Integration 11C

1 a $\int \cot^2 x\,dx = \int (\operatorname{cosec}^2 x - 1)dx$

$= -\cot x - x + c$

b $\int \cos^2 x\,dx = \int \frac{1}{2}(1+\cos 2x)dx$

$= \frac{1}{2}x + \frac{1}{4}\sin 2x + c$

c $\int \sin 2x \cos 2x\,dx = \int \frac{1}{2}\sin 4x\,dx$

$= -\frac{1}{8}\cos 4x + c$

d $\int (1+\sin x)^2 dx = \int (1 + 2\sin x + \sin^2 x)dx$

But $\cos 2x = 1 - 2\sin^2 x$

$\therefore \sin^2 x = \frac{1}{2} - \frac{1}{2}\cos 2x$

$\therefore \int (1+\sin x)^2 dx = \int \left(\frac{3}{2} + 2\sin x - \frac{1}{2}\cos 2x\right)dx$

$= \frac{3}{2}x - 2\cos x - \frac{1}{4}\sin 2x + c$

e $\int \tan^2 3x\,dx = \int (\sec^2 3x - 1)dx$

$= \frac{1}{3}\tan 3x - x + c$

f $\int (\cot x - \operatorname{cosec} x)^2\,dx$

$= \int (\cot^2 x - 2\cot x \operatorname{cosec} x + \operatorname{cosec}^2 x)dx$

$= \int (2\operatorname{cosec}^2 x - 1 - 2\cot x \operatorname{cosec} x)\,dx$

$= -2\cot x - x + 2\operatorname{cosec} x + c$

g $\int (\sin x + \cos x)^2 dx$

$= \int (\sin^2 x + 2\sin x \cos x + \cos^2 x)dx$

$= \int (1 + \sin 2x)dx$

$= x - \frac{1}{2}\cos 2x + c$

h $\int \sin^2 x \cos^2 x\,dx = \int \left(\frac{1}{2}\sin 2x\right)^2 dx$

$= \int \frac{1}{4}\sin^2 2x\,dx$

$= \int \frac{1}{4}\left(\frac{1}{2} - \frac{1}{2}\cos 4x\right)dx$

$= \int \left(\frac{1}{8} - \frac{1}{8}\cos 4x\right)dx$

$= \frac{1}{8}x - \frac{1}{32}\sin 4x + c$

i $\frac{1}{\sin^2 x \cos^2 x} = \frac{1}{\left(\frac{1}{2}\sin 2x\right)^2} = 4\operatorname{cosec}^2 2x$

$\therefore \int \frac{1}{\sin^2 x \cos^2 x}dx = \int 4\operatorname{cosec}^2 2x\,dx$

$= -2\cot 2x + c$

j $\int (\cos 2x - 1)^2 dx$

$= \int (\cos^2 2x - 2\cos 2x + 1)dx$

$= \int \left(\frac{1}{2}\cos 4x + \frac{1}{2} - 2\cos 2x + 1\right)dx$

$= \int \left(\frac{1}{2}\cos 4x + \frac{3}{2} - 2\cos 2x\right)dx$

$= \frac{1}{8}\sin 4x + \frac{3}{2}x - \sin 2x + c$

2 a $\int \left(\frac{1-\sin x}{\cos^2 x}\right)dx = \int (\sec^2 x - \tan x \sec x)dx$

$= \tan x - \sec x + c$

b $\int \left(\frac{1+\cos x}{\sin^2 x}\right)dx = \int (\operatorname{cosec}^2 x + \cot x \operatorname{cosec} x)dx$

$= -\cot x - \operatorname{cosec} x + c$

c $\int \frac{\cos 2x}{\cos^2 x}dx = \int \frac{2\cos^2 x - 1}{\cos^2 x}dx$

$= \int (2 - \sec^2 x)dx$

$= 2x - \tan x + c$

d $\int \frac{\cos^2 x}{\sin^2 x}\,dx = \int \cot^2 x\,dx$

$= \int (\operatorname{cosec}^2 x - 1)\,dx$

$= -\cot x - x + c$

e $I = \int \frac{(1+\cos x)^2}{\sin^2 x}\,dx = \int \frac{1+2\cos x+\cos^2 x}{\sin^2 x}\,dx$

$= \int (\operatorname{cosec}^2 x + 2\cot x \operatorname{cosec} x + \cot^2 x)\,dx$

But $\operatorname{cosec}^2 x = 1 + \cot^2 x$

$\Rightarrow \cot^2 x = \operatorname{cosec}^2 x - 1$

$\therefore I = \int (2\operatorname{cosec}^2 x - 1 + 2\cot x \operatorname{cosec} x)\,dx$

$= -2\cot x - x - 2\operatorname{cosec} x + c$

f $\int (\cot x - \tan x)^2\,dx$

$= \int (\cot^2 x - 2\cot x \tan x + \tan^2 x)\,dx$

$= \int (\operatorname{cosec}^2 x - 1 - 2 + \sec^2 x - 1)\,dx$

$= \int (\operatorname{cosec}^2 x - 4 + \sec^2 x)\,dx$

$= -\cot x - 4x + \tan x + c$

g $\int (\cos x - \sin x)^2\,dx$

$= \int (\cos^2 x - 2\cos x \sin x + \sin^2 x)\,dx$

$= \int (1 - \sin 2x)\,dx$

$= x + \frac{1}{2}\cos 2x + c$

h $\int (\cos x - \sec x)^2\,dx$

$= \int (\cos^2 x - 2\cos x \sec x + \sec^2 x)\,dx$

$= \int \left(\frac{1}{2}\cos 2x + \frac{1}{2} - 2 + \sec^2 x\right)dx$

$= \int \left(\frac{1}{2}\cos 2x - \frac{3}{2} + \sec^2 x\right)dx$

$= \frac{1}{4}\sin 2x - \frac{3}{2}x + \tan x + c$

i $\int \frac{\cos 2x}{1-\cos^2 2x}\,dx = \int \frac{\cos 2x}{\sin^2 2x}\,dx$

$= \int \cot 2x \operatorname{cosec} 2x\,dx$

$= -\frac{1}{2}\operatorname{cosec} 2x + c$

3 $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin^2 x\,dx = \frac{1}{2}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (1-\cos 2x)\,dx$

$= \frac{1}{2}\left[x - \frac{1}{2}\sin 2x\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}}$

$= \frac{1}{2}\left(\left(\frac{\pi}{2} - 0\right) - \left(\frac{\pi}{4} - \frac{1}{2}\right)\right) = \frac{1}{2}\left(\frac{\pi}{4} + \frac{1}{2}\right)$

$= \frac{2+\pi}{8}$

4 a $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1}{\sin^2 x \cos^2 x}\,dx = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{4}{\sin^2 2x}\,dx$

$= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} 4\operatorname{cosec}^2 2x\,dx = \left[-2\cot 2x\right]_{\frac{\pi}{6}}^{\frac{\pi}{3}}$

$= \frac{2}{\sqrt{3}} + \frac{2}{\sqrt{3}} = \frac{4\sqrt{3}}{3}$

b $\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} (\sin x - \operatorname{cosec} x)^2\,dx$

$= \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} (\sin^2 x - 2 + \operatorname{cosec}^2 x)\,dx$

$= \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \left(\frac{1}{2}(1-\cos 2x) - 2 + \operatorname{cosec}^2 x\right)dx$

$= \left[\frac{x}{2} - \frac{1}{4}\sin 2x - 2x - \cot x\right]_{\frac{\pi}{6}}^{\frac{\pi}{4}}$

$= \left(\frac{\pi}{8} - \frac{1}{4} - \frac{\pi}{2} - 1\right) - \left(\frac{\pi}{12} - \frac{\sqrt{3}}{8} - \frac{\pi}{3} - \sqrt{3}\right)$

$= \frac{27\sqrt{3} - 30 - 3\pi}{24}$

$= \frac{9\sqrt{3} - 10 - \pi}{8}$

c $\int_0^{\frac{\pi}{4}} \frac{(1+\sin x)^2}{\cos^2 x}\,dx$

$= \int_0^{\frac{\pi}{4}} \frac{(1+2\sin x+\sin^2 x)}{\cos^2 x}\,dx$

$= \int_0^{\frac{\pi}{4}} \left(2\sec^2 x + 2\sec x \tan x - 1\right)dx$

$= \left[2\tan x + 2\sec x - x\right]_0^{\frac{\pi}{4}}$

$= \left(2 + 2\sqrt{2} - \frac{\pi}{4}\right) - 2 = 2\sqrt{2} - \frac{\pi}{4}$

d $\int_{\frac{3\pi}{8}}^{\frac{\pi}{2}} \frac{\sin 2x}{1-\sin^2 2x}\,dx$

$= \int_{\frac{3\pi}{8}}^{\frac{\pi}{2}} \frac{\sin 2x}{\cos^2 2x}\,dx$

$= \int_{\frac{3\pi}{8}}^{\frac{\pi}{2}} \sec 2x \tan 2x\,dx$

$= \left[\frac{1}{2}\sec 2x\right]_{\frac{3\pi}{8}}^{\frac{\pi}{2}}$

$= -\frac{1}{2} + \frac{\sqrt{2}}{2} = \frac{\sqrt{2}-1}{2}$

5 a $\sin(3x+2x) = \sin 3x \cos 2x + \cos 3x \sin 2x$
$\sin(3x-2x) = \sin 3x \cos 2x - \cos 3x \sin 2x$

Adding the above,
$\sin 5x + \sin x = 2\sin 3x \cos 2x$

b $\int \sin 3x \cos 2x\,dx = \frac{1}{2}\int (\sin 5x + \sin x)\,dx$

$= \frac{1}{2}\left(-\frac{1}{5}\cos 5x - \cos x\right) + c$

$= -\frac{1}{10}\cos 5x - \frac{1}{2}\cos x + c$

6 a $f(x) = 5\sin^2 x + 7\cos^2 x$

$= 5\sin^2 x + 7 - 7\sin^2 x$

$= 7 - 2\sin^2 x$

$= 7 - 2\left(\frac{1}{2}(1-\cos 2x)\right)$

$= 7 - 1 + 2\cos 2x$

$= \cos 2x + 6$

6 b $\int_0^{\frac{\pi}{4}} f(x)\,dx = \int_0^{\frac{\pi}{4}} (\cos 2x + 6)\,dx$

$= \left[\frac{1}{2}\sin 2x + 6x\right]_0^{\frac{\pi}{4}}$

$= \frac{1}{2}(1+3\pi)$

7 a $\cos^4 x \equiv \left(\cos^2 x\right)^2 \equiv \left(\frac{1}{2}(\cos 2x + 1)\right)^2$

$\equiv \frac{1}{4}\left(\cos^2 2x + 2\cos 2x + 1\right)$

$\equiv \frac{1}{4}\left(\frac{1}{2}\left(\cos 4x + 1\right) + 2\cos 2x + 1\right)$

$\equiv \frac{1}{8}\cos 4x + \frac{1}{2}\cos 2x + \frac{3}{8}$

b $\int \cos^4 x\,dx = \int \left(\frac{1}{8}\cos 4x + \frac{1}{2}\cos 2x + \frac{3}{8}\right)dx$

$= \frac{1}{32}\sin 4x + \frac{1}{4}\sin 2x + \frac{3x}{8} + c$