Numerical methods, Mixed exercise 10

1 a $f(x) = x^3 - 6x - 2$

$f(x) = 0 \Rightarrow x^3 = 6x + 2$

$x^2 = 6 + \frac{2}{x}$

$x = \pm\sqrt{6 + \frac{2}{x}}$

$a = 6,\ b = 2$

b $x_{n+1} = \sqrt{6 + \frac{2}{x_n}}$

$x_0 = 2 \Rightarrow x_1 = \sqrt{6 + \frac{2}{2}} = \sqrt{7} = 2.64575...$

$x_2 = \sqrt{6 + \frac{2}{2.64575...}} = 2.59921...$

$x_3 = \sqrt{6 + \frac{2}{2.59921...}} = 2.60181...$

$x_4 = \sqrt{6 + \frac{2}{2.60181...}} = 2.60167...$

To 4 d.p., the values are $x_1 = 2.6458$, $x_2 = 2.5992$, $x_3 = 2.6018$, $x_4 = 2.6017$.

c $f(2.6015) = 2.6015^3 - 6 \times 2.6015 - 2$

$= -0.0025...$

$f(2.6025) = 2.6025^3 - 6 \times 2.6025 - 2$

$= 0.0117...$

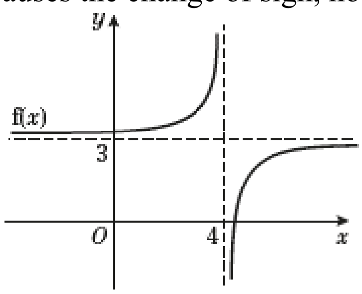
There is a change of sign in this interval so $\alpha = 2.602$ correct to 3 d.p.

2 a $f(x) = \frac{1}{4-x} + 3$

$f(3.9) = \frac{1}{0.1} + 3 = 13$

$f(4.1) = -\frac{1}{0.1} + 3 = -7$

b There is an asymptote at $x = 4$ which causes the change of sign, not a root.



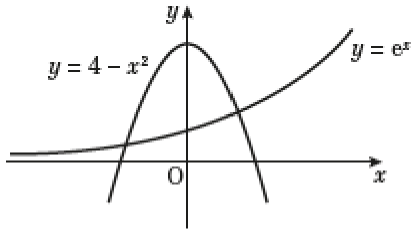
c $f(x) = 0 \Rightarrow \frac{1}{4-x} + 3 = 0$

$\frac{1}{x-4} = 3$

$1 = 3x - 12 \Rightarrow x = \frac{13}{3}$

So $\alpha = \frac{13}{3}$.

3 a



b There is one positive and one negative root of the equation $p(x) = q(x)$ at the points of intersection.

$p(x) = q(x) \Rightarrow 4 - x^2 = e^x$

i.e. $x^2 + e^x - 4 = 0$

c $x^2 = 4 - e^x$

$x = \pm(4 - e^x)^{\frac{1}{2}}$

d $x_{n+1} = \pm(4 - e^{x_n})^{\frac{1}{2}}$

$x_0 = -2 \Rightarrow x_1 = -(4 - e^{-2})^{\frac{1}{2}} = -1.96587...$

$x_2 = -(4 - e^{-1.96587...})^{\frac{1}{2}} = -1.96467...$

$x_3 = -(4 - e^{-1.96467...})^{\frac{1}{2}} = -1.96463...$

$x_4 = -(4 - e^{-1.96463...})^{\frac{1}{2}} = -1.96463...$

To 4 d.p., the values are $x_1 = -1.9659$, $x_2 = -1.9647$, $x_3 = -1.9646$, $x_4 = -1.9646$.

e $x_0 = 1.4 \Rightarrow 4 - e^{1.4} < 0$

There can be no square root of a negative number.

4 a $g(x) = x^5 - 5x - 6$
$g(1) = 1 - 5 - 6 = -10$
$g(2) = 32 - 10 - 6 = 16$

There is a change of sign in the interval, so there must be a root in the interval, since f is continuous over the interval.

b $g(x) = 0 \Rightarrow x^5 = 5x + 6$

$x = (5x+6)^{\frac{1}{5}}$

$p = 5, q = 6, r = 5$

c $x_{n+1} = (5x_n + 6)^{\frac{1}{5}}$

$x_0 = 1 \Rightarrow x_1 = (5+6)^{\frac{1}{5}} = 1.61539...$

$x_2 = (5 \times 1.61539... + 6)^{\frac{1}{5}} = 1.69707...$

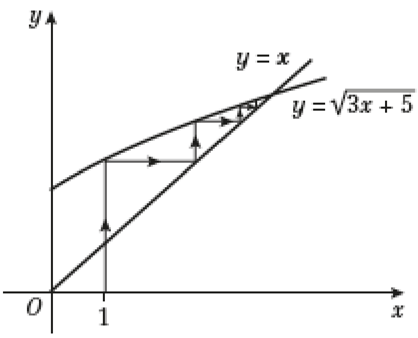
$x_3 = (5 \times 1.69707... + 6)^{\frac{1}{5}} = 1.70681...$

To 4 d.p., the values are $x_1 = 1.6154$, $x_2 = 1.6971$, $x_3 = 1.7068$.

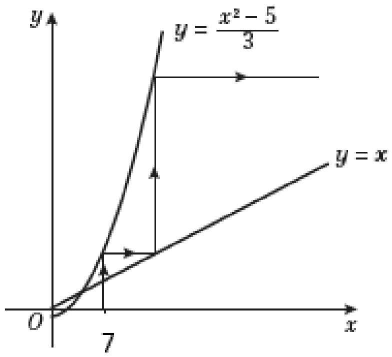
d $g(1.7075) = 1.7075^5 - 5 \times 1.7075 - 6$
$= -0.0229...$
$g(1.7085) = 1.7085^5 - 5 \times 1.7085 - 6$
$= 0.0146...$
The sign change implies there is a root in this interval so $\alpha = 1.708$ correct to 3 d.p.

5 a $g(x) = x^2 - 3x - 5$
$g(x) = 0 \Rightarrow x^2 - 3x - 5 = 0$
$x^2 = 3x + 5$
$x = \sqrt{3x+5}$

b, c



d



$g(x) = 0 \Rightarrow x^2 - 3x - 5 = 0$
$3x = x^2 - 5$
$x = \dfrac{x^2 - 5}{3}$

6 a $f(x) = 5x - 4\sin x - 2$
$f(1.1) = 5(1.1) - 4\sin(1.1) - 2$
$= -0.0648...$
$f(1.15) = 5(1.15) - 4\sin(1.15) - 2$
$= -0.0989...$

$f(1.1) < 0$ and $f(1.15) > 0$ so there is a change of sign, which implies there is a root between $x = 1.1$ and $x = 1.15$.

b $5x - 4\sin x - 2 = 0$

$5x - 2 = 4\sin x$ Add 4 sin x to each side.

$5x = 4\sin x + 2$ Add 2 to each side.

$\dfrac{5x}{5} = \dfrac{4\sin x}{5} + \dfrac{2}{5}$ Divide each term by 5.

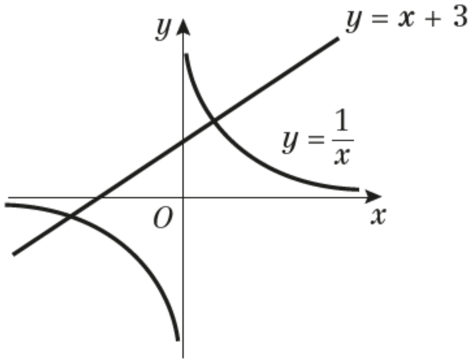
$x = \dfrac{4}{5}\sin x + \dfrac{2}{5}$ Simplify.

So $p = \dfrac{4}{5}$ and $q = \dfrac{2}{5}$.

c $x_0 = 1.1 \Rightarrow$
$x_1 = 0.8\sin(1.1) + 0.4 = 1.1129...$
$x_2 = 0.8\sin(1.1129...) + 0.4 = 1.1176...$
$x_3 = 0.8\sin(1.1176...) + 0.4 = 1.1192...$
$x_4 = 0.8\sin(1.1192...) + 0.4 = 1.1198...$

To 3 d.p., the values are $x_1 = 1.113$, $x_2 = 1.118$, $x_3 = 1.119$, $x_4 = 1.120$.

7 a



b The line meets the curve at two points, so there are two values of x that satisfy the equation $\frac{1}{x} = x + 3$.

So $\frac{1}{x} = x + 3$ has two roots.

c $\frac{1}{x} = x + 3 \Rightarrow 0 = x + 3 - \frac{1}{x}$

Let $f(x) = x + 3 - \frac{1}{x}$

$f(0.30) = (0.30) + 3 - \frac{1}{0.30} = -0.0333...$

$f(0.31) = (0.31) + 3 - \frac{1}{0.31} = 0.0841...$

$f(0.30) < 0$ and $f(0.31) > 0$ so there is a change of sign, which implies there is a root between $x = 0.30$ and $x = 0.31$.

d $\frac{1}{x} = x + 3$

$\frac{1}{x} \times x = x \times x + 3 \times x$ Multiply by x.

$1 = x^2 + 3x$

So $x^2 + 3x - 1 = 0$

e Using $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ with $a = 1, b = 3, c = -1$

$x = \frac{-(3) \pm \sqrt{(3)^2 - 4(1)(-1)}}{2(1)}$

$= \frac{-3 \pm \sqrt{9 + 4}}{2} = \frac{-3 \pm \sqrt{13}}{2}$

So $x = \frac{-3 + \sqrt{13}}{2} = 0.3027...$

The positive root is 0.303 to 3 d.p.

8 a $g(x) = x^3 - 7x^2 + 2x + 4$

$g'(x) = 3x^2 - 14x + 2$

b Using $x_0 = 6.6$,

$x_1 = x_0 - \frac{g(x_0)}{g'(x_0)}$

$= 6.6 - \frac{g(6.6)}{g'(6.6)}$

$= 6.6 - \frac{6.6^3 - 7(6.6^2) + 2(6.6) + 4}{3(6.6^2) - 14(6.6) + 2}$

$= 6.606$ correct to 3 d.p.

c $g(1) = 0 \Rightarrow x - 1$ is a factor of $g(x)$

$g(x) = (x - 1)(x^2 - 6x - 4)$

$(x - 1)(x^2 - 6x - 4) = 0$

Other two roots of $g(x)$ are given by

$\frac{6 \pm \sqrt{36 + 16}}{2} = \frac{6 \pm \sqrt{52}}{2} = 3 \pm \sqrt{13}$

d Percentage error:

$\frac{6.606 - \left(3 + \sqrt{13}\right)}{3 + \sqrt{13}} \times 100 = 0.007\%$

9 a $f(x) = 2\sec x + 2x - 3$

$f(0.4) = 2\sec 0.4 + 0.8 - 3 = -0.0285...$

$f(0.5) = 2\sec 0.5 + 1 - 3 = 0.2789...$

The sign change implies there is a root in this interval.

9 b $f'(x) = 2\sec x \tan x + 2$

Using $x_0 = 0.4$,

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$
$$= 0.4 - \frac{f(0.4)}{f'(0.4)}$$
$$= 0.4 - \frac{-0.0285...}{2\sec 0.4 \tan 0.4 + 2}$$
$$= 0.4097...$$

$\alpha = 0.410$ correct to 3 d.p.

c $f(-1.1895) =$
$2\sec(-1.1895) + 2\times(-1.1895) - 3$
$= -0.0044...$

$f(-1.1905) =$
$2\sec(-1.1905) + 2\times(-1.1905) - 3$
$= 0.0069...$

There is a change of sign in this interval, so there is a root $\beta = 1.191$ correct to 3 d.p.

10 a $e^{0.8x} - \frac{1}{3-2x} = 0$

$e^{0.8x} = \frac{1}{3-2x}$ Add $\frac{1}{3-2x}$ to each side.

$(3-2x)e^{0.8x} = \frac{1}{3-2x} \times (3-2x)$

Multiply each side by $(3 - 2x)$.

$(3-2x)e^{0.8x} = 1$ Simplify.

$\frac{(3-2x)e^{0.8x}}{e^{0.8x}} = \frac{1}{e^{0.8x}}$ Divide each side by $e^{0.8x}$.

$3-2x = e^{-0.8x}$ Simplify (remember $\frac{1}{e^a} = e^{-a}$).

$3 = e^{-0.8x} + 2x$ Add $2x$ to each side.

$2x = 3 - e^{-0.8x}$ Subtract $e^{-0.8x}$ from each side.

$\frac{2x}{2} = \frac{3}{2} - \frac{e^{-0.8x}}{2}$ Divide each term by 2.

$x = 1.5 - 0.5e^{-0.8x}$ Simplify.

b $x_0 = 1.3$

$x_1 = 1.5 - 0.5e^{-0.8(1.3)} = 1.32327...$
$x_2 = 1.5 - 0.5e^{-0.8(1.32327..)} = 1.32653...$
$x_3 = 1.5 - 0.5e^{-0.8(1.32653...)} = 1.32698...$

So $x_3 = 1.327$ correct to 3 d.p.

10 c $e^{0.8x} - \frac{1}{3-2x} = 0$

$e^{0.8x} = \frac{1}{3-2x}$ Add $\frac{1}{3-2x}$ to each side.

$0.8x = \ln\left(\frac{1}{3-2x}\right)$ Take logs.

$0.8x = -\ln(3-2x)$ Simplify using $\ln\left(\frac{1}{c}\right) = -\ln c$.

$\frac{0.8x}{0.8} = -\frac{\ln(3-2x)}{0.8}$ Divide each side by 0.8.

$x = -1.25\ln(3-2x)$ Simplify $\left(\frac{1}{0.8} = 1.25\right)$.

So $p = -1.25$

d $x_0 = -2.6$

$x_1 = -1.25\ln[3-2(-2.6)]$
$= -2.63016...$
$x_2 = -1.25\ln[3-2(-2.63016...)]$
$= -2.63933...$
$x_3 = -1.25\ln[3-2(-2.63933...)]$
$= -2.64210...$
So $x_3 = -2.64$ (2 d.p.)

11 a $y = x^x \Rightarrow \ln y = x\ln x$

$\frac{1}{y}\frac{dy}{dx} = x \times \frac{1}{x} + \ln x = 1 + \ln x$

$\frac{dy}{dx} = y(1+\ln x) = x^x(1+\ln x)$

b $f(x) = x^x - 2$
$f(1.4) = 1.4^{1.4} - 2 = -0.3983...$
$f(1.6) = 1.6^{1.6} - 2 = 0.1212...$
The sign change implies there is a root in this interval.

c $f'(x) = x^x(1+\ln x)$
$f(1.5) = 1.5^{1.5} - 2 = -0.16288...$
$f'(1.5) = 1.5^{1.5}(1+\ln 1.5) = 2.58200...$

$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$
$= 1.5 - \frac{f(1.5)}{f'(1.5)}$
$= 1.5 + \frac{0.16288...}{2.58200...}$
$= 1.56308...$

To a second approximation, $\alpha = 1.5631$, to 4 d.p.

d $f(1.55955) = 1.55955^{1.55955} - 2$
$= -0.00017...$
$f(1.55965) = 1.55965^{1.55965} - 2$
$= 0.00011...$
There is a change of sign in this interval so $\alpha = 1.5596$ to 4 d.p.

12 a $f(x) = \cos(4x) - \frac{1}{2}x$
$f(1.3) = \cos 5.2 - 0.65 = -0.181...$
$f(1.4) = \cos 5.6 - 0.7 = 0.0755...$
The sign change implies there is a root in this interval.

b $f'(x) = -4\sin(4x) - \frac{1}{2}$
At B, $f'(x) = 0$
$\sin 4x = -\frac{1}{8} \Rightarrow$
$4x = -0.1253...,\ 3.2669...,\ 6.1579...,$ etc
From the graph $0 < x < 1$ so $4x = 3.2669$
So $x = 0.81673...$
$f(0.81673...) = \cos(3.2669...) - 0.40803...$
$= -1.4005...$
B has coordinates (0.817, −1.401).

12 c $x_{n+1} = \frac{1}{4}\arccos\left(\frac{1}{2}x_n\right)$

$x_0 = 0.4 \Rightarrow x_1 = \frac{1}{4}\arccos(0.2) = 0.34235...$

$x_2 = \frac{1}{4}\arccos(0.17117...) = 0.34969...$

$x_3 = \frac{1}{4}\arccos(0.17484...) = 0.34876...$

$x_4 = \frac{1}{4}\arccos(0.17438...) = 0.34887...$

To 4 d.p., the values are $x_1 = 0.3424$, $x_2 = 0.3497$, $x_3 = 3488$, $x_4 = 0.3489$.

d $\text{f}(1.7) = \cos 6.8 - 0.85 = 0.01939...$

$\text{f}'(1.7) = -4\sin 6.8 - \frac{1}{2} = -2.4765...$

$x_1 = x_0 - \frac{\text{f}(x_0)}{\text{f}'(x_0)}$

$= 1.7 - \frac{\text{f}(1.7)}{\text{f}'(1.7)}$

$= 1.7 + \frac{0.01939...}{2.4765...}$

$= 1.7078...$

To 3 d.p., the second approximation is 1.708.

12 e $\text{f}(1.7075) = \cos 6.83 - 0.85375$

$= 0.000435...$

$\text{f}(1.7085) = \cos 6.834 - 0.85425$

$= -0.00215...$

There is a change of sign so there is a root of 1.708 correct to 3 decimal places in this interval.

Challenge

a $\text{f}(x) = x^6 + x^3 - 7x^2 - x + 3$

$\text{f}'(x) = 6x^5 + 3x^2 - 14x - 1$

$\text{f}''(x) = 30x^4 + 6x - 14$

i $\text{f}''(x) = 0 \Rightarrow 6x = 14 - 30x^4$

$3x = 7 - 15x^4$

$x = \frac{7 - 15x^4}{3}$

ii $\text{f}''(x) = 0 \Rightarrow 6x = 14 - 30x^4$

$15x^4 + 3x = 7$

$x(15x^3 + 3) = 7$

$x = \frac{7}{15x^3 + 3}$

iii $\text{f}''(x) = 0 \Rightarrow 6x = 14 - 30x^4$

$15x^4 = 7 - 3x$

$x^4 = \frac{7 - 3x}{15}$

$x = \sqrt[4]{\frac{7 - 3x}{15}}$

b As B is a point of inflection $\text{f}''(x) = 0$.
Using $x_0 = 1$ in part **iii**

$x_1 = \sqrt[4]{\frac{4}{15}} = 0.7186...$

$x_2 = \sqrt[4]{\frac{7 - 3\times 0.7186...}{15}} = 0.7538...$

$x_3 = \sqrt[4]{\frac{7 - 3\times 0.7538...}{15}} = 0.7496...$

$x_4 = \sqrt[4]{\frac{7 - 3\times 0.7496...}{15}} = 0.7501...$

$x_5 = \sqrt[4]{\frac{7 - 3\times 0.7501...}{15}} = 0.7501...$

Correct to 3 d.p., an approximation for the x-coordinate of B is 0.750.

c A has a negative x-coordinate. Formula **iii** gives the positive fourth root, so cannot be used to find a negative root.

d As A is a point of inflection, $f''(x) = 0$.

$f''(0) = -14$

$f''(-1) = 30(-1^4) + 6(-1) - 14 = 10$

There is a change of sign, so the x-coordinate of the root A lies in the interval [−1, 0].

$f'''(x) = 120x^3 + 6$

Using the Newton–Raphson formula:

$$x_1 = x_0 - \frac{f''(x_0)}{f'''(x_0)}$$

Using $x_0 = -0.9$

$$x_1 = -0.9 - \frac{f''(-0.9)}{f'''(-0.9)}$$

$$= -0.9 - \frac{30(-0.9)^4 + 6(-0.9) - 14}{120(-0.9)^3 + 6}$$

$$= -0.9 - \frac{19.683 - 5.4 - 14}{-87.48 + 6}$$

$$= -0.9 - \frac{0.283}{81.48} = -0.89652...$$

$$x_2 = -0.89652... - \frac{f''(-0.89652...)}{f'''(-0.89652...)}$$

$$= -0.89650...$$

The x-coordinate of A is −0.897 correct to 3 d.p.