Equations and inequalities, Mixed Exercise 3

1 a $2kx - y = 4$ (**1**)
$4kx + 3y = -2$ (**2**)
Multiply equation (**1**) by 2 to give
$4kx - 2y = 8$ (**3**)
Subtract equation (**2**) from equation (**3**)
$-5y = 10$
$y = -2$

b Using (**1**), $2kx + 2 = 4$:
$2kx = 2$
$x = \frac{1}{k}$

2 Rearrange $x + 2y = 3$ to give:
$x = 3 - 2y$
Substitute into $x^2 - 4y^2 = -33$:
$$(3-2y)^2 - 4y^2 = -33$$
$$9 - 12y + 4y^2 - 4y^2 = -33$$
$$-12y = -33 - 9$$
$$-12y = -42$$
$$y = \tfrac{7}{2}$$
Substitute into $x = 3 - 2y$:
$x = 3 - 7 = -4$
So the solution is $x = -4, y = \frac{7}{2}$

3 a Rearrange $x - 2y = 1$ to give:
$x = 1 + 2y$
Substitute into $3xy - y^2 = 8$:
$$3y(1+2y) - y^2 = 8$$
$$3y + 6y^2 - y^2 = 8$$
$$5y^2 + 3y - 8 = 0$$

b $(5y+8)(y-1) = 0$
$y = -\frac{8}{5}$ or $y = 1$
Substitute into $x = 1 + 2y$.
When $y = -\frac{8}{5}$, $x = 1 - \frac{16}{5} = -\frac{11}{5}$
When $y = 1$, $x = 1 + 2 = 3$

3 b So the solutions are
$\left(-\frac{11}{5}, -\frac{8}{5}\right)$ and $(3, 1)$

4 a Rearrange $x + y = 2$ to give:
$y = 2 - x$
$$x^2 + x(2 - x) - (2 - x)^2 = -1$$
$$x^2 + 2x - x^2 - 4 + 4x - x^2 + 1 = 0$$
$$-x^2 + 6x - 3 = 0$$
$$x^2 - 6x + 3 = 0$$

b Using the quadratic formula:
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$x = \frac{6 \pm \sqrt{36 - 12}}{2}$$
$$x = \frac{6 \pm \sqrt{24}}{2}$$
$$x = \frac{6 \pm \sqrt{4 \times 6}}{2}$$
$$x = \frac{6 \pm 2\sqrt{6}}{2}$$
$$x = 3 \pm \sqrt{6}$$

Substitute into $y = 2 - x$.
$$y = 2 - \left(3 \pm \sqrt{6}\right)$$
$$y = -1 \pm \sqrt{6}$$

5 a $9 = 3^2$, so $3^x = \left(3^2\right)^{y-1}$
$\Rightarrow 3^x = 3^{2(y-1)}$

Equate powers:
$x = 2(y - 1)$
$\Rightarrow x = 2y - 2$

5 b $x = 2y - 2$

Substitute into $x^2 = y^2 + 7$:

$$(2y-2)^2 = y^2 + 7$$
$$4y^2 - 8y + 4 = y^2 + 7$$
$$4y^2 - y^2 - 8y + 4 - 7 = 0$$
$$3y^2 - 8y - 3 = 0$$
$$(3y+1)(y-3) = 0$$
$$y = -\tfrac{1}{3} \text{ or } y = 3$$

Substitute into $x = 2y - 2$.

When $y = -\frac{1}{3}$, $x = -\frac{2}{3} - 2 = -2\frac{2}{3}$

When $y = 3$, $x = 6 - 2 = 4$

The solutions are:

$x = -\frac{8}{3}, y = -\frac{1}{3}$ and $x = 4, y = 3$

6 Rearrange $x + 2y = 3$ to give:

$x = 3 - 2y$

Substitute into $x^2 - 2y + 4y^2 = 18$:

$$(3-2y)^2 - 2y + 4y^2 = 18$$
$$9 - 12y + 4y^2 - 2y + 4y^2 = 18$$
$$8y^2 - 14y + 9 - 18 = 0$$
$$8y^2 - 14y - 9 = 0$$
$$(4y-9)(2y+1) = 0$$
$$y = \tfrac{9}{4} \text{ or } y = -\tfrac{1}{2}$$

Substitute into $x = 3 - 2y$.

When $y = \frac{9}{4}$, $x = 3 - \frac{9}{2} = -\frac{3}{2}$

When $y = -\frac{1}{2}$, $x = 3 + 1 = 4$

The solutions are: $x = -\frac{3}{2}, y = \frac{9}{4}$

and $x = 4$, $y = -\frac{1}{2}$

7 a Rearrange $-\dfrac{k}{2}x + y = 1$ and substitute into the quadratic equation:

$$y = 1 + \frac{k}{2}x$$

7 a $kx^2 - x\left(1 + \dfrac{k}{2}x\right) + (k+1)x = 1$

$$kx^2 - x - \frac{k}{2}x^2 + kx + x - 1 = 0$$
$$kx^2 + 2kx - 2 = 0$$

Using the discriminant for one solution:

$$b^2 - 4ac = 0$$
$$(2k)^2 - 4(k)(-2) = 0$$
$$4k^2 + 8k = 0$$
$$4k(k+2) = 0$$

k is non-zero, so $k = -2$

b Substituting into $kx^2 + 2kx - 2 = 0$ gives:

$$-2x^2 + 4x - 2 = 0$$
$$-x^2 - 2x - 1 = 0$$
$$x^2 + 2x + 1 = 0$$
$$(x+1)^2 = 0$$
$$x = -1$$

Substitute for x and k:

$$y = 1 + \frac{k}{2}x = 1 - (-1) = 2$$

Therefore, the coordinates are $(-1, 2)$.

8 The sloping ceiling can be modelled by $h = \frac{15}{2} - \frac{1}{5}x$

If the ball hits the ceiling, then

$$\tfrac{15}{2} - \tfrac{1}{5}x = -\tfrac{3}{10}x^2 + \tfrac{5}{2}x + \tfrac{3}{2}$$
$$-\tfrac{3}{10}x^2 + \tfrac{27}{10}x - 6 = 0$$
$$-3x^2 + 27x - 60 = 0$$
$$x^2 - 9x + 20 = 0$$

Using the discriminant:

$b^2 - 4ac = 81 - 80 = 1$

As $1 > 0$, there are two solutions.

Therefore, the model does predict that the ball hits the ceiling.

9 a $3x - x > 13 + 8$

$2x > 21$

$x > 10\frac{1}{2}$

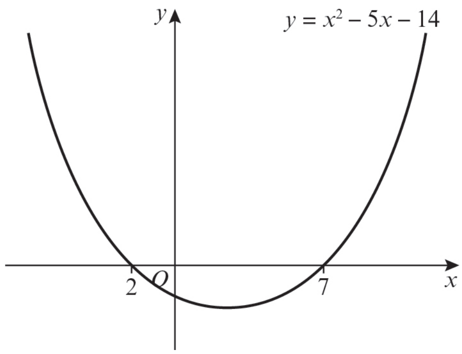
In set notation, the solution is

$\{x : x > \frac{21}{2}\}$

b $x^2 - 5x - 14 = 0$

$(x+2)(x-7) = 0$

$x = -2$ or $x = 7$



$x^2 - 5x - 14 > 0$ when $x < -2$

or $x > 7$

In set notation, the solution is

$\{x : x < -2\} \cup \{x : x > 7\}$

10 Multiplying out the brackets:

$x^2 - 5x + 4 < 2x - 8$

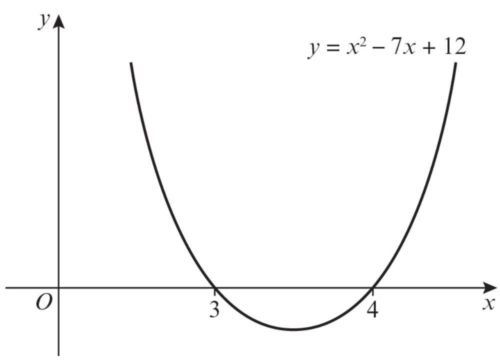
$x^2 - 5x - 2x + 4 + 8 < 0$

$x^2 - 7x + 12 < 0$

$x^2 - 7x + 12 = 0$

$(x-3)(x-4) = 0$

$x = 3$ or $x = 4$



$x^2 - 7x + 12 < 0$ when $3 < x < 4$

11 a $x^2 + x - 2 = 18$

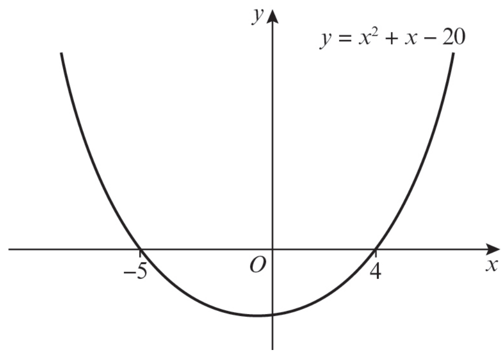
$x^2 + x - 20 = 0$

$(x+5)(x-4) = 0$

$x = -5$ or $x = 4$

b $(x - 1)(x + 2) > 18$

$\Rightarrow x^2 + x - 20 > 0$



$x^2 + x - 20 > 0$ when $x < -5$ or $x > 4$

In set notation, the solution is

$\{x : x < -5\} \cup \{x : x > 4\}$

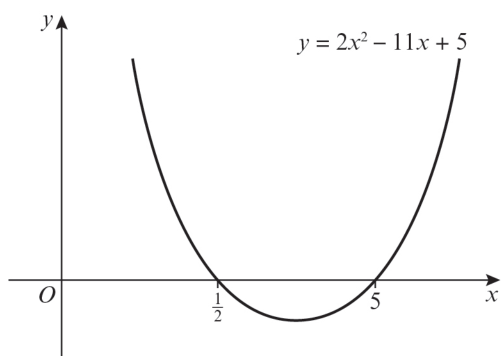
12 a $6x - 2x < 3 + 7$

$4x < 10$

$x < \frac{5}{2}$

b $(2x-1)(x-5) = 0$

$x = \frac{1}{2}$ or $x = 5$



$2x^2 - 11x + 5 < 0$ when $\frac{1}{2} < x < 5$

12 c $5 < \frac{20}{x}$

Multiply both sides by x^2

$5x^2 < 20x$

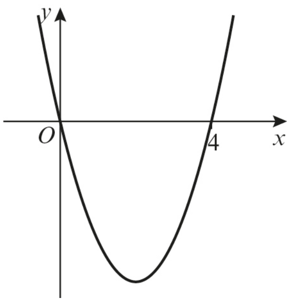
$5x^2 - 20x < 0$

Solve the quadratic to find the critical values:

$5x^2 - 20x = 0$

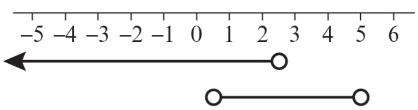
$5x(x - 4) = 0$

$x = 0$ or $x = 4$



The solution is $0 < x < 4$

d



Intersection is $\frac{1}{2} < x < \frac{5}{2}$

13 $\frac{8}{x^2} + 1 \le \frac{9}{x}$

Multiply both sides by x^2:

$8 + x^2 \le 9x$

$x^2 - 9x + 8 \le 0$

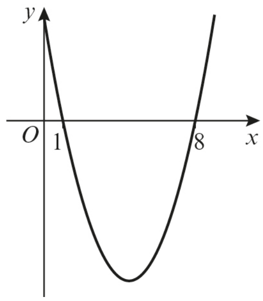
Solve the quadratic to find the critical values:

$x^2 - 9x + 8 = 0$

$(x - 1)(x - 8) = 0$

$x = 1$ or $x = 8$

13



The solution is $1 \le x \le 8$

14 $a = k, b = 8, c = 5$

Using the discriminant $b^2 - 4ac \ge 0$:

$8^2 - 4k \times 5 \ge 0$

$64 - 20k \ge 0$

$64 \ge 20k$

$\frac{64}{20} \ge k$

$k \le \frac{16}{5}$

15 $a = 2,\ b = 4k,\ c = -5k$

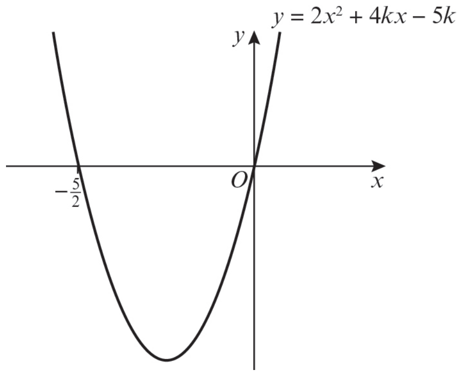
Using the discriminant $b^2 - 4ac < 0$:

$(4k)^2 - 4(2)(-5k) < 0$

$16k^2 + 40k < 0$

$8k(2k + 5) < 0$

$k = 0$ or $k = -\frac{5}{2}$



$-\frac{5}{2} < k < 0$

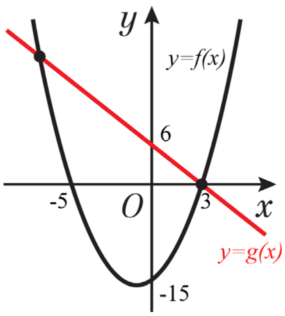
16 a $y = x^2 + 2x - 15$

$y = (x + 5)(x - 3)$

$0 = (x + 5)(x - 3)$

$x = -5$ or $x = 3$

When $x = 0$, $y = -15$



b $x^2 + 2x - 15 = 6 - 2x$

$x^2 + 4x - 21 = 0$

$(x + 7)(x - 3) = 0$

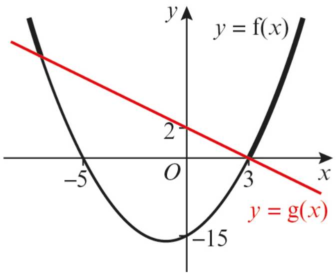
$x = -7$ or $x = 3$

When $x = -7$, $y = 20$

When $x = 3$, $y = 0$

The points of intersection are $(-7, 20)$ and $(3, 0)$.

c



From the graph and the calculated points of intersection, the required values are

$x < -7$ or $x > 3$.

17 $2x^2 + 3x - 15 = 8 + 2x$

$2x^2 + x - 23 = 0$

$$x = \frac{-1 \pm \sqrt{185}}{4} = \tfrac{1}{4}\left(-1 \pm \sqrt{185}\right)$$

$$\tfrac{1}{4}\left(-1 - \sqrt{185}\right) < x < \tfrac{1}{4}\left(-1 + \sqrt{185}\right)$$

18 $y = x^2 + 4x - 12$

$x^2 + 4x - 12 = 0$

$(x + 6)(x - 2) = 0$

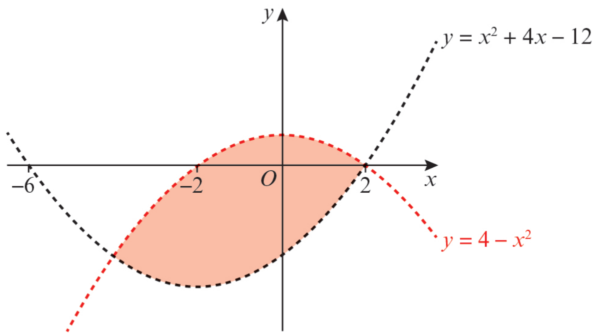
$x = -6$ or $x = 2$

$y = 4 - x^2$

$4 - x^2 = 0$

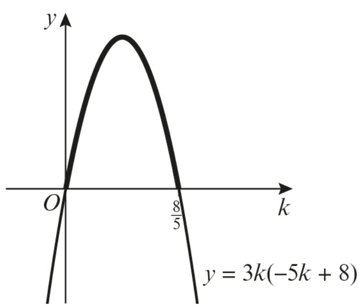
$(2 + x)(2 - x) = 0$

$x = -2$ or $x = 2$



Challenge

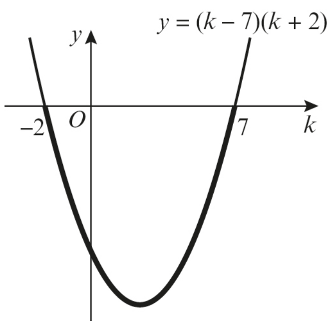
1 $2kx^2 + 5kx + 5k - 3 = 0$
Using the discriminant:
$b^2 - 4ac \geq 0$ for real roots.
$(5k)^2 - 4(2k)(5k - 3) \geq 0$
$25k^2 - 40k^2 + 24k \geq 0$
$-15k^2 + 24k \geq 0$
$3k(-5k + 8) \geq 0$
$3k(-5k + 8) = 0$
$k = 0$ or $k = \frac{8}{5}$



$0 < k \leq \frac{8}{5}$

2 $2x - k = 3x^2 + 2kx + 5$
$3x^2 + 2kx - 2x + 5 + k = 0$
$3x^2 + (2k - 2)x + 5 + k = 0$
If the line and parabola do not intersect then there are no solutions.

Using the discriminant:
$b^2 - 4ac < 0$
$(2k - 2)^2 - 4(3)(5 + k) < 0$
$4k^2 - 8k + 4 - 60 - 12k < 0$
$4k^2 - 20k - 56 < 0$
$k^2 - 5k - 14 < 0$
$k^2 - 5k - 14 = 0$
$(k - 7)(k + 2) = 0$
$k = 7$ or $k = -2$



The line and the parabola do not intersect in the interval $-2 < k < 7$