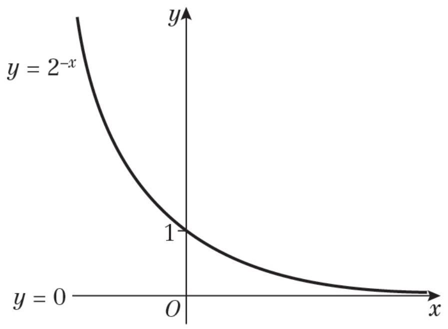
Exponentials and logarithms, Mixed Exercise 14

1 a $y = 2^{-x} = (2^{-1})^x = (\frac{1}{2})^x$



b $y = 5e^x - 1$

The graph is a translation by the vector $\begin{pmatrix} 0 \\ -1 \end{pmatrix}$ and a vertical stretch scale factor 5 of the graph $y = e^x$.

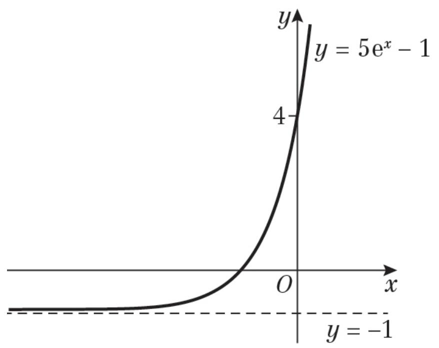
The graph crosses the y-axis when $x = 0$.

$y = 5 \times e^0 - 1$

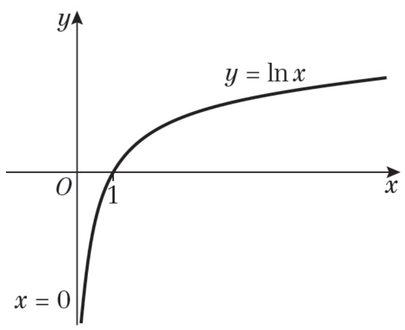
$y = 4$

The graph crosses the y-axis at (0, 4).

Asymptote is at $y = -1$.



c $y = \ln x$



2 a $\log_a(p^2q) = \log_a(p^2) + \log_a q$

$= 2\log_a p + \log_a q$

2 b $\log_a(pq) = \log_a p + \log_a q$

So

$\log_a p + \log_a q = 5 \quad$ **(1)**

$2\log_a p + \log_a q = 9 \quad$ **(2)**

Subtract **(1)** from **(2)**:

$\log_a p = 4$

So $\log_a q = 1$

3 a $p = \log_q 16$

$= \log_q(2^4)$

$= 4\log_q 2$

$\log_q 2 = \frac{p}{4}$

b $\log_{q+1}(8q) = \log_q 8 + \log_q q$

$= \log_q(2^3) + \log_q q$

$= 3\log_q 2 + \log_q q$

$= 3 \times \frac{p}{4} + 1$

$= \frac{3p}{4} + 1$

4 a $4^x = 23$

$\log_4 23 = x$

$x = 2.26$

b $7^{(2x+1)} = 1000$

$\log_7 1000 = 2x + 1$

$2x = \log_7 1000 - 1$

$x = \frac{1}{2}\log_7 1000 - \frac{1}{2}$

$= 1.27$

c $10^x = 6^{x+2}$

$\log(10^x) = \log(6^{x+2})$

$x \log 10 = (x + 2) \log 6$

$x \log 10 - x \log 6 = 2 \log 6$

$x(\log 10 - \log 6) = 2 \log 6$

$x = \frac{2\log 6}{\log 10 - \log 6}$

$= 7.02$

5 a $4^x - 2^{x+1} - 15 = 0$

$2^{2x} - 2 \times 2^x - 15 = 0$

$(2^x)^2 - 2 \times 2^x - 15 = 0$

Let $u = 2^x$

$u^2 - 2u - 15 = 0$

b $(u+3)(u-5) = 0$

So $u = -3$ or $u = 5$

If $u = -3$, $2^x = -3$. No solution.

If $u = 5$, $2^x = 5$

$\log 2^x = \log 5$

$x \log 2 = \log 5$

$x = \dfrac{\log 5}{\log 2}$

$= 2.32$ (2 d.p.)

6 $\log_2(x+10) - \log_2(x-5) = 4$

$\log_2\left(\dfrac{x+10}{x-5}\right) = 4$

$\dfrac{x+10}{x-5} = 2^4$

$16x - 80 = x + 10$

$15x = 90$

$x = 6$

7 a $y = e^{-x}$

$\dfrac{dy}{dx} = -e^{-x}$

b $y = e^{11x}$

$\dfrac{dy}{dx} = 11e^{11x}$

c $y = 6e^{5x}$

$\dfrac{dy}{dx} = 5 \times 6e^{5x} = 30e^{5x}$

8 a $\ln(2x-5) = 8$ (inverse of ln)

$2x - 5 = e^8$ $(+5)$

$2x = e^8 + 5$ $(\div 2)$

$x = \dfrac{e^8 + 5}{2}$

8 b $e^{4x} = 5$ (inverse of e)

$4x = \ln 5$ $(\div 4)$

$x = \dfrac{\ln 5}{4}$

c $24 - e^{-2x} = 10$ $(+e^{-2x})$

$24 = 10 + e^{-2x}$ (-10)

$14 = e^{-2x}$ (inverse of e)

$\ln(14) = -2x$ $(\div -2)$

$-\frac{1}{2}\ln(14) = x$

$x = -\frac{1}{2}\ln(14)$

d $\ln(x) + \ln(x-3) = 0$

$\ln(x(x-3)) = 0$

$x(x-3) = e^0$

$x(x-3) = 1$

$x^2 - 3x - 1 = 0$

$x = \dfrac{3 \pm \sqrt{9+4}}{2}$

$= \dfrac{3 \pm \sqrt{13}}{2}$

$= \dfrac{3 + \sqrt{13}}{2}$

(x cannot be negative because of initial equation)

e $e^x + e^{-x} = 2$

$e^x + \dfrac{1}{e^x} = 2$ $(\times\ e^x)$

$(e^x)^2 + 1 = 2e^x$

$(e^x)^2 - 2e^x + 1 = 0$

$(e^x - 1)^2 = 0$

$e^x = 1$

$x = \ln 1 = 0$

f $\ln 2 + \ln x = 4$

$\ln 2x = 4$

$2x = e^4$

$x = \dfrac{e^4}{2}$

9 $P = 100 + 850e^{-\frac{t}{2}}$

a New price is when $t = 0$

Substitute $t = 0$ into $P = 100 + 850e^{-\frac{t}{2}}$ to give:

$P = 100 + 850e^{-\frac{0}{2}} \quad (e^0 = 1)$

$= 100 + 850 = 950$

The new price is £950

b After 3 years $t = 3$.

Substitute $t = 3$ into $P = 100 + 850e^{-\frac{t}{2}}$ to give:

$P = 100 + 850e^{-\frac{3}{2}} = 289.66$

Price after 3 years is £290 (to nearest £)

c It is worth less than £200 when $P < 200$

Substitute $P = 200$ into $P = 100 + 850e^{-\frac{t}{2}}$ to give:

$200 = 100 + 850e^{-\frac{t}{2}}$

$100 = 850e^{-\frac{t}{2}}$

$\frac{100}{850} = e^{-\frac{t}{2}}$

$\ln\left(\frac{100}{850}\right) = -\frac{t}{2}$

$t = -2\ln\left(\frac{100}{850}\right)$

$t = 4.28$

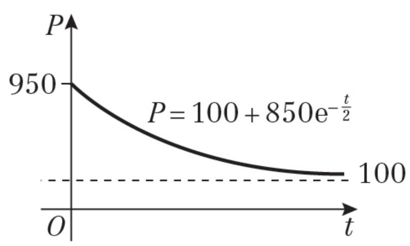
It is worth less than £200 after 4.28 years.

d As $t \to \infty$, $e^{-\frac{t}{2}} \to 0$

Hence, $P \to 100 + 850 \times 0 = 100$

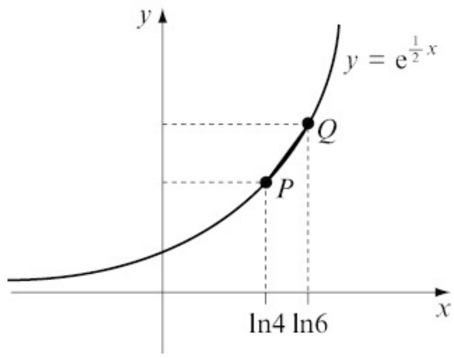
The computer will be worth £100 eventually.

e



9 f A good model. The computer will always be worth something.

10 a



Q has y-coordinate $e^{\frac{1}{2}\ln 16} = e^{\ln 16^{\frac{1}{2}}} = 16^{\frac{1}{2}} = 4$

P has y-coordinate $e^{\frac{1}{2}\ln 4} = e^{\ln 4^{\frac{1}{2}}} = 4^{\frac{1}{2}} = 2$

Gradient of the line $PQ = \frac{\text{change in } y}{\text{change in } x}$

$= \frac{4-2}{\ln 16 - \ln 4}$

$= \frac{2}{\ln\left(\frac{16}{4}\right)}$

$= \frac{2}{\ln 4}$

Using $y = mx + c$, the equation of the line PQ is:

$y = \frac{2}{\ln 4}x + c$

$(\ln 4, 2)$ lies on the line so

$y = \frac{2}{\ln 4}x + c$

$2 = 2 + c$

$c = 0$

Equation of PQ is $y = \frac{2x}{\ln 4}$

b The line passes through the origin as $c = 0$.

c Length from $(\ln 4, 2)$ to $(\ln 16, 4)$ is

$\sqrt{(\ln 16 - \ln 4)^2 + (4-2)^2}$

$= \sqrt{\left(\ln\frac{16}{4}\right)^2 + 2^2}$

$= \sqrt{(\ln 4)^2 + 4} = 2.43$

11 a $T = 55e^{-\frac{t}{8}} + 20$

t is the time in minutes and time cannot be negative as you can't go back in time.

b The starting temperature of the cup of tea is when $t = 0$

$T = 55e^{-\frac{0}{8}} + 20 = 75°C$

c When $T = 50°C$

$55e^{-\frac{t}{8}} + 20 = 50$

$55e^{-\frac{t}{8}} = 30$

$e^{-\frac{t}{8}} = \frac{30}{55}$

$\ln\left(e^{-\frac{t}{8}}\right) = \ln\left(\frac{30}{55}\right)$

$-\frac{t}{8} = \ln\left(\frac{30}{55}\right)$

$t = -8\ln\left(\frac{30}{55}\right)$

$= 4.849...$

≈ 5 minutes

d The exponential term will always be positive, so the overall temperature will be greater than 20°C.

12 a As $S = aV^b$

$\log S = \log (aV^b)$

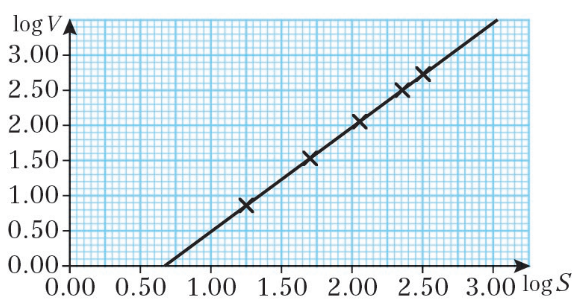
$\log S = \log a + \log (V^b)$

$\log S = \log a + b \log V$

b

$\log S$	1.26	1.70	2.05	2.35	2.50
$\log V$	0.86	1.53	2.05	2.49	2.72

c



12 d b is the gradient $= \frac{2.72 - 0.86}{2.5 - 1.26}$

$= \frac{1.86}{1.24} = 1.5$

Intercept $= \log a$

$\log a = -1.05$

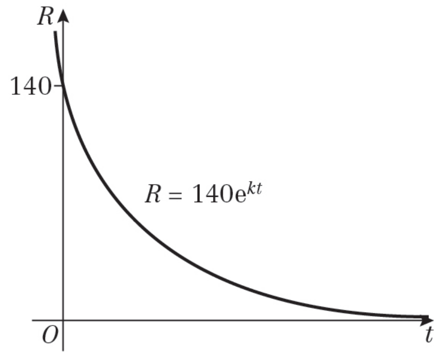
$10^{-1.05} = a$

$a = 0.0891...$

$a \approx 0.09$

13 a kt must be negative as the model is for decay, not growth. As t (for time) is always positive, k must be negative.

b



c $R = 140e^{kt}$

When $R = 70$ and $t = 30$

$70 = 140e^{30k}$

$\frac{1}{2} = e^{30k}$

$\ln\frac{1}{2} = \ln\left(e^{30k}\right)$

$= 30k$

$k = \frac{1}{30}\ln\frac{1}{2}$

$= \frac{1}{30}\ln\left(2^{-1}\right)$

$= -\frac{1}{30}\ln 2$

So $c = -\frac{1}{30}$

14 a $V = e^{0.4x} - 1$

When $x = 5$

$V = e^{0.4\times5} - 1$

$= e^2 - 1$

$= 7.389... - 1$

$= 6.389...$

≈ 6.4 million views

b $V = e^{0.4x} - 1$

$\frac{dV}{dx} = 0.4e^{0.4x}$

14 c When $x = 100$

$\frac{dV}{dx} = 0.4e^{0.4 \times 100}$

$= 0.4e^{40}$

$= 9.415... \times 10^{16}$ million

So 9.4×10^{22} new views per day

d This number is greater than the population of the world, so the model is not valid after 100 days.

15 a $M = \frac{2}{3}\log_{10}(S) - 10.7$

When $S = 2.24 \times 10^{22}$

$M = \frac{2}{3}\log_{10}(2.24 \times 10^{22}) - 10.7$

$= \frac{2}{3}(\log_{10} 2.24 + \log 10^{22}) - 10.7$

$= \frac{2}{3}(0.3502... + 22) - 10.7$

$= 4.2$

b i When $M = 6$

$6 = \frac{2}{3}\log_{10}(S) - 10.7$

$16.7 = \frac{2}{3}\log_{10}(S)$

$25.05 = \log_{10}(S)$

$10^{25.05} = S$

$S = 1.12 \times 10^{25}$ dyne cm

ii When $M = 7$

$7 = \frac{2}{3}\log_{10}(S) - 10.7$

$17.7 = \frac{2}{3}\log_{10}(S)$

$26.55 = \log_{10}(S)$

$10^{26.55} = S$

$S = 3.55 \times 10^{26}$ dyne cm

c $\frac{3.55 \times 10^{26}}{1.12 \times 10^{25}} = 31.6...$

≈ 32 times

16 a The student goes wrong in line 2, where the subtraction should be a division (as in line 2 below).

16 b The full working should have looked like this:

$\log_2 x - \frac{1}{2}\log_2(x+1) = 1$

$\log_2 x - \log_2\left((x+1)^{\frac{1}{2}}\right) = 1$

$\log_2 x - \log_2(\sqrt{x+1}) = 1$

$\log_2 \frac{x}{\sqrt{x+1}} = 1$

$\frac{x}{\sqrt{x+1}} = 2^1$

$x = 2\sqrt{x+1}$ (square)

$x^2 = 4x + 4$

$x^2 - 4x - 4 = 0$ (use quadratic formula)

$x = 2 + 2\sqrt{2}$

($x \neq 2 - 2\sqrt{2}$ because log cannot take negative input values)

Challenge

a $y = 9^x = (3^2)^x = 3^{2x}$

So $\log_3 y = 2x$

b As $y = 9^x$

$\log_9 y = \log_9(9^x)$

$\log_9 y = x\log_9 9$

$\log_9 y = 1$, so $\log_9 y = x$

$2x = 2\log_9 y$ and from **a**, $2x = \log_3 y$

So $\log_3 y = 2\log_9 y$

$\log_3 y = \log_9 y^2$

c Using $\log_3 y = \log_9 y^2$

$\log_3(2-3x) = \log_9(2-3x)^2$

$= \log_9(4 - 12x + 9x^2)$

So $\log_9(4 - 12x + 9x^2) = \log_9(6x^2 - 19x + 2)$

Therefore $4 - 12x + 9x^2 = 6x^2 - 19x + 2$

$3x^2 + 7x + 2 = 0$

$(3x + 1)(x + 2) = 0$

$x = -\frac{1}{3}$ or $x = -2$

17 $9^x - 11(3^x) + 18 = 0$

$(3^x)^2 - 11(3^x) + 18 = 0$

Let $u = 3^x$

$u^2 - 11u + 18 = 0$

$(u-9)(u-2) = 0$

$u = 2$ or $u = 9$

When $u = 9$

$3^x = 9$

$\ln(3^x) = \ln 9$

$x = \frac{\ln 9}{\ln 3} = 2$

When $u = 2$

$3^x = 2$

$\ln(3^x) = \ln 2$

$x = \frac{\ln 2}{\ln 3} = 0.631$ (3 s.f.)

So $x = 2$ or $x = 0.631$ (3 s.f.)