Algebraic expressions, Mixed Exercise 1

1 a $y^3 \times y^5$
$= y^{3+5}$
$= y^8$

b $3x^2 \times 2x^5$
$= 3 \times 2 \times x^{2+5}$
$= 6x^7$

c $(4x^2)^3 \div 2x^5$
$= 4^3x^{2\times 3} \div 2x^5$
$= 64x^6 \div 2x^5$
$= 32x^{6-5}$
$= 32x$

d $4b^2 \times 3b^3 \times b^4$
$= 4 \times 3 \times b^{2+3+4}$
$= 12b^9$

2 a $(x + 3)(x - 5)$
$= x^2 - 5x + 3x - 15$
$= x^2 - 2x - 15$

b $(2x - 7)(3x + 1)$
$= 6x^2 + 2x - 21x - 7$
$= 6x^2 - 19x - 7$

c $(2x + 5)(3x - y + 2)$
$= 2x(3x - y + 2) + 5(3x - y + 2)$
$= 6x^2 - 2xy + 4x + 15x - 5y + 10$
$= 6x^2 - 2xy + 19x - 5y + 10$

3 a $x(x + 4)(x - 1)$
$= (x^2 + 4x)(x - 1)$
$= x^3 - x^2 + 4x^2 - 4x$
$= x^3 + 3x^2 - 4x$

b $(x + 2)(x - 3)(x + 7)$
$= (x^2 - 3x + 2x - 6)(x + 7)$
$= (x^2 - x - 6)(x + 7)$
$= x^2(x + 7) - x(x + 7) - 6(x + 7)$
$= x^3 + 7x^2 - x^2 - 7x - 6x - 42$
$= x^3 + 6x^2 - 13x - 42$

c $(2x + 3)(x - 2)(3x - 1)$
$= (2x^2 - 4x + 3x - 6)(3x - 1)$
$= (2x^2 - x - 6)(3x - 1)$
$= 2x^2(3x - 1) - x(3x - 1) - 6(3x - 1)$
$= 6x^3 - 2x^2 - 3x^2 + x - 18x + 6$
$= 6x^3 - 5x^2 - 17x + 6$

4 a $3(5y + 4)$
$= 15y + 12$

b $5x^2(3 - 5x + 2x^2)$
$= 15x^2 - 25x^3 + 10x^4$

c $5x(2x + 3) - 2x(1 - 3x)$
$= 10x^2 + 15x - 2x + 6x^2$
$= 16x^2 + 13x$

d $3x^2(1 + 3x) - 2x(3x - 2)$
$= 3x^2 + 9x^3 - 6x^2 + 4x$
$= 9x^3 - 3x^2 + 4x$

5 a $3x^2 + 4x$
$= x(3x + 4)$

b $4y^2 + 10y$
$= 2y(2y + 5)$

c $x^2 + xy + xy^2$
$= x(x + y + y^2)$

d $8xy^2 + 10x^2y$
$= 2xy(4y + 5x)$

6 a $x^2 + 3x + 2$
$= x^2 + x + 2x + 2$
$= x(x + 1) + 2(x + 1)$
$=(x + 1)(x + 2)$

b $3x^2 + 6x$
$= 3x(x + 2)$

c $x^2 - 2x - 35$
$= x^2 - 7x + 5x - 35$
$= x(x - 7) + 5(x - 7)$
$= (x - 7)(x + 5)$

d $2x^2 - x - 3$
$= 2x^2 - 3x + 2x - 3$
$= x(2x - 3) + (2x - 3)$
$= (2x - 3)(x + 1)$

e $5x^2 - 13x - 6$
$= 5x^2 + 2x - 15x - 6$
$= x(5x + 2) - 3(5x + 2)$
$= (5x + 2)(x - 3)$

6 f $6 - 5x - x^2$
$= 6 + x - 6x - x^2$
$= (6 + x) - x(6 + x)$
$= (6 + x)(1 - x)$

7 a $2x^3 + 6x$
$= 2x(x^2 + 3)$

b $x^3 - 36x$
$= x(x^2 - 36)$
$= x(x^2 - 6^2)$
$= x(x + 6)(x - 6)$

c $2x^3 + 7x^2 - 15x$
$= x(2x^2 + 7x - 15)$
$= x(2x^2 - 3x + 10x - 15)$
$= x(x(2x - 3) + 5(2x - 3))$
$= x(2x - 3)(x + 5)$

8 a $9x^3 \div 3x^{-3}$
$= 3x^{3-(-3)}$
$= 3x^6$

b $\left(4^{\frac{3}{2}}\right)^{\frac{1}{3}}$
$= 4^{\frac{3}{2}\times\frac{1}{3}}$
$= 4^{\frac{1}{2}}$
$= \sqrt{4}$
$= \pm 2$

c $3x^{-2} \times 2x^4$
$= 6x^{-2+4}$
$= 6x^2$

d $3x^{\frac{1}{3}} \div 6x^{\frac{2}{3}}$
$= \frac{1}{2}x^{\frac{1}{3}-\frac{2}{3}}$
$= \frac{1}{2}x^{-\frac{1}{3}}$ or $\frac{1}{2\left(\sqrt[3]{x}\right)}$

9 a $\left(\frac{8}{27}\right)^{\frac{2}{3}}$
$= \left(\frac{\sqrt[3]{8}}{\sqrt[3]{27}}\right)^2$
$= \left(\frac{2}{3}\right)^2$
$= \frac{4}{9}$

b $\left(\frac{225}{289}\right)^{\frac{3}{2}}$
$= \left(\frac{\sqrt{225}}{\sqrt{289}}\right)^3$
$= \pm\frac{15^3}{17^3}$
$= \pm\frac{3375}{4913}$

10 a $\frac{3}{\sqrt{63}}$
$= \frac{3}{\sqrt{9}\times\sqrt{7}}$
$= \frac{3}{3\sqrt{7}}$
$= \frac{1}{\sqrt{7}}$
$= \frac{\sqrt{7}}{7}$

b $\sqrt{20} + 2\sqrt{45} - \sqrt{80}$
$= \sqrt{4}\sqrt{5} + 2\sqrt{9}\sqrt{5} - \sqrt{16}\sqrt{5}$
$= 2\sqrt{5} + 2\times 3\sqrt{5} - 4\sqrt{5}$
$= \sqrt{5}\,(2 + 6 - 4)$
$= 4\sqrt{5}$

11 a When $x = 25$,
$35x^2 + 2x - 48$
$= 35 \times 25^2 + 2 \times 25 - 48$
$= 21\,877$

b $35x^2 + 2x - 48$
$= 35x^2 + 42x - 40x - 48$
$= 7x(5x + 6) - 8(5x + 6)$
$= (5x + 6)(7x - 8)$
When $x = 25$, $5x + 6 = 131$
and $7x - 8 = 167$;
both 131 and 167 are prime numbers.

12 a $\sqrt{2}\left(3+\sqrt{5}\right)$

$= 3\sqrt{2}+\sqrt{2}\sqrt{5}$

$= 3\sqrt{2}+\sqrt{10}$

b $\left(2-\sqrt{5}\right)\left(5+\sqrt{3}\right)$

$= 10+2\sqrt{3}-5\sqrt{5}-\sqrt{3}\sqrt{5}$

$= 10+2\sqrt{3}-5\sqrt{5}-\sqrt{15}$

c $\left(6-\sqrt{2}\right)\left(4-\sqrt{7}\right)$

$= 24-6\sqrt{7}-4\sqrt{2}+\sqrt{2}\sqrt{7}$

$= 24-6\sqrt{7}-4\sqrt{2}+\sqrt{14}$

13 a $\dfrac{1}{\sqrt{3}}$

$= \dfrac{1\times\sqrt{3}}{\sqrt{3}\times\sqrt{3}}$

$= \dfrac{\sqrt{3}}{3}$

b $\dfrac{1}{\sqrt{2}-1}$

$= \dfrac{\sqrt{2}+1}{\left(\sqrt{2}-1\right)\left(\sqrt{2}+1\right)}$

$= \dfrac{\sqrt{2}+1}{2-1}$

$= \sqrt{2}+1$

c $\dfrac{3}{\sqrt{3}-2}$

$= \dfrac{3\left(\sqrt{3}+2\right)}{\left(\sqrt{3}-2\right)\left(\sqrt{3}+2\right)}$

$= \dfrac{3\sqrt{3}+6}{3-4}$

$= \dfrac{3\sqrt{3}+6}{-1}$

$= -3\sqrt{3}-6$

d $\dfrac{\sqrt{23}-\sqrt{37}}{\sqrt{23}+\sqrt{37}}$

$= \dfrac{\left(\sqrt{23}-\sqrt{37}\right)\left(\sqrt{23}-\sqrt{37}\right)}{\left(\sqrt{23}+\sqrt{37}\right)\left(\sqrt{23}-\sqrt{37}\right)}$

$= \dfrac{23-2\sqrt{23}\sqrt{37}+37}{23-37}$

$= \dfrac{60-2\sqrt{851}}{-14}$

$= \dfrac{30-\sqrt{851}}{-7}$

e $\dfrac{1}{\left(2+\sqrt{3}\right)^2}$

$= \dfrac{1}{\left(2+\sqrt{3}\right)\left(2+\sqrt{3}\right)}$

$= \dfrac{1}{4+2\sqrt{3}+2\sqrt{3}+\sqrt{9}}$

$= \dfrac{1}{4+2\sqrt{3}+2\sqrt{3}+\sqrt{9}}$

$= \dfrac{1\times\left(7-4\sqrt{3}\right)}{\left(7+4\sqrt{3}\right)\left(7-4\sqrt{3}\right)}$

$= \dfrac{7-4\sqrt{3}}{49-28\sqrt{3}+28\sqrt{3}-48}$

$= \dfrac{7-4\sqrt{3}}{1}$

$= 7-4\sqrt{3}$

f $\dfrac{1}{\left(4-\sqrt{7}\right)^2}$

$= \dfrac{1}{\left(4-\sqrt{7}\right)\left(4-\sqrt{7}\right)}$

$= \dfrac{1}{16-4\sqrt{7}-4\sqrt{7}+\sqrt{49}}$

$= \dfrac{1}{23-8\sqrt{7}}$

13 f $\dfrac{1}{\left(4-\sqrt{7}\right)^2}$

$= \dfrac{1\times\left(23+8\sqrt{7}\right)}{\left(23-8\sqrt{7}\right)\left(23+8\sqrt{7}\right)}$

$= \dfrac{23+8\sqrt{7}}{529+184\sqrt{7}-184\sqrt{7}-448}$

$= \dfrac{23+8\sqrt{7}}{81}$

14 a $x^3 - x^2 - 17x - 15$

$= (x + 3)(x^2 + bx + c)$

$= x^3 + bx^2 + cx + 3x^2 + 3bx + 3c$

$= x^3 + bx^2 + 3x^2 + 3bx + cx + 3c$

$= x^3 - x^2 - 17x - 15$

Equating the coefficients gives $b + 3 = -1$,

$3b + c = -17$, $3c = -15$

Using $b + 3 = -1$, $b = -4$

Using $3c = -15$, $c = -5$

b $x^3 - x^2 - 17x - 15$

$= (x + 3)(x^2 - 4x - 5)$

$= (x + 3)(x^2 - 5x + x - 5)$

$= (x + 3)(x(x - 5)+(x - 5))$

$= (x + 3)(x - 5)(x + 1)$

15 a $y^{\frac{1}{3}}$

$= \left(\dfrac{1}{64}x^3\right)^{\frac{1}{3}}$

$= \dfrac{1}{\sqrt[3]{64}}x^{3\times\frac{1}{3}}$

$= \dfrac{x}{4}$

b $4y^{-1}$

$= 4\left(\dfrac{1}{64}x^3\right)^{-1}$

$= 4\times\dfrac{1}{\frac{1}{64}}x^{3\times(-1)}$

$= 256x^{-3}$

16 $\dfrac{5}{\sqrt{75}-\sqrt{50}}$

$= \dfrac{5}{\sqrt{25\times3}-\sqrt{25\times2}}$

$= \dfrac{5}{5\sqrt{3}-5\sqrt{2}}$

$= \dfrac{1}{\sqrt{3}-\sqrt{2}}$

$= \dfrac{1\times(\sqrt{3}+\sqrt{2})}{(\sqrt{3}-\sqrt{2})(\sqrt{3}+\sqrt{2})}$

$= \dfrac{\sqrt{3}+\sqrt{2}}{\sqrt{9}+\sqrt{2}\sqrt{3}-\sqrt{2}\sqrt{3}-\sqrt{4}}$

$= \dfrac{\sqrt{3}+\sqrt{2}}{1}$

$= \sqrt{3}+\sqrt{2}$

17 $\left(\sqrt{11}-5\right)\left(5-\sqrt{11}\right)$

$= 5\sqrt{11}-\sqrt{121}-25+5\sqrt{11}$

$= 10\sqrt{11}-36$

18 $x - 64x^3$

$= x(1 - 64x^2)$

$= x[\,1^2 - (8x)^2\,]$

$= x(1 + 8x)(1 - 8x)$

19 27^{2x+1}

$= (3^3)^{2x+1}$

$= 3^{3(2x+1)}$

$= 3^{6x+3}$

$= 3^y$

$y = 6x + 3$

20 $8 + x\sqrt{12} = \dfrac{8x}{\sqrt{3}}$

$8\sqrt{3} + x\sqrt{12}\sqrt{3} = 8x$

$8\sqrt{3} + x\sqrt{36} = 8x$

$8\sqrt{3} + 6x = 8x$

$8\sqrt{3} = 2x$

$x = 4\sqrt{3}$

21 Area $= \sqrt{12}$ cm^2, length $= \left(1+\sqrt{3}\right)$cm

$$\text{Width} = \frac{\sqrt{12}}{1+\sqrt{3}}$$
$$= \frac{\sqrt{12}\times\left(1-\sqrt{3}\right)}{\left(1+\sqrt{3}\right)\left(1-\sqrt{3}\right)}$$
$$= \frac{\sqrt{12}-\sqrt{36}}{1-\sqrt{3}+\sqrt{3}-\sqrt{9}}$$
$$= \frac{\sqrt{4\times3}-6}{-2}$$
$$= \frac{2\sqrt{3}-6}{-2}$$
$$= \frac{-\sqrt{3}+3}{1}$$
$$= 3-\sqrt{3} \text{ cm}$$

22
$$\frac{\left(2-\sqrt{x}\right)^2}{\sqrt{x}}$$
$$= \frac{(2-x^{\frac{1}{2}})^2}{x^{\frac{1}{2}}}$$
$$= \frac{(2-x^{\frac{1}{2}})(2-x^{\frac{1}{2}})}{x^{\frac{1}{2}}}$$
$$= \frac{4-2x^{\frac{1}{2}}-2x^{\frac{1}{2}}\quad x^{\frac{1}{2}+\frac{1}{2}}}{x^{\frac{1}{2}}}$$
$$= x^{-\frac{1}{2}}(4-4x^{\frac{1}{2}}+x)$$
$$= 4x^{-\frac{1}{2}}-4x^{-\frac{1}{2}+\frac{1}{2}}+x^{1-\frac{1}{2}}$$
$$= 4x^{-\frac{1}{2}}-4x^{0}+x^{\frac{1}{2}}$$
$$= 4x^{-\frac{1}{2}}-4+x^{\frac{1}{2}}$$
$$= \frac{4}{\sqrt{x}}-4+\sqrt{x}$$

23 a
$$243\sqrt{3}$$
$$= 3^5\times3^{\frac{1}{2}}$$
$$= 3^{5+\frac{1}{2}}$$
$$= 3^{\frac{11}{2}}$$
$$a = \frac{11}{2}$$

b From part a: $3^x\times27^y = 243\sqrt{3} = 3^{\frac{11}{2}}$
$$3^x\times3^{3y} = 3^{\frac{11}{2}}$$
$$3^{x+3y} = 3^{\frac{11}{2}}$$
So $x+3y = \frac{11}{2}$
$$3y = \frac{11-2x}{2}$$
$$y = \frac{11-2x}{6}$$

24
$$\frac{4x^3+x^{\frac{5}{2}}}{\sqrt{x}}$$
$$= \frac{4x^3+x^{\frac{5}{2}}}{x^{\frac{1}{2}}}$$
$$= x^{-\frac{1}{2}}\left(4x^3+x^{\frac{5}{2}}\right)$$
$$= 4x^{-\frac{1}{2}+3}+x^{-\frac{1}{2}+\frac{5}{2}}$$
$$= 4x^{\frac{5}{2}}+x^2$$
$$= 4x^a+x^b$$
$$a = \frac{5}{2},\ b = 2$$

Challenge

a $\left(\sqrt{a}+\sqrt{b}\right)\left(\sqrt{a}-\sqrt{b}\right)$

$= a-\sqrt{a}\sqrt{b}+\sqrt{a}\sqrt{b}-b$

$= a-b$

b Rationalising the denominators:

$$\frac{1}{\sqrt{1}+\sqrt{2}}+\frac{1}{\sqrt{2}+\sqrt{3}}+\frac{1}{\sqrt{3}+\sqrt{4}}+\ldots+\frac{1}{\sqrt{24}+\sqrt{25}}$$

$=$

$$\frac{\sqrt{1}-\sqrt{2}}{\left(\sqrt{1}+\sqrt{2}\right)\left(\sqrt{1}-\sqrt{2}\right)}+\frac{\sqrt{2}-\sqrt{3}}{\left(\sqrt{2}+\sqrt{3}\right)\left(\sqrt{\ }-\sqrt{3}\right)}+\frac{\sqrt{3}-\sqrt{4}}{\left(\sqrt{3}+\sqrt{4}\right)\left(\sqrt{\ }-\sqrt{4}\right)}\quad\ldots+\frac{\sqrt{24}-\sqrt{25}}{\left(\sqrt{24}+\sqrt{25}\right)\left(\sqrt{24}-\sqrt{25}\right)}$$

$$= \frac{\sqrt{1}-\sqrt{2}}{1-2}+\frac{\sqrt{2}-\sqrt{3}}{2-3}+\frac{\sqrt{3}-\sqrt{4}}{3-4}+\ldots+\frac{\sqrt{24}-\sqrt{25}}{24-25}$$

$$= -\left(\sqrt{1}-\sqrt{2}\right)-\left(\sqrt{2}-\sqrt{3}\right)-\left(\sqrt{3}-\sqrt{4}\right)-\ldots-\left(\sqrt{24}-\sqrt{25}\right)$$

$$= -\sqrt{1}+\sqrt{2}-\sqrt{2}+\sqrt{3}-\sqrt{3}+\sqrt{4}-\ldots-\sqrt{24}+\sqrt{25}$$

$$= -\sqrt{1}+\sqrt{25}$$

$= -1 + 5$

$= 4$