

## Integration 11C

**1 a**  $\int \cot^2 x \, dx = \int (\operatorname{cosec}^2 x - 1) \, dx$   
 $= -\cot x - x + c$

**b**  $\int \cos^2 x \, dx = \int \frac{1}{2}(1 + \cos 2x) \, dx$   
 $= \frac{1}{2}x + \frac{1}{4}\sin 2x + c$

**c**  $\int \sin 2x \cos 2x \, dx = \int \frac{1}{2}\sin 4x \, dx$   
 $= -\frac{1}{8}\cos 4x + c$

**d**  $\int (1 + \sin x)^2 \, dx = \int (1 + 2\sin x + \sin^2 x) \, dx$

But  $\cos 2x = 1 - 2\sin^2 x$

$$\therefore \sin^2 x = \frac{1}{2} - \frac{1}{2}\cos 2x$$

$$\begin{aligned} \therefore \int (1 + \sin x)^2 \, dx &= \int \left( \frac{3}{2} + 2\sin x - \frac{1}{2}\cos 2x \right) \, dx \\ &= \frac{3}{2}x - 2\cos x - \frac{1}{4}\sin 2x + c \end{aligned}$$

**e**  $\int \tan^2 3x \, dx = \int (\sec^2 3x - 1) \, dx$

$$= \frac{1}{3}\tan 3x - x + c$$

**f**  $\int (\cot x - \operatorname{cosec} x)^2 \, dx$

$$\begin{aligned} &= \int (\cot^2 x - 2\cot x \operatorname{cosec} x + \operatorname{cosec}^2 x) \, dx \\ &= \int (2\operatorname{cosec}^2 x - 1 - 2\cot x \operatorname{cosec} x) \, dx \end{aligned}$$

$$= -2\cot x - x + 2\operatorname{cosec} x + c$$

**g**  $\int (\sin x + \cos x)^2 \, dx$

$$\begin{aligned} &= \int (\sin^2 x + 2\sin x \cos x + \cos^2 x) \, dx \\ &= \int (1 + \sin 2x) \, dx \\ &= x - \frac{1}{2}\cos 2x + c \end{aligned}$$

**h**  $\int \sin^2 x \cos^2 x \, dx = \int \left( \frac{1}{2}\sin 2x \right)^2 \, dx$

$$\begin{aligned} &= \int \frac{1}{4}\sin^2 2x \, dx \\ &= \int \frac{1}{4} \left( \frac{1}{2} - \frac{1}{2}\cos 4x \right) \, dx \\ &= \int \left( \frac{1}{8} - \frac{1}{8}\cos 4x \right) \, dx \\ &= \frac{1}{8}x - \frac{1}{32}\sin 4x + c \end{aligned}$$

**i**  $\frac{1}{\sin^2 x \cos^2 x} = \frac{1}{(\frac{1}{2}\sin 2x)^2} = 4\operatorname{cosec}^2 2x$

$$\begin{aligned} \therefore \int \frac{1}{\sin^2 x \cos^2 x} \, dx &= \int 4\operatorname{cosec}^2 2x \, dx \\ &= -2\cot 2x + c \end{aligned}$$

**j**  $\int (\cos 2x - 1)^2 \, dx$

$$\begin{aligned} &= \int (\cos^2 2x - 2\cos 2x + 1) \, dx \\ &= \int \left( \frac{1}{2}\cos 4x + \frac{1}{2} - 2\cos 2x + 1 \right) \, dx \\ &= \int \left( \frac{1}{2}\cos 4x + \frac{3}{2} - 2\cos 2x \right) \, dx \\ &= \frac{1}{8}\sin 4x + \frac{3}{2}x - \sin 2x + c \end{aligned}$$

**2 a**  $\int \left( \frac{1 - \sin x}{\cos^2 x} \right) \, dx = \int (\sec^2 x - \tan x \sec x) \, dx$

$$= \tan x - \sec x + c$$

**b**  $\int \left( \frac{1 + \cos x}{\sin^2 x} \right) \, dx = \int (\operatorname{cosec}^2 x + \cot x \operatorname{cosec} x) \, dx$

$$= -\cot x - \operatorname{cosec} x + c$$

**c**  $\int \frac{\cos 2x}{\cos^2 x} \, dx = \int \frac{2\cos^2 x - 1}{\cos^2 x} \, dx$

$$\begin{aligned} &= \int (2 - \sec^2 x) \, dx \\ &= 2x - \tan x + c \end{aligned}$$

**d** 
$$\begin{aligned} \int \frac{\cos^2 x}{\sin^2 x} dx &= \int \cot^2 x dx \\ &= \int (\operatorname{cosec}^2 x - 1) dx \\ &= -\cot x - x + c \end{aligned}$$

**e** 
$$\begin{aligned} I &= \int \frac{(1+\cos x)^2}{\sin^2 x} dx = \int \frac{1+2\cos x+\cos^2 x}{\sin^2 x} dx \\ &= \int (\operatorname{cosec}^2 x + 2\cot x \operatorname{cosec} x + \cot^2 x) dx \end{aligned}$$

But  $\operatorname{cosec}^2 x = 1 + \cot^2 x$   
 $\Rightarrow \cot^2 x = \operatorname{cosec}^2 x - 1$   
 $\therefore I = \int (2\operatorname{cosec}^2 x - 1 + 2\cot x \operatorname{cosec} x) dx$   
 $= -2\cot x - x - 2\operatorname{cosec} x + c$

**f** 
$$\begin{aligned} \int (\cot x - \tan x)^2 dx &= \int (\cot^2 x - 2\cot x \tan x + \tan^2 x) dx \\ &= \int (\operatorname{cosec}^2 x - 1 - 2 + \sec^2 x - 1) dx \\ &= \int (\operatorname{cosec}^2 x - 4 + \sec^2 x) dx \\ &= -\cot x - 4x + \tan x + c \end{aligned}$$

**g** 
$$\begin{aligned} \int (\cos x - \sin x)^2 dx &= \int (\cos^2 x - 2\cos x \sin x + \sin^2 x) dx \\ &= \int (1 - \sin 2x) dx \\ &= x + \frac{1}{2} \cos 2x + c \end{aligned}$$

**h** 
$$\begin{aligned} \int (\cos x - \sec x)^2 dx &= \int (\cos^2 x - 2\cos x \sec x + \sec^2 x) dx \\ &= \int \left( \frac{1}{2} \cos 2x + \frac{1}{2} - 2 + \sec^2 x \right) dx \\ &= \int \left( \frac{1}{2} \cos 2x - \frac{3}{2} + \sec^2 x \right) dx \\ &= \frac{1}{4} \sin 2x - \frac{3}{2}x + \tan x + c \end{aligned}$$

**i** 
$$\begin{aligned} \int \frac{\cos 2x}{1 - \cos^2 2x} dx &= \int \frac{\cos 2x}{\sin^2 2x} dx \\ &= \int \cot 2x \operatorname{cosec} 2x dx \\ &= -\frac{1}{2} \operatorname{cosec} 2x + c \end{aligned}$$

**3** 
$$\begin{aligned} \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin^2 x dx &= \frac{1}{2} \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (1 - \cos 2x) dx \\ &= \frac{1}{2} \left[ x - \frac{1}{2} \sin 2x \right]_{\frac{\pi}{4}}^{\frac{\pi}{2}} \\ &= \frac{1}{2} \left( \left( \frac{\pi}{2} - 0 \right) - \left( \frac{\pi}{4} - \frac{1}{2} \right) \right) = \frac{1}{2} \left( \frac{\pi}{4} + \frac{1}{2} \right) \\ &= \frac{2+\pi}{8} \end{aligned}$$

**4 a** 
$$\begin{aligned} \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1}{\sin^2 x \cos^2 x} dx &= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{4}{\sin^2 2x} dx \\ &= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} 4\operatorname{cosec}^2 2x dx = [-2\cot 2x]_{\frac{\pi}{6}}^{\frac{\pi}{3}} \\ &= \frac{2}{\sqrt{3}} + \frac{2}{\sqrt{3}} = \frac{4\sqrt{3}}{3} \end{aligned}$$

**b** 
$$\begin{aligned} \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} (\sin x - \operatorname{cosec} x)^2 dx &= \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} (\sin^2 x - 2 + \operatorname{cosec}^2 x) dx \\ &= \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \left( \frac{1}{2}(1 - \cos 2x) - 2 + \operatorname{cosec}^2 x \right) dx \\ &= \left[ \frac{x}{2} - \frac{1}{4} \sin 2x - 2x - \cot x \right]_{\frac{\pi}{6}}^{\frac{\pi}{4}} \\ &= \left( \frac{\pi}{8} - \frac{1}{4} - \frac{\pi}{2} - 1 \right) - \left( \frac{\pi}{12} - \frac{\sqrt{3}}{8} - \frac{\pi}{3} - \sqrt{3} \right) \\ &= \frac{27\sqrt{3} - 30 - 3\pi}{24} \\ &= \frac{9\sqrt{3} - 10 - \pi}{8} \end{aligned}$$

**c** 
$$\begin{aligned} \int_0^{\frac{\pi}{4}} \frac{(1 + \sin x)^2}{\cos^2 x} dx &= \int_0^{\frac{\pi}{4}} \frac{(1 + 2\sin x + \sin^2 x)}{\cos^2 x} dx \\ &= \int_0^{\frac{\pi}{4}} (2\sec^2 x + 2\sec x \tan x - 1) dx \\ &= [2\tan x + 2\sec x - x]_0^{\frac{\pi}{4}} \\ &= \left( 2 + 2\sqrt{2} - \frac{\pi}{4} \right) - 2 = 2\sqrt{2} - \frac{\pi}{4} \end{aligned}$$

$$\begin{aligned}
 \mathbf{d} \quad & \int_{\frac{3\pi}{8}}^{\frac{\pi}{2}} \frac{\sin 2x}{1 - \sin^2 2x} dx \\
 &= \int_{\frac{3\pi}{8}}^{\frac{\pi}{2}} \frac{\sin 2x}{\cos^2 2x} dx \\
 &= \int_{\frac{3\pi}{8}}^{\frac{\pi}{2}} \sec 2x \tan 2x dx \\
 &= \left[ \frac{1}{2} \sec 2x \right]_{\frac{3\pi}{8}}^{\frac{\pi}{2}} \\
 &= -\frac{1}{2} + \frac{\sqrt{2}}{2} = \frac{\sqrt{2}-1}{2}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{7} \quad \mathbf{a} \quad & \cos^4 x \equiv (\cos^2 x)^2 \equiv \left( \frac{1}{2}(\cos 2x + 1) \right)^2 \\
 &\equiv \frac{1}{4}(\cos^2 2x + 2 \cos 2x + 1) \\
 &\equiv \frac{1}{4} \left( \frac{1}{2}(\cos 4x + 1) + 2 \cos 2x + 1 \right) \\
 &\equiv \frac{1}{8} \cos 4x + \frac{1}{2} \cos 2x + \frac{3}{8}
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{b} \quad & \int \cos^4 x dx = \int \left( \frac{1}{8} \cos 4x + \frac{1}{2} \cos 2x + \frac{3}{8} \right) dx \\
 &= \frac{1}{32} \sin 4x + \frac{1}{4} \sin 2x + \frac{3x}{8} + c
 \end{aligned}$$

**5 a**  $\sin(3x+2x) = \sin 3x \cos 2x + \cos 3x \sin 2x$   
 $\sin(3x-2x) = \sin 3x \cos 2x - \cos 3x \sin 2x$

Adding the above,  
 $\sin 5x + \sin x = 2 \sin 3x \cos 2x$

$$\begin{aligned}
 \mathbf{b} \quad & \int \sin 3x \cos 2x dx = \frac{1}{2} \int (\sin 5x + \sin x) dx \\
 &= \frac{1}{2} \left( -\frac{1}{5} \cos 5x - \cos x \right) + c \\
 &= -\frac{1}{10} \cos 5x - \frac{1}{2} \cos x + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{6} \quad \mathbf{a} \quad & f(x) = 5 \sin^2 x + 7 \cos^2 x \\
 &= 5 \sin^2 x + 7 - 7 \sin^2 x \\
 &= 7 - 2 \sin^2 x \\
 &= 7 - 2 \left( \frac{1}{2}(1 - \cos 2x) \right) \\
 &= 7 - 1 + 2 \cos 2x \\
 &= \cos 2x + 6
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{6} \quad \mathbf{b} \quad & \int_0^{\frac{\pi}{4}} f(x) dx = \int_0^{\frac{\pi}{4}} (\cos 2x + 6) dx \\
 &= \left[ \frac{1}{2} \sin 2x + 6x \right]_0^{\frac{\pi}{4}} \\
 &= \frac{1}{2}(1 + 3\pi)
 \end{aligned}$$