Integration, Mixed Exercise 13

1 a $\int (x+1)(2x-5)\ \mathrm{d}x = \int (2x^2-3x-5)\ \mathrm{d}x$
$= 2\frac{x^3}{3} - 3\frac{x^2}{2} - 5x + c$
$= \frac{2}{3}x^3 - \frac{3}{2}x^2 - 5x + c$

b $\int \left(x^{\frac{1}{3}} + x^{-\frac{1}{3}} \right) \mathrm{d}x = \frac{x^{\frac{4}{3}}}{\frac{4}{3}} + \frac{x^{\frac{2}{3}}}{\frac{2}{3}} + c$
$= \frac{3}{4}x^{\frac{4}{3}} + \frac{3}{2}x^{\frac{2}{3}} + c$

2 $f'(x) = x^2 - 3x - \frac{2}{x^2} = x^2 - 3x - 2x^{-2}$
So $f(x) = \frac{x^3}{3} - 3\frac{x^2}{2} - 2\frac{x^{-1}}{-1} + c$
So $f(x) = \frac{1}{3}x^3 - \frac{3}{2}x^2 + \frac{2}{x} + c$
But $f(1) = 1 \Rightarrow \frac{1}{3} - \frac{3}{2} + 2 + c = 1$
So $c = \frac{1}{6}$
So the equation is $y = \frac{1}{3}x^3 - \frac{3}{2}x^2 + \frac{2}{x} + \frac{1}{6}$

3 a $\int (8x^3 - 6x^2 + 5)\ \mathrm{d}x = 8\frac{x^4}{4} - 6\frac{x^3}{3} + 5x + c$
$= 2x^4 - 2x^3 + 5x + c$

b $\int (5x+2)x^{\frac{1}{2}}\ \mathrm{d}x = \int \left(5x^{\frac{3}{2}} + 2x^{\frac{1}{2}} \right) \mathrm{d}x$
$= 5\frac{x^{\frac{5}{2}}}{\frac{5}{2}} + 2\frac{x^{\frac{3}{2}}}{\frac{3}{2}} + c$
$= 2x^{\frac{5}{2}} + \frac{4}{3}x^{\frac{3}{2}} + c$

4 $y = \frac{(x+1)(2x-3)}{\sqrt{x}}$
$y = (2x^2 - x - 3)x^{-\frac{1}{2}}$
$y = 2x^{\frac{3}{2}} - x^{\frac{1}{2}} - 3x^{-\frac{1}{2}}$
$\int y\ \mathrm{d}x = \int \left(2x^{\frac{3}{2}} - x^{\frac{1}{2}} - 3x^{-\frac{1}{2}} \right) \mathrm{d}x$
$= 2\frac{x^{\frac{5}{2}}}{\frac{5}{2}} - \frac{x^{\frac{3}{2}}}{\frac{3}{2}} - 3\frac{x^{\frac{1}{2}}}{\frac{1}{2}} + c$
$= \frac{4}{5}x^{\frac{5}{2}} - \frac{2}{3}x^{\frac{3}{2}} - 6x^{\frac{1}{2}} + c$

5 $\frac{\mathrm{d}x}{\mathrm{d}t} = (t+1)^2 = t^2 + 2t + 1$
$\Rightarrow x = \frac{t^3}{3} + 2\frac{t^2}{2} + t + c$
But $x = 0$ when $t = 2$.
So $0 = \frac{8}{3} + 4 + 2 + c$
$\Rightarrow c = -\frac{26}{3}$
So $x = \frac{1}{3}t^3 + t^2 + t - \frac{26}{3}$
When $t = 3$, $x = \frac{27}{3} + 9 + 3 - \frac{26}{3}$
So $x = \frac{37}{3}$ or $12\frac{1}{3}$

6 a $y^{\frac{1}{2}} = x^{\frac{1}{3}} + 3$
So $y = \left(x^{\frac{1}{3}} + 3 \right)^2$
So $y = \left(x^{\frac{1}{3}} \right)^2 + 6x^{\frac{1}{3}} + 9$
So $y = x^{\frac{2}{3}} + 6x^{\frac{1}{3}} + 9$
$(A = 6, B = 9)$

b $\int y\ \mathrm{d}x = \int \left(x^{\frac{2}{3}} + 6x^{\frac{1}{3}} + 9 \right) \mathrm{d}x$
$= \frac{x^{\frac{5}{3}}}{\frac{5}{3}} + 6\frac{x^{\frac{4}{3}}}{\frac{4}{3}} + 9x + c$
$= \frac{3}{5}x^{\frac{5}{3}} + \frac{9}{2}x^{\frac{4}{3}} + 9x + c$

7 a $y^{\frac{1}{2}} = 3x^{\frac{1}{4}} - 4x^{-\frac{1}{4}}$
$y = (3x^{\frac{1}{4}} - 4x^{-\frac{1}{4}})^2$
$= 9x^{\frac{1}{2}} - 24 + 16x^{-\frac{1}{2}}$
$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{9}{2}x^{-\frac{1}{2}} - 8x^{-\frac{3}{2}}$

b $\int \left(9x^{\frac{1}{2}} - 24 + 16x^{-\frac{1}{2}} \right) \mathrm{d}x$
$= \frac{9x^{\frac{3}{2}}}{\frac{3}{2}} - 24x + \frac{16x^{\frac{1}{2}}}{\frac{1}{2}} + c$
$= 6x^{\frac{3}{2}} - 24x + 32x^{\frac{1}{2}} + c$

8 $\int \left(\frac{a}{3x^3} - ab \right) \mathrm{d}x = \int \left(\frac{a}{3}x^{-3} - ab \right) \mathrm{d}x$
$= \frac{a}{3} \times \frac{x^{-2}}{-2} - abx + c$

8 $\int\left(\frac{a}{3x^3}-ab\right)\,dx=-\frac{a}{6x^2}-abx+c$

$= -\frac{2}{3x^2}+14x+c$

Equating coefficients $-\frac{a}{6}=-\frac{2}{3}$ and $-ab=14$

$a=4,\ b=-3.5$

9 $f'(t)=-9.8t$

$f(t)=-\frac{9.8t^2}{2}+c$
$=-4.9t^2+c$

$f(0)=-4.9(0)^2+c$
$=70$
$c=70$

$f(t)=-4.9t^2+70$

$f(3)=-4.9(3)^2+70$
$=25.9$

The height of the rock above the ground after 3 seconds is 25.9 m.

10 a $f(t)=\int(5+2t)\,dt$

$=5t+\frac{2t^2}{2}+c$

$=5t+t^2+c$

As $f(0)=0,\ 5(0)+0^2+c=0$
$c=0$

$f(t)=5t+t^2$

b When $f(t)=100,\ 5t+t^2=100$
$t^2+5t-100=0$

Using the formula

$t=\frac{-5\pm\sqrt{5^2-4(1)(-100)}}{2(1)}$

$t=\frac{-5\pm\sqrt{425}}{2}$

$t=7.8$ or $t=-12.8$
As $t>0$, $t=7.8$ seconds

11 a $2=5+2x-x^2$
$\Rightarrow x^2-2x-3=0$
$\Rightarrow (x-3)(x+1)=0$
$\Rightarrow x=-1(A),3(B)$

b Area of $R=\int_{-1}^{3}(5+2x-x^2-2)\,dx$
$=\int_{-1}^{3}(3+2x-x^2)\,dx$
$=\left(3x+x^2-\tfrac{1}{3}x^3\right)_{-1}^{3}$
$=\left(9+9-\tfrac{27}{3}\right)-\left(-3+1+\tfrac{1}{3}\right)$
$=9+2-\tfrac{1}{3}$
$=10\tfrac{2}{3}$

12 a $\left(x^{\frac{1}{2}}-4\right)\left(x^{-\frac{1}{2}}-1\right)$
$=1-4x^{-\frac{1}{2}}-x^{\frac{1}{2}}+4=5-4x^{-\frac{1}{2}}-x^{\frac{1}{2}}$

$\int\left(x^{\frac{1}{2}}-4\right)\left(x^{-\frac{1}{2}}-1\right)\,dx=5x-\frac{4x^{\frac{1}{2}}}{\frac{1}{2}}-\frac{x^{\frac{3}{2}}}{\frac{3}{2}}+c$
$=5x-8x^{\frac{1}{2}}-\tfrac{2}{3}x^{\frac{3}{2}}+c$

b $\int_{1}^{4}\left(x^{\frac{1}{2}}-4\right)\left(x^{-\frac{1}{2}}-1\right)\,dx=\left(5x-8x^{\frac{1}{2}}-\tfrac{2}{3}x^{\frac{3}{2}}\right)_{1}^{4}$
$=\left(20-8\times2-\tfrac{2}{3}\times2^3\right)-\left(5-8-\tfrac{2}{3}\right)$
$=4-\tfrac{16}{3}+3+\tfrac{2}{3}$
$=7-\tfrac{14}{3}$
$=\tfrac{7}{3}$ or $2\tfrac{1}{3}$

13 a $(x-3)^2=x^2-6x+9$
So $x(x-3)^2=x^3-6x^2+9x$
$y=0\Rightarrow x=0$ or 3 (twice)
So A is the point $(3,0)$.

b $\frac{dy}{dx}=0\Rightarrow 0=3x^2-12x+9$
$\Rightarrow 0=3(x^2-4x+3)$
$\Rightarrow 0=3(x-3)(x-1)$
$\Rightarrow 0=1$ or 3
$x=3$ at A, the minimum, so B is $(1,4)$
(Found by substituting $x=1$ into original equation.)

13 c Area of $R = \int_0^3 \left(x^3 - 6x^2 + 9x\right) \, \mathrm{d}x$

$$= \left(\tfrac{1}{4}x^4 - 2x^3 + \tfrac{9}{2}x^2\right)_0^3$$

$$= \left(\tfrac{81}{4} - 2 \times 27 + \tfrac{9}{2} \times 9\right) - (0)$$

$$= 6\tfrac{3}{4}$$

14 a $y = 3x^{\frac{1}{2}} - 4x^{-\frac{1}{2}}$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \tfrac{3}{2}x^{-\frac{1}{2}} + \tfrac{1}{2} \times 4x^{-\frac{3}{2}}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \tfrac{3}{2}x^{-\frac{1}{2}} + 2x^{-\frac{3}{2}}$$

b $\int y \, \mathrm{d}x = \int \left(3x^{\frac{1}{2}} - 4x^{-\frac{1}{2}}\right) \mathrm{d}x$

$$= \frac{3x^{\frac{3}{2}}}{\frac{3}{2}} - \frac{4x^{\frac{1}{2}}}{\frac{1}{2}} + c$$

$$= 2x^{\frac{3}{2}} - 8x^{\frac{1}{2}} + c$$

c $\int_1^3 y \, \mathrm{d}x = \left(2x^{\frac{3}{2}} - 8x^{\frac{1}{2}}\right)_1^3$

$$= \left(2 \times 3\sqrt{3} - 8\sqrt{3}\right) - (2 - 8)$$

$$= -2\sqrt{3} + 6$$

$$= 6 - 2\sqrt{3}$$

So $A = 6$ and $B = -2$

15 a $y = 12x^{\frac{1}{2}} - x^{\frac{3}{2}}$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 6x^{-\frac{1}{2}} - \tfrac{3}{2}x^{\frac{1}{2}}$$

$$= \tfrac{3}{2}x^{-\frac{1}{2}}(4 - x)$$

b $\frac{\mathrm{d}y}{\mathrm{d}x} = 0 \Rightarrow x = 4, y = 12 \times 2 - 2^3 = 16$

So B is the point $(4, 16)$.

c Area $= \int_0^{12} \left(12x^{\frac{1}{2}} - x^{\frac{3}{2}}\right) \mathrm{d}x$

$$= \left(\frac{12x^{\frac{3}{2}}}{\frac{3}{2}} - \frac{x^{\frac{5}{2}}}{\frac{5}{2}}\right)_0^{12}$$

$$= \left(8x^{\frac{3}{2}} - \tfrac{2}{5}x^{\frac{5}{2}}\right)_0^{12}$$

$$= \left(8 \times \sqrt{12^3} - \tfrac{2}{5}\sqrt{12^5}\right) - (0)$$

$$= 133 \text{ (3 s.f.)}$$

16 a $x(8 - x) = 12$

$$\Rightarrow 8x - x^2 = 12$$

$$\Rightarrow 0 = x^2 - 8x + 12$$

$$\Rightarrow 0 = (x - 6)(x - 2)$$

$$\Rightarrow x = 2 \text{ or } x = 6$$

M is on the same line as L.

So M is the point $(6, 12)$.

b Area $= \int_6^8 \left(8x - x^2\right) \mathrm{d}x$

$$= \left(4x^2 - \frac{x^3}{3}\right)_6^8$$

$$= \left(4 \times 64 - \tfrac{512}{3}\right) - \left(4 \times 36 - \tfrac{216}{3}\right)$$

$$= 256 - 170\tfrac{2}{3} - 144 + 72$$

$$= 13\tfrac{1}{3}$$

17 a A is the point $(1, 0)$, B is the point $(5, 0)$.

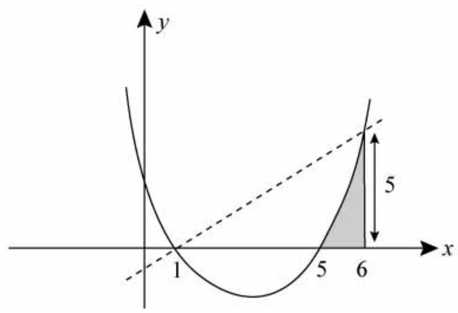
$$x - 1 = (x - 1)(x - 5)$$

$$\Rightarrow 0 = (x - 1)(x - 5 - 1)$$

$$\Rightarrow 0 = (x - 1)(x - 6)$$

$$\Rightarrow x = 1, x = 6$$

So C is the point $(6, 5)$.

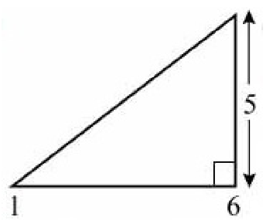


b Drop a perpendicular from C to the x-axis to a point D.

The area of the shaded region is

Area of triangle $ABD - \int_5^6 (x - 1)(x - 5) \, \mathrm{d}x$

$= \text{Area of } ABD - \int_5^6 \left(x^2 - 6x + 5\right) \mathrm{d}x$



17 b Area $=\left(\frac{1}{2}\times5\times5\right)-\int_5^6\left(x^2-6x+5\right)\,\mathrm{d}x$

$=12\frac{1}{2}-\left[\frac{1}{3}x^3-3x^2+5x\right]_5^6$

$=12\frac{1}{2}-\left[(72-108+30)-\left(41\frac{2}{3}-75+25\right)\right]$

$=12\frac{1}{2}-\left[(-6)-\left(-8\frac{1}{3}\right)\right]$

$=12\frac{1}{2}-2\frac{1}{3}$

$=10\frac{1}{6}$

18 a For the point A, which lies on the line and the curve

$4q+25=p+40-16$

$\Rightarrow\ 4q=p-1 \qquad \mathbf{(1)}$

For the point B, which lies on the line and the curve

$8q+25=p+80-64$

$\Rightarrow\ 8q=p-9 \qquad \mathbf{(2)}$

Subtracting $\mathbf{(2)}-\mathbf{(1)}$

$\Rightarrow 4q=-8$

$\Rightarrow q=-2$

Substituting into $\mathbf{(1)}$

$\Rightarrow p=1+4q$

$\Rightarrow p=-7$

b At A, $y=4q+25=17$

So C is given by

$17=-7+10x-x^2$

$x^2-10x+24=0$

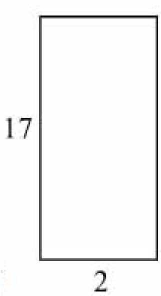
$(x-6)(x-4)=0$

$x=4,\ x=6$

So C is the point $(6,17)$

c The required area is

$\int_4^6(-7+10x-x^2)\,\mathrm{d}x-\text{area of rectangle}$



18 c Area $=\left(-7x+5x^2-\frac{1}{3}x^3\right)_4^6-34$

$=(-42+180-72)-\left(-28+80-\frac{64}{3}\right)-34$

$=\frac{4}{3}$ or $1\frac{1}{3}$

19 $\int\left(\frac{9}{x^2}-8\sqrt{x}+4x-5\right)\mathrm{d}x$

$=\int(9x^{-2}-8x^{\frac{1}{2}}+4x-5)\,\mathrm{d}x$

$=\frac{9x^{-1}}{-1}-\frac{8x^{\frac{3}{2}}}{\frac{3}{2}}+\frac{4x^2}{2}-5x+c$

$=-\frac{9}{x}-\frac{16x^{\frac{3}{2}}}{3}+2x^2-5x+c$

20 $A^2=\int_4^9\left(\frac{3}{\sqrt{x}}-A\right)\mathrm{d}x$

$=\int_4^9\left(3x^{-\frac{1}{2}}-A\right)\mathrm{d}x$

$=\left[\frac{3x^{\frac{1}{2}}}{\frac{1}{2}}-Ax\right]_4^9$

$=\left[6x^{\frac{1}{2}}-Ax\right]_4^9$

$=\left(6(9)^{\frac{1}{2}}-A(9)\right)-\left(6(4)^{\frac{1}{2}}-A(4)\right)$

$=(18-9A)-(12-4A)$

$0=(A+6)(A-1)$

$A=-6$ or $A=1$

21 a $\mathrm{f}'(x)=\frac{(2-x^2)^3}{x^2}$

$=\frac{(2-x^2)(2-x^2)(2-x^2)}{x^2}$

$=\frac{(4-4x^2+x^4)(2-x^2)}{x^2}$

$=x^{-2}(8-12x^2+6x^4-x^6)$

$=8x^{-2}-12+6x^2-x^4$

So $A=6$ and $B=-1$

b $\mathrm{f}''(x)=-16x^{-3}+12x-4x^3$

c $\mathrm{f}(x)=\int(8x^{-2}-12+6x^2-x^4)\,\mathrm{d}x$

$=\frac{8x^{-1}}{-1}-12x+\frac{6x^3}{3}-\frac{x^5}{5}+c$

$=-\frac{8}{x}-12x+2x^3-\frac{x^5}{5}+c$

21 c When $x = -2$ and $y = 9$

$$-\frac{8}{-2} - 12(-2) + 2(-2)^3 - \frac{(-2)^5}{5} + c = 9$$

$$4 + 24 - 16 + \tfrac{32}{5} + c = 9$$

$$c = -\tfrac{47}{5}$$

$$f(x) = -\frac{8}{x} - 12x + 2x^3 - \frac{x^5}{5} - \frac{47}{5}$$

22 a $y = 3 - 5x - 2x^2$

When $y = 0$, $3 - 5x - 2x^2 = 0$

$(3 + x)(1 - 2x) = 0$

$x = -3$ or $x = \tfrac{1}{2}$

The points are $A(-3, 0)$ and $B(\tfrac{1}{2}, 0)$.

b $\int_{-3}^{\frac{1}{2}} (3 - 5x - 2x^2)\ \mathrm{d}x$

$$= \left[3x - \frac{5x^2}{2} - \frac{2x^3}{3}\right]_{-3}^{\frac{1}{2}}$$

$$= \left(3\left(\tfrac{1}{2}\right) - \frac{5\left(\frac{1}{2}\right)^2}{2} - \frac{2\left(\frac{1}{2}\right)^3}{3}\right) - \left(3(-3) - \frac{5(-3)^2}{2} - \frac{2(-3)^3}{3}\right)$$

$$= \left(\tfrac{3}{2} - \tfrac{5}{8} - \tfrac{1}{12}\right) - \left(-9 - \tfrac{45}{2} + \tfrac{54}{3}\right)$$

$$= 14\tfrac{7}{24}$$

23 a $(x - 4)(2x + 3) = 0$

$x = 4$ or $x = -\tfrac{3}{2}$

The points are $A(-\tfrac{3}{2}, 0)$ and $B(4, 0)$.

b $R = \int_{-\frac{3}{2}}^{4} (x - 4)(2x + 3)\ \mathrm{d}x$

$$= \int_{-\frac{3}{2}}^{4} (2x^2 - 5x - 12)\ \mathrm{d}x$$

$$= \left[\frac{2x^3}{3} - \frac{5x^2}{2} - 12x\right]_{-\frac{3}{2}}^{4}$$

$$= \left(\frac{2(4)^3}{3} - \frac{5(4)^2}{2} - 12(4)\right) - \left(\frac{2(-\frac{3}{2})^3}{3} - \frac{5(-\frac{3}{2})^2}{2} - 12(-\tfrac{3}{2})\right)$$

$$= \left(\tfrac{128}{3} - 40 - 48\right) - \left(-\tfrac{9}{4} - \tfrac{45}{8} + 18\right)$$

$$= -55\tfrac{11}{24}$$

Area $= 55\tfrac{11}{24}$

24 a $x(x - 3)(x + 2) = 0$

$x = 0$, $x = 3$ or $x = -2$

The points are $A(-2, 0)$ and $B(3, 0)$.

b $\int_{-2}^{0} x(x-3)(x+2)\ \mathrm{d}x - \int_{0}^{3} x(x-3)(x+2)\ \mathrm{d}x$

$$= \int_{-2}^{0} (x^3 - x^2 - 6x)\ \mathrm{d}x - \int_{0}^{3} (x^3 - x^2 - 6x)\ \mathrm{d}x$$

$$\int_{-2}^{0} (x^3 - x^2 - 6x)\ \mathrm{d}x = \left[\frac{x^4}{4} - \frac{x^3}{3} - \frac{6x^2}{2}\right]_{-2}^{0}$$

$$= \left[\frac{x^4}{4} - \frac{x^3}{3} - 3x^2\right]_{-2}^{0}$$

$$= \left(\frac{0^4}{4} - \frac{0^3}{3} - 3(0)^2\right) - \left(\frac{(-2)^4}{4} - \frac{(-2)^3}{3} - 3(-2)^2\right)$$

$$= 0 - \left(4 + \tfrac{8}{3} - 12\right)$$

$$= 5\tfrac{1}{3}$$

$$\int_{0}^{3} (x^3 - x^2 - 6x)\ \mathrm{d}x = \left[\frac{x^4}{4} - \frac{x^3}{3} - 3x^2\right]_{0}^{3}$$

$$= \left(\frac{3^4}{4} - \frac{3^3}{3} - 3(3)^2\right) - \left(\frac{0^4}{4} - \frac{0^3}{3} - 3(0)^2\right)$$

$$= \left(\tfrac{81}{4} - 9 - 27\right)$$

$$= -15\tfrac{3}{4}$$

Total area is $5\tfrac{1}{3} - (-15\tfrac{3}{4}) = 21\tfrac{1}{12}$

Challenge

To find the points of intersection:

$x^2 - 5x + 7 = \frac{1}{2}x^2 - \frac{5}{2}x + 7$
$2x^2 - 10x + 14 = x^2 - 5x + 14$
$x^2 - 5x = 0$
$x(x - 5) = 0$
$x = 0$ or $x = 5$

Area R =
(area under the curve $y = \frac{1}{2}x^2 - \frac{5}{2}x + 7$)
– (area under the curve $y = x^2 - 5x + 7$)

Area under the curve: $y = \frac{1}{2}x^2 - \frac{5}{2}x + 7$:

$$\text{Area under the curve} = \int_0^5 \left(\tfrac{1}{2}x^2 - \tfrac{5}{2}x + 7\right) \, dx$$
$$= \left[\frac{1}{3} \times \frac{x^3}{2} - \frac{1}{2} \times \frac{5x^2}{2} + 7x\right]_0^5$$
$$= \left[\frac{x^3}{6} - \frac{5x^2}{4} + 7x\right]_0^5$$
$$= \left(\frac{5^3}{6} - \frac{5(5)^2}{4} + 7(5)\right) - \left(\frac{0^3}{6} - \frac{5(0)^2}{4} + 7(0)\right)$$
$$= \left(\tfrac{125}{6} - \tfrac{125}{4} + 35\right)$$
$$24\tfrac{7}{12}$$

Area under the curve: $y = x^2 - 5x + 7$:

$$\text{Area under the curve} = \int_0^5 (x^2 - 5x + 7) \, dx$$
$$= \left[\frac{x^3}{3} - \frac{5x^2}{2} + 7x\right]_0^5$$
$$= \left(\frac{5^3}{3} - \frac{5(5)^2}{2} + 7(5)\right) - \left(\frac{0^3}{3} - \frac{5(0)^2}{2} + 7(0)\right)$$
$$= \left(\tfrac{125}{3} - \tfrac{125}{2} + 35\right)$$
$$= 14\tfrac{1}{6}$$

$$R = 24\tfrac{7}{12} - 14\tfrac{1}{6}$$
$$= 10\tfrac{5}{12}$$