This is not in the interval)

As $\sin X$ is −ve, X is in the third and fourth quadrants.

Read off solutions in the interval

$-10° < X \le 350°$

   $X = 240°, 300°$

So $\theta = X + 10°$

   $= 250°, 310°$

2 e Solve $\cos X° = -1$ where $X = (50° + 2\theta)$.

The interval for X is

$2(0) + 50° \le X \le 2(360°) + 50°$

i.e. $50° \le X \le 770°$

First solution is $\cos^{-1}(-1) = 180°$

Second solution in interval

   $= 180° + 360° = 540°$

$\theta = \dfrac{X - 50°}{2}$

So $\theta = 65°, 245°$

2 f Solve $\tan X° = -0.51$

where $X = (3\theta + 25°)$.

The interval for X is

$3(-90°) + 25 \le X \le 3(180°) + 25°$

i.e. $-245° \le X \le 565°$

First solution is $\tan^{-1}(-0.51) = -27.0°$

Other solutions $= -27.0° \pm n180°$, where

n is an integer

Solution in range are therefore:

$X = -207°, -27.0°, 153°, 333°$ and $513°$

$$\theta = \frac{X - 25°}{3}$$

So $\theta = -77.3°, -17.3°, 42.7°, 103°, 163°$

3 a Let $X = 3\theta$

So $3 \sin X = 2 \cos X$

$$\frac{\sin X}{\cos X} = \frac{2}{3}$$

$$\tan X = \frac{2}{3}$$

As $X = 3\theta$, then as $0° \le \theta \le 180°$

So $3 \times 0° \le X \le 3 \times 180°$

So the interval for X is $0° \le X \le 540°$.

$X = 33.7°, 213.7°, 393.7°$

i.e. $3\theta = 33.7°, 213.7°, 393.7°$

So $\theta = 11.2°, 71.2°, 131.2°$

b Let $X = \theta + 45°$

So $4 \sin X = 5 \cos X$

$$\frac{\sin X}{\cos X} = \frac{5}{4}$$

$$\tan X = \frac{5}{4}$$

As $X = \theta + 45°$, then as $0° \le \theta \le 450°$

so $0 + 45 \le X \le 450° + 45$

So the interval for X is $45° \le X \le 495°$.

$X = 51.3°, 231.3°, 411.3°$

i.e. $\theta + 45° = 51.3°, 231.3°, 411.3°$

So $\theta = 6.3°, 186.3°, 366.3°$

c Let $X = 2x$

$2 \sin X - 7 \cos X = 0$

$2 \sin X = 7 \cos X$

$$\frac{\sin X}{\cos X} = \frac{7}{2}$$

$$\tan X = \frac{7}{2}$$

As $X = 2x$, then as $0° \le x \le 180°$

So $2 \times 0° \le X \le 2 \times 180°$

c So the interval for X is $0° \le X \le 360°$.

$X = 74.05°, 254.05°$

i.e. $2x = 74.05°, 254.05°$

So $x = 37.0°, 127.0°$

d Let $X = \theta - 60°$

So $\sqrt{3} \sin X + \cos X = 0$

$\sqrt{3} \sin X = -\cos X$
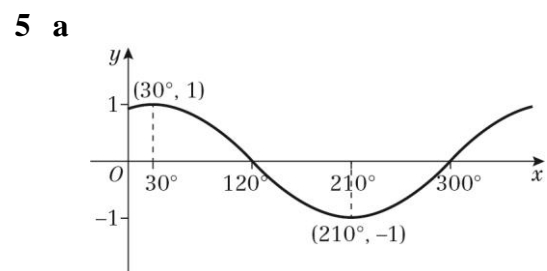
$$\frac{\sin X}{\cos X} = \frac{-1}{\sqrt{3}}$$

$$\tan X = \frac{-1}{\sqrt{3}}$$

As $X = \theta - 60°$, then as $-180° \le \theta \le 180°$

So $-180° - 60 \le X \le 180° - 60$

So the interval for X is $-240° \le X \le 120°$.

$X = -30°, -210°$

i.e. $\theta - 60° = -210°, -30°$

So $\theta = -150°, 30°$

4 a Let $X = x + 20°$

So $\sin X = \frac{1}{2}$

As $X = x + 20°$, then as $0 \le x \le 180°$

So $0 + 20 \le x \le 180° + 20$

So the interval for X is $20° \le X \le 200°$.

$X = 30°, 150°$

i.e. $x + 20° = 30°, 150°$

So $x = 10°, 130°$

b Let $X = 2x$

So $\cos X = -0.8$

As $X = 2x$, then as $0 \le x \le 180°$

So $2 \times 0 \le X \le 2 \times 180°$

So the interval for X is $0 \le X \le 360°$

$X = 143.13°, 216.87°$

i.e. $2x = 143.13°, 216.87°$

So $x = 71.6°, 108.4°$

5 a

b $\left(0°, \dfrac{\sqrt{3}}{2}\right)$, $(120°, 0)$, $(300°, 0)$

5 c Let $X = x + 60°$

So $\sin X = 0.55$

As $X = x + 60°$, then as $0° \leq x \leq 360°$

So $0° + 60° \leq x \leq 360° + 60°$

So the interval for X is $60° \leq X \leq 420°$.

$X = 33.4°, 146.6°, 393.4°$

i.e. $x + 60° = 33.4°, 146.6°, 393.4°$

So $x = 86.6°, 333.4°$

6 a $4\sin x = 3\cos x$

$$\frac{\sin x}{\cos x} = \frac{3}{4}$$

$$\tan x = \frac{3}{4}$$

b Let $X = 2\theta$

So $\tan X = \dfrac{3}{4}$

As $X = 2\theta$, then as $0° \leq \theta \leq 360°$

So $2 \times 0° \leq X \leq 2 \times 360°$

So the interval for X is $0° \leq X \leq 720°$.

$X = 36.87°, 216.87°, 396.87°, 576.87°$

i.e. $2\theta = 36.87°, 216.87°, 396.87°, 576.87°$

So $\theta = 18.4°, 108.4°, 198.4°, 288.4°$

7 a $\tan 60k° = -\dfrac{1}{\sqrt{3}}$

Let $X = 60k°$

So $\tan X = -\dfrac{1}{\sqrt{3}}$

$X = -30°, 150°, 330°,$

I.e. $60k° = -30°, 150°, 330°,$

$k = -0.5, 2.5,$

$k > 0$ so $k = 2.5$

b No because when $X = 330°$, $k = 5.5$.

As k increases, the period of the tan graph increases.

8 Let $X = 3x - 45°$

So $\sin X = \frac{1}{2}$

As $X = 3x - 45°$, then as $0° \leq x \leq 180°$

So $3 \times 0° - 45° \leq x \leq 3 \times 180° - 45°$

So the interval for X is $-45° \leq X \leq 495°$.

$X = 30°, 150°, 390°$

i.e. $3x - 45° = 30°, 150°, 390°$

So $x = 25°, 65°, 145°$