Exponentials and logarithms 14G

1 a When $e^x = 6$
$\ln(e^x) = \ln 6$
$x = \ln 6$

b When $e^{2x} = 11$
$\ln(e^{2x}) = \ln 11$
$2x = \ln 11$
$x = \frac{1}{2}\ln 11$

c When $e^{-x+3} = 20$
$\ln(e^{-x+3}) = \ln 20$
$-x + 3 = \ln 20$
$x = 3 - \ln 20$

d When $3e^{4x} = 1$
$e^{4x} = \frac{1}{3}$
$\ln(e^{4x}) = \ln \frac{1}{3}$
$4x = \ln \frac{1}{3}$
$x = \frac{1}{4}\ln \frac{1}{3}$

e When $e^{2x+6} = 3$
$\ln(e^{2x+6}) = \ln 3$
$2x + 6 = \ln 3$
$x = \ln 3 - 6$
$x = \frac{1}{2}\ln 3 - 3$

f When $e^{5-x} = 19$
$\ln(e^{5-x}) = \ln 19$
$5 - x = \ln 19$
$x = 5 - \ln 19$

2 a When $\ln x = 2$
$e^{\ln x} = e^2$
$x = e^2$

b When $\ln(4x) = 1$
$e^{\ln(4x)} = e^1$
$4x = e^1$
$x = \frac{e}{4}$

c When $\ln(2x + 3) = 4$
$e^{\ln(2x+3)} = e^4$
$2x + 3 = e^4$
$2x = e^4 - 3$
$x = \frac{1}{2}e^4 - \frac{3}{2}$

2 d When $2\ln(6x - 2) = 5$
$\ln(6x - 2) = \frac{5}{2}$
$e^{\ln(6x-2)} = e^{\frac{5}{2}}$
$6x - 2 = e^{\frac{5}{2}}$
$6x = e^{\frac{5}{2}} + 2$
$x = \frac{1}{6}(e^{\frac{5}{2}} + 2)$

e When $\ln(18 - x) = \frac{1}{2}$
$e^{\ln(18-x)} = e^{\frac{1}{2}}$
$18 - x = e^{\frac{1}{2}}$
$x = 18 - e^{\frac{1}{2}}$

f When $\ln(x^2 - 7x + 11) = 0$
$e^{\ln(x^2-7x+11)} = e^0$
$x^2 - 7x + 11 = 1$
$x^2 - 7x + 10 = 0$
$(x - 2)(x - 5) = 0$
$x = 2$ or $x = 5$

3 a $e^{2x} - 8e^x + 12 = 0$
Let $u = e^x$
$u^2 - 8u + 12 = 0$
$(u - 2)(u - 6) = 0$
$u = 2$ or $u = 6$
$e^x = 2$ or $e^x = 6$

When $e^x = 2$
$\ln(e^x) = \ln 2$
$x = \ln 2$
When $e^x = 6$
$\ln(e^x) = \ln 6$
$x = \ln 6$

$x = \ln 2$ or $x = \ln 6$

b $e^{4x} - 3e^{2x} + 2 = 0$
Let $u = e^{2x}$
$u^2 - 3u + 2 = 0$
$(u - 1)(u - 2) = 0$
$u = 1$ or $u = 2$
$e^{2x} = 1$ or $e^{2x} = 2$

3 b When $e^{2x} = 1$
$\ln(e^{2x}) = \ln 1$
$2x = 0$
$x = 0$
When $e^{2x} = 2$
$\ln(e^{2x}) = \ln 2$
$2x = \ln 2$
$x = \frac{1}{2}\ln 2$

$x = 0$ or $x = \frac{1}{2}\ln 2$

c $(\ln x)^2 + 2\ln x - 15 = 0$
Let $u = \ln x$
$u^2 + 2u - 15 = 0$
$(u + 5)(u - 3) = 0$
$u = -5$ or $u = 3$

When $\ln x = -5$
$e^{\ln x} = e^{-5}$
$x = e^{-5}$
When $\ln x = 3$
$e^{\ln x} = e^3$
$x = e^3$

$x = e^{-5}$ or $x = e^3$

d $e^x - 5 + 4e^{-x} = 0$
Multiply each term by e^x
$e^{2x} - 5e^x + 4 = 0$
Let $u = e^x$
$u^2 - 5u + 4 = 0$
$(u - 1)(u - 4) = 0$
$u = 1$ or $u = 4$
$e^x = 1$ or $e^x = 4$

When $e^x = 1$
$\ln(e^x) = \ln 1$
$x = 0$
When $e^x = 4$
$\ln(e^x) = \ln 4$
$x = \ln 4$

$x = 0$ or $x = \ln 4$

e $3e^{2x} - 16e^x + 5 = 0$
Let $u = e^x$
$3u^2 - 16u + 5 = 0$
$(3u - 1)(u - 5) = 0$
$u = \frac{1}{3}$ or $u = 5$
$e^x = \frac{1}{3}$ or $e^x = 5$

e When $e^x = \frac{1}{3}$
$\ln(e^x) = \ln \frac{1}{3}$
$x = \ln \frac{1}{3}$
When $e^x = 5$
$\ln(e^x) = \ln 5$
$x = \ln 5$

$x = \ln \frac{1}{3}$ or $x = \ln 5$

f $(\ln x)^2 - 4\ln x - 12 = 0$
Let $u = \ln x$
$u^2 - 4u - 12 = 0$
$(u + 2)(u - 6) = 0$
$u = -2$ or $u = 6$

When $\ln x = -2$
$e^{\ln x} = e^{-2}$
$x = e^{-2}$
When $\ln x = 6$
$e^{\ln x} = e^6$
$x = e^6$

$x = e^{-2}$ or $x = e^6$

4 $e^x - 7 + 12e^{-x} = 0$
Multiply each term by e^x
$e^{2x} - 7e^x + 12 = 0$
Let $u = e^x$
$u^2 - 7u + 12 = 0$
$(u - 3)(u - 4) = 0$
$u = 3$ or $u = 4$
$e^x = 3$ or $e^x = 4$

When $e^x = 3$
$\ln(e^x) = \ln 3$
$x = \ln 3$
When $e^x = 4$
$\ln(e^x) = \ln 4$
$x = \ln 2^2$
$x = 2\ln 2$

$x = \ln 3$ or $x = 2\ln 2$

5 a When $\ln(8x - 3) = 2$
$e^{\ln(8x-3)} = e^2$
$8x - 3 = e^2$
$8x = e^2 + 3$
$x = \frac{1}{8}(e^2 + 3)$

5 b When $e^{5(x-8)} = 3$
$\ln(e^{5(x-8)}) = \ln 3$
$5(x-8) = \ln 3$
$x - 8 = \frac{1}{5}\ln 3$
$x = \frac{1}{5}\ln 3 + 8$

c $e^{10x} - 8e^{5x} + 7 = 0$
Let $u = e^{5x}$
$u^2 - 8u + 7 = 0$
$(u-1)(u-7) = 0$
$u = 1$ or $u = 7$
$e^{5x} = 1$ or $e^{5x} = 7$

When $e^{5x} = 1$
$\ln(e^{5x}) = \ln 1$
$5x = 0$
$x = 0$
When $e^{5x} = 7$
$\ln(e^{5x}) = \ln 7$
$5x = \ln 7$
$x = \frac{1}{5}\ln 7$

$x = 0$ or $x = \frac{1}{5}\ln 7$

d When $(\ln x - 1)^2 = 4$
$(\ln x)^2 - 2\ln x - 3 = 0$
Let $u = \ln x$
$u^2 - 2u - 3 = 0$
$(u+1)(u-3) = 0$
$u = -1$ or $u = 3$

When $\ln x = -1$
$e^{\ln x} = e^{-1}$
$x = e^{-1}$
When $\ln x = 3$
$e^{\ln x} = e^3$
$x = e^3$

$x = e^{-1}$ or $x = e^3$

6 When $3^x e^{4x-1} = 5$
$\ln(3^x e^{4x-1}) = \ln 5$
$\ln(3^x) + \ln(e^{4x-1}) = \ln 5$
$x\ln 3 + 4x - 1 = \ln 5$
$x\ln 3 + 4x = 1 + \ln 5$
$x(\ln 3 + 4) = 1 + \ln 5$
$$x = \frac{1+\ln 5}{4+\ln 3}$$

7 a $D = 6$ when $t = 0$ so 6 is the initial concentration of the drug in mg/l.

7 b $D = 6e^{\frac{-t}{10}}$
When $t = 2$
$D = 6e^{\frac{-2}{10}}$
$D = 4.91$ mg/l (3 s.f.)

c When $6e^{\frac{-t}{10}} = 3$
$e^{\frac{-t}{10}} = \frac{1}{2}$
$\ln e^{\frac{-t}{10}} = \ln\frac{1}{2}$
$\frac{-t}{10} = \ln\frac{1}{2}$
$t = -10\ln\frac{1}{2}$
$t = 6.931471...$
$t = 6$ hours and 55.888... minutes
$t = 6$ hours and 56 minutes

8 a A is where $x = 0$
Substitute $x = 0$ into
$y = 3 + \ln(4-x)$ to give
$y = 3 + \ln 4$
$A = (0, 3 + \ln 4)$

b B is where $y = 0$
Substitute $y = 0$ into
$y = 3 + \ln(4-x)$ to give
$0 = 3 + \ln(4-x)$
$-3 = \ln(4-x)$
$e^{-3} = 4 - x$
$x = 4 - e^{-3}$
$B = (4 - e^{-3}, 0)$

9 a When $t = 0$, $V = 27\,000$,
so $27\,000 = Ae^{k \times 0} = A$
When $t = 5$, $V = 18\,000$,
so $18\,000 = Ae^{5k}$

Substituting in $A = 27\,000$

$$18000 = 27000e^{5k}$$

$$\frac{18000}{27000} = e^{5k}$$

$$\frac{2}{3} = e^{5k}$$

$$\ln\left(\frac{2}{3}\right) = \ln\left(e^{5k}\right)$$

$$\ln\left(\frac{2}{3}\right) = 5k$$

$k = -0.08109\ldots = -0.0811$ (3 s.f.)

So $A = 27\,000$, $k = -0.0811$ (3 s.f.)

b According to the model,
when $t = 8$, $V = 14\,100$ (3 s.f.)
so model is reliable.

10 a Consider linear model in the form:

$P = mt + c$

When $t = 0$, $P = 7.6$, so $c = 7.6$

When $t = 20$, $P = 12.1$, so $12.1 = 20m + 7.6$

Solve to find m:

$$12.1 = 20m + 7.6$$

$$m = \frac{12.1 - 7.6}{20} = 0.225t$$

Linear model: $P = 0.225t + 7.6$

b Consider exponential model in the form:

$P = ab^t$

When $t = 0$, $P = 7.6$, so $a = 7.6$

When $t = 20$, $P = 12.1$, so $12.1 = 7.6b^{20}$

Solve to find b:

$$12.1 = 7.6b^{20}$$

$$\frac{12.1}{7.6} = b^{20}$$

$$\ln\left(\frac{12.1}{7.6}\right) = 20\ln b$$

$$\ln\left(\frac{12.1}{7.6}\right)^{\frac{1}{20}} = \ln b$$

$$b = 1.0235 \text{ (4 d.p.)}$$

c When $t = 50$, linear model predicts 18.85 million people, and exponential model predicts 24.3 million people. Exponential model is best supported by the given fact.

Challenge

$g(0) = Ae^{B \times 0} + C = 5$
$A + C = 5$
As $y = 2$ is an asymptote, $C = 2$
$A = 3$ and $g(6) = 3e^{B \times 6} + 2 = 10$

$$3e^{6B} = 8$$

$$e^{6B} = \tfrac{8}{3}$$

$$\ln(e^{6B}) = \ln\tfrac{8}{3}$$

$$6B = \ln\tfrac{8}{3}$$

$$B = \frac{\ln\frac{8}{3}}{6}$$