Integration 11J

1 a $\frac{dy}{dx}=(1+y)(1-2x)$

$\Rightarrow \int \frac{1}{1+y}dy=\int(1-2x)dx$

$\Rightarrow \ln|1+y|=x-x^2+c$

$\Rightarrow 1+y=e^{(x-x^2+c)}$

$\Rightarrow 1+y=A\,e^{x-x^2}, \quad (A=e^c)$

$\Rightarrow y=A\,e^{x-x^2}-1$

b $\frac{dy}{dx}=y\tan x$

$\Rightarrow \int \frac{1}{y}dy=\int \tan x\,dx$

$\Rightarrow \ln|y|=\ln|\sec x|+c$

$\Rightarrow \ln|y|=\ln|k\sec x|, \quad (c=\ln k)$

$\Rightarrow y=k\sec x$

c $\cos^2 x\frac{dy}{dx}=y^2\sin^2 x$

$\Rightarrow \int \frac{1}{y^2}dy=\int \frac{\sin^2 x}{\cos^2 x}dx$

$\Rightarrow \int \frac{1}{y^2}dy=\int \tan^2 x\,dx=\int(\sec^2 x-1)dx$

$\Rightarrow -\frac{1}{y}=\tan x-x+c$

$\Rightarrow y=\frac{-1}{\tan x-x+c}$

d $\frac{dy}{dx}=2e^{x-y}=2e^xe^{-y}$

$\Rightarrow \int \frac{1}{e^{-y}}dy=\int 2e^x dx$

i.e. $\Rightarrow \int e^y dy=\int 2e^x dx$

$\Rightarrow e^y=2e^x+c$

$\Rightarrow y=\ln|2e^x+c|$

2 a $\frac{dy}{dx}=\sin x\cos^2 x$

$\Rightarrow \int dy=\int \sin x\cos^2 x\,dx$

$\Rightarrow y=-\frac{\cos^3 x}{3}+c$

$y=0, x=\frac{\pi}{3}\Rightarrow 0=-\frac{\left(\frac{1}{8}\right)}{3}+c\Rightarrow c=\frac{1}{24}$

$\therefore\ y=\frac{1}{24}-\frac{1}{3}\cos^3 x$

b $\frac{dy}{dx}=\sec^2 x\sec^2 y$

$\Rightarrow \int \frac{1}{\sec^2 y}dy=\int \sec^2 x\,dx$

$\Rightarrow \int \cos^2 y\,dy=\int \sec^2 x\,dx$

$\Rightarrow \int\left(\frac{1}{2}+\frac{1}{2}\cos 2y\right)dy=\int \sec^2 x\,dx$

$\Rightarrow \frac{1}{2}y+\frac{1}{4}\sin 2y=\tan x+c$

or $\sin 2y+2y=4\tan x+k$

$y=0, x=\frac{\pi}{4}\Rightarrow 0=4+k\Rightarrow k=-4$

$\therefore \sin 2y+2y=4\tan x-4$

c $\frac{dy}{dx}=2\cos^2 y\cos^2 x$

$\Rightarrow \int \frac{1}{\cos^2 y}dy=\int 2\cos^2 x\,dx$

$\Rightarrow \int \sec^2 y\,dy=\int(1+\cos 2x)dx$

$\Rightarrow \tan y=x+\frac{1}{2}\sin 2x+c$

$x=0, \quad y=\frac{\pi}{4}\Rightarrow 1=0+c$

$\therefore\ \tan y=x+\frac{1}{2}\sin 2x+1$

2 d $\sin y \cos x \frac{dy}{dx} = \frac{\cos y}{\cos x}$

$\tan y \frac{dy}{dx} = \sec^2 x$

$\int \tan y \, dy = \int \sec^2 x \, dx$

$-\ln|\cos y| = \tan x + c$

$x = 0, y = 0 \Rightarrow 0 = 0 + c \Rightarrow c = 0$

$-\ln|\cos y| = \tan x$

$\cos y = e^{-\tan x}$

$y = \arccos\left(e^{-\tan x}\right)$

3 a $x^2 \frac{dy}{dx} = y + xy$

$x^2 \frac{dy}{dx} = y(1+x)$

$\frac{1}{y}\frac{dy}{dx} = \frac{1+x}{x^2} = \frac{1}{x^2} + \frac{1}{x}$

$\int \frac{1}{y} \, dy = \int \left(\frac{1}{x^2} + \frac{1}{x}\right) dx$

$\ln y = -\frac{1}{x} + \ln x + \ln A$

$y = e^{-\frac{1}{x} + \ln x + \ln A} = e^{-\frac{1}{x}} \times e^{\ln x} \times e^{\ln A}$

$y = Axe^{-\frac{1}{x}}$

b $y = e^4, x = -1 \Rightarrow e^4 = -Ae$

$A = -e^3$

$y = -e^3 xe^{-\frac{1}{x}} = -xe^{\left(\frac{3x-1}{x}\right)}$

4 $(2y + 2yx)\frac{dy}{dx} = 1 - y^2$

$2y(1+x)\frac{dy}{dx} = 1 - y^2$

$\int \frac{2y}{1-y^2} \, dy = \int \frac{1}{1+x} \, dx$

$\ln k - \ln\left|1 - y^2\right| = \ln|1+x|$

$\ln\left|\frac{k}{1-y^2}\right| = \ln|1+x|$

$\frac{k}{1-y^2} = 1 + x$

$x = 0, y = 0 \Rightarrow k = 1$

$\frac{1}{1-y^2} = 1 + x$

$1 - y^2 = \frac{1}{x+1}$

$y^2 = 1 - \frac{1}{x+1} = \frac{x}{x+1}$

$y = \sqrt{\frac{x}{x+1}}$

5 $e^{x+y}\frac{dy}{dx} = 2x + xe^y$

$e^x e^y \frac{dy}{dx} = x(2 + e^y)$

$\frac{e^y}{2+e^y}\frac{dy}{dx} = xe^{-x}$

$\int \frac{e^y}{2+e^y} \, dy = \int xe^{-x} \, dx$

$\ln\left|2 + e^y\right| = \int xe^{-x} \, dx$

Let $u = x \Rightarrow \frac{du}{dx} = 1$

$\frac{dv}{dx} = e^{-x} \Rightarrow v = -e^{-x}$

So $\ln\left|2 + e^y\right| = -xe^{-x} + \int e^{-x} \, dx$

$\ln\left|2 + e^y\right| = -xe^{-x} - e^{-x} + c$

6 $(1-x^2)\frac{dy}{dx} = xy + y$

$(1-x^2)\frac{dy}{dx} = y(x+1)$

$\frac{1}{y}\frac{dy}{dx} = \frac{x+1}{(1-x^2)} = \frac{x+1}{(1-x)(1+x)} = \frac{1}{1-x}$

$\int \frac{1}{y}\,dy = \int \frac{1}{1-x}dx$

$\ln y = \ln k - \ln|1-x| = \ln\left|\frac{k}{1-x}\right|$

$y = \frac{k}{1-x}$

$y = 6, x = 0.5 \Rightarrow 6 = 2k \Rightarrow k = 3$

$y = \frac{3}{1-x}$

7 $(1+x^2)\frac{dy}{dx} = x - xy^2$

$(1+x^2)\frac{dy}{dx} = x(1-y^2)$

$\int \frac{1}{1-y^2}\,dy = \int \frac{x}{1+x^2}\,dx$

$\frac{1}{2}\int\left(\frac{1}{1+y} + \frac{1}{1-y}\right)dy = \int \frac{x}{1+x^2}\,dx$

$\frac{1}{2}\left(\ln|1+y| - \ln|1-y|\right) = \frac{1}{2}\ln|1+x^2| + c$

$x = 0, y = 2 \Rightarrow c = \frac{1}{2}\ln 3$

$\left(\ln|1+y| - \ln|1-y|\right) = \ln|3(1+x^2)|$

$\frac{1+y}{y-1} = 3 + 3x^2$

$1 + y = 3y + 3x^2y - 3 - 3x^2$

$1 + 3 + 3x^2 = 3y + 3x^2y - y$

$3(x^2+1) + 1 = y(3(x^2+1) - 1)$

$y = \frac{3(1+x^2)+1}{3(1+x^2)-1}$

8 $\frac{dy}{dx} = xe^{-y}$

$\int e^y\,dy = \int x\,dx$

$e^y = \frac{x^2}{2} + c$

$x = 4, y = \ln 2 \Rightarrow 2 = 8 + c \Rightarrow c = -6$

$e^y = \frac{x^2-12}{2}$

$y = \ln\left|\frac{x^2-12}{2}\right|$

9 $\frac{dy}{dx} = \cos^2 y + \cos 2x\cos^2 y$

$\frac{dy}{dx} = \cos^2 y(1+\cos 2x)$

$\int \sec^2 y\,dy = \int(1+\cos 2x)\,dx$

$\tan y = x + \frac{1}{2}\sin 2x + c$

$x = \frac{\pi}{4}, y = \frac{\pi}{4} \Rightarrow 1 = \frac{\pi}{4} + \frac{1}{2}\sin\frac{\pi}{2} + c$

$c = \frac{1}{2} - \frac{\pi}{4} = \frac{2-\pi}{4}$

$\tan y = x + \frac{1}{2}\sin 2x + \frac{2-\pi}{4}$

10 $\frac{dy}{dx} = xy\sin x$

$\int \frac{1}{y}\,dy = \int x\sin x\,dx$

$\ln|y| = -x\cos x + \int \cos x\,dx$

$\ln|y| = -x\cos x + \sin x + c$

$x = \frac{\pi}{2}, y = 1 \Rightarrow c = -1$

$\ln|y| = -x\cos x + \sin x - 1$

11 a $I = \int \frac{3x+4}{x}\,dx$

$I = \int 3 + \frac{4}{x}\,dx = 3x + 4\ln x + c$

11 b $\dfrac{dy}{dx} = \dfrac{3x\sqrt{y} + 4\sqrt{y}}{x} = \sqrt{y}\,\dfrac{3x+4}{x}$

$$\int \frac{1}{\sqrt{y}}\, dy = 3x + 4\ln x + c \text{ (from } \mathbf{a}\text{)}$$

$$2\sqrt{y} = 3x + 4\ln x + c$$

$$x = 1, y = 16 \Rightarrow 8 = 3 + c \Rightarrow c = 5$$

$$\sqrt{y} = \frac{3}{2}x + 2\ln x + \frac{5}{2}$$

$$y = \left(\frac{3}{2}x + 2\ln x + \frac{5}{2}\right)^2$$

12 a $\dfrac{8x-18}{(3x-8)(x-2)} = \dfrac{A}{3x-8} + \dfrac{B}{x-2}$

$$8x - 18 = A(x-2) + B(3x-8)$$

$$x = 2: -2 = -2B \Rightarrow B = 1$$

$$x = \frac{8}{3}: \frac{64}{3} - 18 = \frac{2}{3}A \Rightarrow A = 5$$

$$\frac{8x-18}{(3x-8)(x-2)} = \frac{5}{3x-8} + \frac{1}{x-2}$$

b $(x-2)(3x-8)\dfrac{dy}{dx} = (8x-18)y$

$$\int \frac{1}{y}\, dy = \int \frac{8x-18}{(3x-8)(x-2)}\, dx$$

$$\ln|y| = \int\left(\frac{5}{3x-8} + \frac{1}{x-2}\right) dx$$

$$\ln|y| = \frac{5}{3}\ln|3x-8| + \ln|x-2| + c$$

c $x = 3, y = 8 \Rightarrow c = \ln 8$

$$\ln|y| = \ln|3x-8|^{\frac{5}{3}} + \ln|8x-2|$$

$$\ln|y| = \ln\left|(3x-8)^{\frac{5}{3}}(8x-2)\right|$$

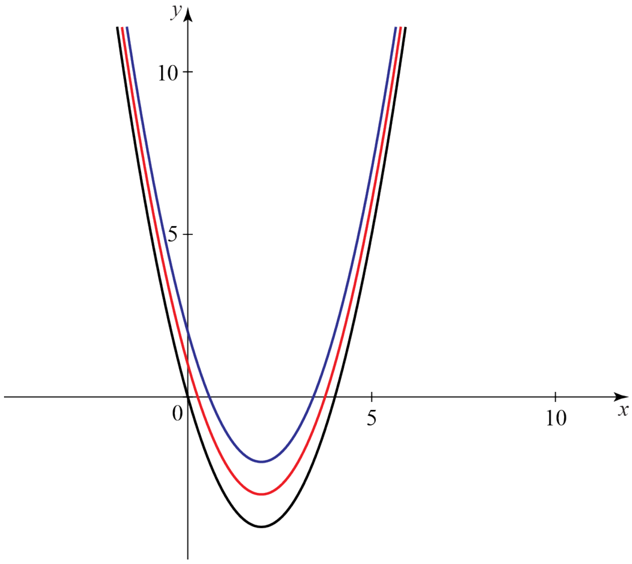
$$y = (3x-8)^{\frac{5}{3}}(8x-2)$$

$$y = 8(x-2)(3x-8)^{\frac{5}{3}}$$

13 a $\dfrac{dy}{dx} = 2x - 4$

$$y = x^2 - 4x + c$$

b



$y = x^2 - 4x, \quad y = x^2 - 4x + 1, \quad y = x^2 - 4x + 2$

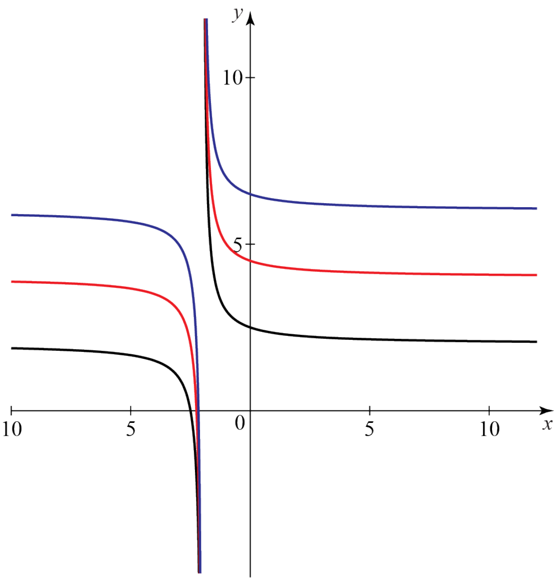
Any graphs of the form $y = x^2 - 4x + c$, where c is any real number.

14 a $\dfrac{dy}{dx} = -\dfrac{1}{(x+2)^2}$

$$y = -\int \frac{1}{(x+2)^2}\, dx$$

$$y = \frac{1}{x+2} + c$$

b



$y = \dfrac{1}{x+2} + 2, \quad y = \dfrac{1}{x+2} + 4, \quad y = \dfrac{1}{x+2} + 6$

Any graphs of the form $y = \dfrac{1}{x+2} + c$, where c is any real number.

14 c $3.1 = \frac{1}{10} + c \Rightarrow c = 3$

$y = \frac{1}{x+2} + 3$

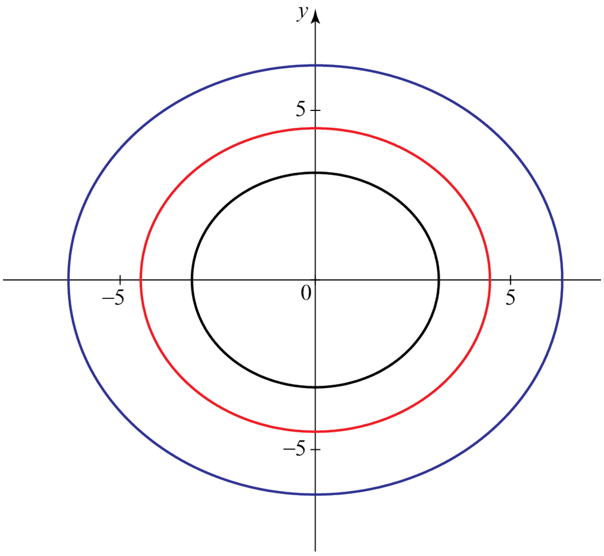
15 a $\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{x}{y}$

$\int y \ \mathrm{d}y = \int x \ \mathrm{d}x$

$\frac{y^2}{2} = -\frac{x^2}{2} + k$

$x^2 + y^2 = c$

15 b



$x^2 + y^2 = 10, \quad x^2 + y^2 = 20, \quad x^2 + y^2 = 40$

Circles with centre (0, 0) and radius $\sqrt{c}$ where c is any positive real number.

c $x^2 + y^2 = 49$