Integration, Mixed exercise 11

1 a $I = \int (2x-3)^7 \, dx$

Consider $y = (2x-3)^8$

$\frac{dy}{dx} = 16(2x-3)^7$

$I = \frac{(2x-3)^8}{16} + c$

b $I = \int x\sqrt{4x-1} \, dx$

Let $u = 4x-1 \Rightarrow \frac{du}{dx} = 4$

$I = \int \frac{u+1}{16}\sqrt{u} \, du$

$= \frac{2u^{\frac{5}{2}}}{80} + \frac{2u^{\frac{3}{2}}}{48} + c$

$= \frac{(4x-1)^{\frac{5}{2}}}{40} + \frac{(4x-1)^{\frac{3}{2}}}{24} + c$

c $I = \int \sin^2 x \cos x \, dx$

Consider $y = \sin^3 x \Rightarrow \frac{dy}{dx} = 3\sin^2 x \cos x$

$I = \frac{1}{3}\sin^3 x + c$

d $I = \int x \ln x \, dx$

Let $u = \ln x \Rightarrow \frac{du}{dx} = \frac{1}{x}$

$\frac{dv}{dx} = x \Rightarrow v = \frac{x^2}{2}$

$I = \frac{x^2}{2}\ln x - \int \frac{x^2}{2} \times \frac{1}{x} \, dx$

$= \frac{x^2}{2}\ln x - \frac{x^2}{4} + c$

e $I = \int \frac{4\sin x \cos x}{4 - 8\sin^2 x} \, dx$

$I = \int \frac{2\sin 2x}{4(1-2\sin^2 x)} \, dx$

$I = \int \frac{2\sin 2x}{4\cos 2x} \, dx$

$= -\frac{1}{4}\ln|\cos 2x| + c$

f $I = \int \frac{1}{3-4x} \, dx$

$= -\frac{1}{4}\ln|3-4x| + c$

2 a $I = \int_{-3}^{0} x(x^2+3)^5 \, dx$

Consider $y = (x^2+3)^6$

$\frac{dy}{dx} = 12x(x^2+3)^5$

$I = \left[\frac{1}{12}(x^2+3)^6\right]_{-3}^{0}$

$= \frac{1}{12}(729 - 2985984)$

$= -\frac{995085}{4}$

b $I = \int_0^{\frac{\pi}{4}} x\sec^2 x \, dx$

Let $u = x \Rightarrow \frac{du}{dx} = 1$

$\frac{dv}{dx} = \sec^2 x \Rightarrow v = \tan x$

$I = [x\tan x]_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} \tan x \, dx$

$= \frac{\pi}{4} + [\ln|\cos x|]_0^{\frac{\pi}{4}} = \frac{\pi}{4} + \ln\frac{1}{\sqrt{2}}$

$= \frac{\pi}{4} - \frac{1}{2}\ln 2$

c $I = \int_1^4 \left(16x^{\frac{3}{2}} - \frac{2}{x}\right) dx$

$= \left[\frac{32}{5}x^{\frac{5}{2}} - 2\ln|x|\right]_1^4$

$= \frac{1024}{5} - 2\ln 4 - \frac{32}{5}$

$= \frac{992}{5} - 2\ln 4$

2 d $I=\int_{\frac{\pi}{12}}^{\frac{\pi}{3}}(\cos x+\sin x)(\cos x-\sin x)\ \mathrm{d}x$

$=\int_{\frac{\pi}{12}}^{\frac{\pi}{3}}(\cos^2 x-\sin^2 x)\ \mathrm{d}x$

$=\int_{\frac{\pi}{12}}^{\frac{\pi}{3}}\cos 2x\ \mathrm{d}x$

$=\left[\frac{1}{2}\sin 2x\right]_{\frac{\pi}{12}}^{\frac{\pi}{3}}$

$=\frac{\sqrt{3}}{4}-\frac{1}{4}$

$=\frac{\sqrt{3}-1}{4}$

e $I=\int_1^4\frac{4}{16x^2+8x-3}\ \mathrm{d}x$

$\frac{4}{16x^2+8x-3}=\frac{4}{(4x+3)(4x-1)}$

$\frac{4}{(4x+3)(4x-1)}=\frac{A}{4x+3}+\frac{B}{4x-1}$

$4=A(4x-1)+B(4x+3)$

$x=\frac{1}{4}\Rightarrow 4=4B\Rightarrow B=1$

$x=-\frac{3}{4}\Rightarrow 4=-4A\Rightarrow A=-1$

$I=\int_1^4\frac{1}{4x-1}-\frac{1}{4x+3}\ \mathrm{d}x$

$=\frac{1}{4}\left[\ln|4x-1|-\ln|4x+3|\right]_1^4$

$=\frac{1}{4}(\ln 15-\ln 19-\ln 3+\ln 7)$

$=\frac{1}{4}\ln\frac{105}{57}$

$=\frac{1}{4}\ln\frac{35}{19}$

f $I=\int_0^{\ln 2}\frac{1}{1+\mathrm{e}^x}\ \mathrm{d}x$

Let $u=1+\mathrm{e}^x\Rightarrow\frac{\mathrm{d}u}{\mathrm{d}x}=\mathrm{e}^x=u-1$

$I=\int_2^3\frac{1}{(u-1)u}\ \mathrm{d}u$

$I=\int_2^3\left(\frac{1}{(u-1)}-\frac{1}{u}\right)\ \mathrm{d}u$

$=\left[\ln|u-1|-\ln|u|\right]_2^3$

$=\ln 2-\ln 3-\ln 1+\ln 2$

$=\ln\frac{4}{3}$

3 a $I=\int_1^{\mathrm{e}}\frac{1}{x^2}\ln x\,\mathrm{d}x$

$u=\ln x\Rightarrow\frac{\mathrm{d}u}{\mathrm{d}x}=\frac{1}{x}$

$\frac{\mathrm{d}v}{\mathrm{d}x}=\frac{1}{x^2}\Rightarrow v=-\frac{1}{x}$

$\therefore I=\left[-\frac{1}{x}\ln x\right]_1^{\mathrm{e}}-\int_1^{\mathrm{e}}\left(-\frac{1}{x^2}\right)\mathrm{d}x$

$=\left(-\frac{1}{\mathrm{e}}\right)-(0)+\left[-\frac{1}{x}\right]_1^{\mathrm{e}}$

$=-\frac{1}{\mathrm{e}}+\left(-\frac{1}{\mathrm{e}}\right)-(-1)$

$=1-\frac{2}{\mathrm{e}}$

3 b $\dfrac{1}{(x+1)(2x-1)} \equiv \dfrac{A}{x+1} + \dfrac{B}{2x-1}$

$\Rightarrow 1 \equiv A(2x-1) + B(x+1)$

$x - \dfrac{1}{2} \Rightarrow 1 = \dfrac{3}{2}B \Rightarrow B = \dfrac{2}{3}$

$x = -1 \Rightarrow 1 = -3A \Rightarrow A = -\dfrac{1}{3}$

$\therefore \int_1^p \dfrac{1}{(x+1)(2x-1)}\,dx = \int_1^p \left(\dfrac{\frac{2}{3}}{2x-1} + \dfrac{-\frac{1}{3}}{x+1} \right) dx$

$= \left[\dfrac{1}{3}\mathrm{1n}\left|2x-1\right| - \dfrac{1}{3}\mathrm{1n}\left|x+1\right| \right]_1^p$

$= \left[\dfrac{1}{3}\mathrm{1n}\left|\dfrac{2x-1}{x+1}\right| \right]_1^p$

$= \dfrac{1}{3}\mathrm{1n}\left(\dfrac{2p-1}{p+1}\right) - \left(\dfrac{1}{3}\mathrm{1n}\,\dfrac{1}{2}\right)$

$= \dfrac{1}{3}\mathrm{1n}\left(\dfrac{4p-2}{p+1}\right)$

4 $\int_{\frac{1}{2}}^{b} \left(\dfrac{2}{x^3} - \dfrac{1}{x^2} \right) dx = \dfrac{9}{4}$

$\left[-\dfrac{1}{x^2} + \dfrac{1}{x} \right]_{\frac{1}{2}}^{b} = \dfrac{9}{4}$

$-\dfrac{1}{b^2} + \dfrac{1}{b} + 4 - 2 = \dfrac{9}{4}$

$\dfrac{b-1}{b^2} = \dfrac{1}{4}$

$b^2 - 4b + 4 = 0$

$(b-2)^2 = 0$

$b = 2$

5 $I = \int_0^{\theta} \cos x \sin^3 x\,dx = \dfrac{9}{64}$

$\left[\dfrac{\sin^4 x}{4} \right]_0^{\theta} = \dfrac{9}{64}$

$\sin^4 \theta = \dfrac{9}{16}$

$\sin \theta = \pm \dfrac{\sqrt{3}}{2}$

$\theta = \dfrac{\pi}{3}$

6 a $t^2 = x + 1 \Rightarrow 2t\,dt = dx$

$\therefore I = \int \dfrac{x}{\sqrt{x+1}}\,dx$

$= \int \dfrac{t^2-1}{t} \times 2t\,dt$

$= \int (2t^2 - 2)\,dt$

$= \dfrac{2}{3}t^3 - 2t + c$

$= \dfrac{2}{3}(x+1)^{\frac{3}{2}} - 2\sqrt{x+1} + c$

$= \dfrac{2}{3}\sqrt{x+1}(x-2) + c$

b $\int_0^3 \dfrac{x}{\sqrt{x+1}}\,dx = \left[\dfrac{2}{3}(x-2)\sqrt{x+1} \right]_0^3$

$= \left(\dfrac{2}{3} \times 2 \right) - \left(-\dfrac{4}{3} \right) = \dfrac{8}{3}$

7 a $I = \int x \sin 8x\,dx$

Let $u = x \Rightarrow \dfrac{du}{dx} = 1$

$\dfrac{dv}{dx} = \sin 8x \Rightarrow v = -\dfrac{1}{8}\cos 8x$

$I = -\dfrac{1}{8}x\cos 8x + \dfrac{1}{8}\int \cos 8x\,dx$

$= -\dfrac{1}{8}x\cos 8x + \dfrac{1}{64}\sin 8x + c$

b $I = \int x^2 \cos 8x\,dx$

Let $u = x^2 \Rightarrow \dfrac{du}{dx} = 2x$

$\dfrac{dv}{dx} = \cos 8x \Rightarrow v = \dfrac{1}{8}\sin 8x$

$I = \dfrac{1}{8}x^2\sin 8x - \dfrac{2}{8}\int x\sin 8x\,dx$

$= \dfrac{1}{8}x^2\sin 8x - \dfrac{2}{8}\left(-\dfrac{1}{8}x\cos 8x + \dfrac{1}{64}\sin 8x \right) + c$

$= \dfrac{1}{8}x^2\sin 8x + \dfrac{1}{32}x\cos 8x - \dfrac{1}{256}\sin 8x + c$

8 a $f(x) \equiv \frac{5x^2 - 8x + 1}{2x(x-1)^2} \equiv \frac{A}{x} + \frac{B}{x-1} + \frac{C}{(x-1)^2}$

$\Rightarrow 5x^2 - 8x + 1 \equiv 2A(x-1)^2$
$+ 2Bx(x-1) + 2Cx$

$x = 0 \Rightarrow 1 = 2A \Rightarrow A = \frac{1}{2}$

$x = 1 \Rightarrow -2 = 2C \Rightarrow C = -1$

Coefficients of x^2: $5 = 2A + 2B \Rightarrow B = 2$

b $\int f(x)\mathrm{d}x = \int \left(\frac{\frac{1}{2}}{x} + \frac{2}{x-1} - \frac{1}{(x-1)^2} \right) \mathrm{d}x$

$= \frac{1}{2}\ln|x| + 2\ln|x-1| + \frac{1}{x-1} + c$

c $\int_4^9 f(x)\mathrm{d}x = \left[\frac{1}{2}\ln|x| + 2\ln|x-1| + \frac{1}{x-1} \right]_4^9$

$= \left[1\text{n}\left|\sqrt{x(x-1)^2}\right| + \frac{1}{x-1} \right]_4^9$

$= \left[1\text{n}(3\times 64) + \frac{1}{8} \right] - \left[1\text{n}(2\times 9) + \frac{1}{3} \right]$

$= 1\text{n}\left(\frac{3\times 64}{2\times 9} \right) + \frac{1}{8} - \frac{1}{3}$

$= 1\text{n}\frac{32}{3} - \frac{5}{24}$

9 a $y = x^{\frac{3}{2}} + 48x^{-1} \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{3}{2}x^{\frac{1}{2}} - 48x^{-2}$

$\frac{\mathrm{d}y}{\mathrm{d}x} = 0 \Rightarrow \frac{3}{2}x^{\frac{1}{2}} = \frac{48}{x^2}$

$\Rightarrow x^{\frac{5}{2}} = \frac{2}{3} \times 48 = 32$

$\Rightarrow x = 4,\ y = 2^3 + 12 = 20$

$\Rightarrow x = 4, y = 20$

b $\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \frac{3}{4}x^{-\frac{1}{2}} + 96x^{-3} > \text{for all } x > 0$

$\therefore$ 20 is a minimum value of y

c Area $= \int_1^4 \left(x^{\frac{3}{2}} + \frac{48}{x} \right) \mathrm{d}x$

$= \left[\frac{2}{5}x^{\frac{5}{2}} + 48\ln|x| \right]_1^4$

$= \left(\frac{2}{5} \times 32 + 48\ln 4 \right) - \left(\frac{2}{5} + 0 \right)$

$= \frac{62}{5} + 48\ln 4$

10 a $I = \int x^2 \ln 2x \,\mathrm{d}x$

Let $u = \ln 2x \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1}{x}$

$\frac{\mathrm{d}v}{\mathrm{d}x} = x^2 \Rightarrow v = \frac{x^3}{3}$

$I = \frac{1}{3}x^3 \ln 2x - \int \frac{x^2}{3}\,\mathrm{d}x$

$= \frac{1}{3}x^3 \ln 2x - \frac{1}{9}x^3 + c$

b $\int_{\frac{1}{2}}^{3} x^2 \ln 2x \,\mathrm{d}x = \left[\frac{1}{3}x^3 \ln 2x - \frac{1}{9}x^3 \right]_{\frac{1}{2}}^{3}$

$= 9\ln 6 - 3 - 0 + \frac{1}{72}$

$= 9\ln 6 - \frac{215}{9}$

11 a $y = (1 + \sin 2x)^2$

$= 1 + 2\sin 2x + \sin^2 2x$

$= 1 + 2\sin 2x + \frac{1}{2} - \frac{1}{2}\cos 4x$

$= \frac{1}{2}(3 + 4\sin 2x - \cos 4x)$

11 b Area of $R = \int_0^{\frac{3\pi}{4}} (1+\sin 2x)^2 \; dx$

$= \frac{1}{2}\int_0^{\frac{3\pi}{4}} (3+4\sin 2x - \cos 4x) \; dx$

$= \frac{1}{2}\left[3x - 2\cos 2x - \frac{1}{4}\sin 4x\right]_0^{\frac{3\pi}{4}}$

$= \left(\frac{9\pi}{8} - 0 - 0\right) - (0-1-0)$

$= \frac{9\pi}{8} + 1$

c $\frac{dy}{dx} = 4\cos 2x + 2\sin 4x$

$\frac{dy}{dx} = 0 \Rightarrow 4\cos 2x + 2\sin 4x = 0$

$4\cos 2x + 4\sin 2x\cos 2x = 0$

$4\cos 2x(1+\sin 2x) = 0$

$2x = \frac{\pi}{2}$ for $x < \frac{3\pi}{4}$

$x = \frac{\pi}{4}, y = \left(1+\sin\frac{\pi}{2}\right)^2 = 4$

Coordinates of $A\left(\frac{\pi}{4}, 4\right)$

12 a $I = \int x e^{-x} dx$

$u = x \Rightarrow \frac{du}{dx} = 1$

$\frac{dv}{dx} = e^{-x} \Rightarrow v = -e^{-x}$

$\therefore I = -xe^{-x} - \int (-e^{-x}) dx$

i.e. $I = -xe^{-x} - e^{-x} + c$

b $e^x \frac{dy}{dx} = \frac{x}{\sin 2y}$

$\Rightarrow \int \sin 2y \, dy = \int xe^{-x} dx$

$\Rightarrow -\frac{1}{2}\cos 2y = -xe^{-x} - e^{-x} + c$

$x = 0, y = \frac{\pi}{4} \Rightarrow 0 = 0 - 1 + c \Rightarrow c = 1$

$\therefore \frac{1}{2}\cos 2y = xe^{-x} + e^{-x} - 1$

or $\cos 2y = 2(xe^{-x} + e^{-x} - 1)$

13 a $I = \int x\sin 2x \; dx$

$u = x \Rightarrow \frac{du}{dx} = 1$

$\frac{dv}{dx} = \sin 2x \Rightarrow v = \frac{-1}{2}\cos 2x$

$\therefore I = -\frac{1}{2}x\cos 2x - \int \frac{-1}{2}\cos 2x \; dx$

$= -\frac{1}{2}x\cos 2x + \frac{1}{4}\sin 2x + c$

b $\frac{dy}{dx} = x\sin 2x\cos^2 y$

$\Rightarrow \int \sec^2 y \, dy = \int x\sin 2x \, dx$

$\Rightarrow \tan y = -\frac{1}{2}x\cos 2x + \frac{1}{4}\sin 2x + c$

$y = 0,\; x = \frac{\pi}{4} \Rightarrow 0 = 0 + \frac{1}{4} + c \Rightarrow c = -\frac{1}{4}$

$\therefore \tan y = -\frac{1}{2}x\cos 2x + \frac{1}{4}\sin 2x - \frac{1}{4}$

14 a $\frac{dy}{dx} = xy^2$

$\Rightarrow \int \frac{1}{y^2} dy = \int x \, dx$

$\Rightarrow -\frac{1}{y} = \frac{x^2}{2} + c$

or $y = \frac{-2}{x^2 + k}$ $\quad (k = 2c)$

b $y = 1, x = 1 \Rightarrow 1 = \frac{-2}{1+k} \Rightarrow k = -3$

$\therefore y = \frac{2}{3 - x^2}$

for $x^2 \neq 3$ and $y > 0$, i.e. $-\sqrt{3} < x < \sqrt{3}$

c When $x = 1, y = 1$, $\frac{dy}{dx}$ is 1

14 d Equation of tangent is:

$$y-1=1(x-1)$$
$$y=x$$

This meets the curve again when:

$$x=\frac{2}{3-x^2}$$
$$3x-x^3=2$$
$$x^3-3x+2=0$$
$$(x-1)(x-1)(x+2)=0$$

Other point is when $x=-2, y=-2$
i.e. (−2, −2)

15 a $I=\int\frac{4x}{(1+2x)^2}\,dx$

$$u=1+2x$$
$$\Rightarrow \frac{du}{2}=dx \text{ and } 4x=2(u-1)$$
$$\therefore I=\int\frac{2(u-1)}{u^2}\times\frac{du}{2}$$
$$=\int\left(\frac{1}{u}-u^{-2}\right)du$$
$$=\ln|u|+\frac{1}{u}+c$$
$$=\ln|1+2x|+\frac{1}{1+2x}+c$$

b $(1+2x)^2\frac{dy}{dx}=\frac{x}{\sin^2 y}$

$$\Rightarrow \int \sin^2 y\,dy=\int\frac{x}{(1+2x)}\,dx$$
$$\Rightarrow \int 4\sin^2 y\,dy=\int\frac{4x}{(1+2x)^2}\,dx$$
$$\Rightarrow \int (2-2\cos 2y)\,dy=I$$
$$\Rightarrow 2y-\sin 2y=\ln|1+2x|+\frac{1}{1+2x}+c$$
$$x=0, y=\frac{\pi}{4}\Rightarrow \frac{\pi}{2}-1=\ln 1+1+c$$
$$\Rightarrow c=\frac{\pi}{2}-2$$
$$\therefore 2y-\sin 2y=\ln|1+2x|+\frac{1}{1+2x}+\frac{\pi}{2}-2$$

16 a $\int xe^{2x}dx$

$$u=x\Rightarrow\frac{du}{dx}=1$$
$$\frac{dv}{dx}=e^{2x}\Rightarrow v=\frac{1}{2}e^{2x}$$
$$\therefore \int xe^{2x}dx=\frac{1}{2}xe^{2x}-\int\frac{1}{2}e^{2x}dx=\frac{1}{2}xe^{2x}-\frac{1}{4}e^{2x}+c$$

$$A_1=-\left[\frac{1}{2}xe^{2x}-\frac{1}{4}e^{2x}\right]_{-\frac{1}{2}}^{0}$$
$$=-\left(\left(0-\frac{1}{4}\right)-\left(-\frac{1}{4}e^{-1}-\frac{1}{4}e^{-1}\right)\right)$$
$$=\frac{1}{4}(1-2e^{-1})$$

$$A_2=\left[\frac{1}{2}xe^{2x}-\frac{1}{4}e^{2x}\right]_{0}^{\frac{1}{2}}$$
$$=\left(\frac{1}{4}e^{1}-\frac{1}{4}e^{1}\right)-\left(0-\frac{1}{4}\right)$$
$$=\frac{1}{4}$$

16 b $\frac{A_1}{A_2} = \frac{\frac{1}{4}\left(1-2e^{-1}\right)}{\frac{1}{4}} = 1-2e^{-1} = \frac{e-2}{e}$

$\therefore A_1 : A_2 = (e-2) : e$

17 a $I = \int x^2 e^{-x}\ dx$

Let $u = x^2 \Rightarrow \frac{du}{dx} = 2x$

$\frac{dv}{dx} = e^{-x} \Rightarrow v = -e^{-x}$

$I = -x^2 e^{-x} + 2\int xe^{-x}\ dx$

Again, let $u = x \Rightarrow \frac{du}{dx} = 1$

$\frac{dv}{dx} = e^{-x} \Rightarrow v = -e^{-x}$

$I = -x^2 e^{-x} - 2xe^{-x} + 2\int e^{-x}\ dx$

$= -x^2 e^{-x} - 2xe^{-x} - 2e^{-x} + c$

$= -e^{-x}\left(x^2 + 2x + 2\right) + c$

b $\frac{dy}{dx} = x^2 e^{3y-x}$

$\frac{dy}{dx} = x^2 e^{3y} e^{-x}$

$\int e^{-3y}\ dx = \int x^2 e^{-x}\ dx$

$-\frac{1}{3}e^{-3y} = -e^{-x}\left(x^2 + 2x + 2\right) + c$

$x = 0, y = 0 \Rightarrow -\frac{1}{3} = -2 + c \Rightarrow c = \frac{5}{3}$

$e^{-3y} = 3e^{-x}\left(x^2 + 2x + 2\right) - 5$

$3y = -\ln\left(3e^{-x}\left(x^2 + 2x + 2\right) - 5\right)$

$y = -\frac{1}{3}\ln\left(3e^{-x}\left(x^2 + 2x + 2\right) - 5\right)$

18 a $y = e^{3x} + 1$

$y = 8 \Rightarrow e^{3h} = 7$

$3h = \ln 7$

$h = \frac{1}{3}\ln 7$

b Area of R is given by

$\int_0^{\frac{1}{3}\ln 7}\left(e^{3x} + 1\right) dx = \left[\frac{1}{3}e^{3x} + x\right]_0^{\frac{1}{3}\ln 7}$

$= \left(\frac{1}{3}e^{\ln 7} + \frac{1}{3}\ln 7\right) - \left(\frac{1}{3} + 0\right)$

$= 2 + \frac{1}{3}\ln 7$

19 a $\frac{x^2}{x^2-1} \equiv A + \frac{B}{x-1} + \frac{C}{x+1}$

$\Rightarrow x^2 \equiv A(x-1)(x+1) + B(x+1) + C(x-1)$

$x = 1 \Rightarrow 1 = 2B \Rightarrow B = \frac{1}{2}$

$x = -1 \Rightarrow 1 = -2C \Rightarrow C = -\frac{1}{2}$.

Coefficients of x^2: $1 = A \Rightarrow A = 1$

b $\frac{dx}{dt} = 2\frac{(x^2-1)}{x^2}$

$\Rightarrow \int \frac{x^2}{x^2-1} dx = \int 2\, dt$

$\Rightarrow \int\left(1 + \frac{\left(\frac{1}{2}\right)}{x-1} - \frac{\left(\frac{1}{2}\right)}{x+1}\right) dx = 2t$

$\Rightarrow x + \frac{1}{2}\ln\left|\frac{x-1}{x+1}\right| = 2t + c$

$x = 2, t = 1 \Rightarrow 2 + \frac{1}{2}\ln\frac{1}{3} = 2 + c \Rightarrow c = \frac{1}{2}\ln\frac{1}{3}$

$\therefore x + \frac{1}{2}\ln\left|\frac{x-1}{x+1}\right| = 2t + \frac{1}{2}\ln\frac{1}{3}$

20 a $y = e^{2x} - e^{-x}$

x	0	0.25	0.5	0.75	1
y	0	0.86992	2.11175	4.00932	7.02118

b Area $\approx \frac{1}{2}h\left(y_0 + 2(y_1 + \ldots) + y_n\right)$

$= \frac{1}{8}(7.02118 + 2 \times 6.99099)$

$= 2.6254$

20 c The curve is convex, so it is an overestimate.

d $\int_0^1 \left(e^{2x} - e^{-x}\right)\, dx = \left[\frac{1}{2}e^{2x} + e^{-x}\right]_0^1$

$= \frac{1}{2}e^2 + e^{-1} - \frac{1}{2} - 1$

$= \frac{1}{2}e^2 + \frac{1}{e} - \frac{3}{2}$

$= \frac{e^3 - 3e + 2}{2e}$

$P = -3,\ Q = 2$

e $\frac{e^3 - 3e + 2}{2e} = 2.5624$

Percentage error

$= \frac{2.5624 - 2.6254}{2.5624} \times 100\% \approx 2.5\%$

21 a $\frac{dv}{dt} = -kV$

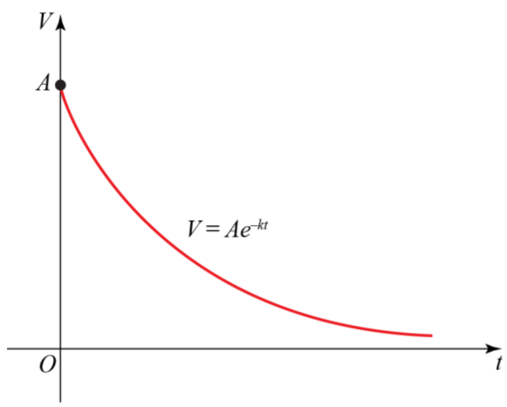
$\Rightarrow \int \frac{1}{V}\, dV = \int -k\, dt$

$\Rightarrow \ln|V| = -kt + C$

$\Rightarrow V = A_1 e^{-kt}$

$t = 0, V = A \Rightarrow V = Ae^{-kt} \qquad (A_1 = A)$

b



c $t = T, V = \frac{1}{2}A \Rightarrow \frac{1}{2}A = Ae^{-kT}$

$\Rightarrow -\ln 2 = -kT$

$\Rightarrow kT = \ln 2$

22 a $\frac{dy}{dx} = \frac{x}{k - y}$

$\int (k - y)\, dy = \int x\, dx$

$-\frac{(k-y)^2}{2} + c = \frac{x^2}{2}$

$x^2 + (y - k)^2 = c$

b Concentric circles with centre (0, 2).

23 a

x	1	1.5	2	2.5
y	1	0.6825	0.5545	0.6454

x	3	3.5	4
y	0.9775	1.5693	2.4361

b Area $= \frac{1}{4}(3.4361 + 2 \times 4.4262)$

$= 3.074$

c If smaller intervals are used and consequently more values, the lines would follow the curve more closely.

23 d Area $= \int_1^4 \frac{1}{5}x^2 \ln x - x + 2$

Now let $I = \int x^2 \ln x \, dx$

Let $u = \ln x \Rightarrow \frac{du}{dx} = \frac{1}{x}$

$\frac{dv}{dx} = x^2 \Rightarrow v = \frac{x^3}{3}$

$I = \frac{1}{3}x^3 \ln x - \int \frac{x^2}{3} \, dx$

$= \frac{1}{3}x^3 \ln x - \frac{1}{9}x^3 + c$

Area $= \left[\frac{1}{15}x^3 \ln x - \frac{1}{45}x^3 - \frac{x^2}{2} + 2x\right]_1^4$

$= \frac{64}{15}\ln 4 - \frac{64}{15} - 8 + 8 - \left(0 - \frac{1}{45} - \frac{1}{2} + 2\right)$

$= -\frac{29}{10} + \frac{64}{15}\ln 4$

e $-\frac{29}{10} + \frac{64}{15}\ln 4 = 3.015$

Percentage error

$= \frac{3.015 - 3.074}{3.015} \times 100\% \approx 2.0\%$

24 a $u = 1 + 2x^2 \Rightarrow du = 4x\,dx \Rightarrow x\,dx = \frac{du}{4}$

So $\int x(1+2x^2)^5 dx = \int \frac{u^5}{4} du = \frac{u^6}{24}$

$+ c_1 = \frac{(1+2x^2)^6}{24} + c_1$

b $\frac{dy}{dx} = x(1+2x^2)^5 \cos^2 2y$

$\Rightarrow \int \sec^2 2y \, dy = \int x(1+2x^2)^5 dx$

$\Rightarrow \frac{1}{2}\tan 2y = \frac{(1+2x^2)^6}{24} + c_2$

$y = \frac{\pi}{8}, x = 0 \Rightarrow \frac{1}{2} = \frac{1}{24} + c_2 \Rightarrow c_2 = \frac{11}{24}$

$\therefore \tan 2y = \frac{(1+2x^2)^6}{12} + \frac{11}{12}$

25 $I = \int \frac{1}{1+x^2} \, dx$

Let $x = \tan u \Rightarrow \frac{dx}{du} = \sec^2 u$

$I = \int \frac{1}{1+\tan^2 u}\sec^2 u \, du$

But $1 + \tan^2 u = \sec^2 u$

So $I = \int du = u + c$

$= \arctan x + c$

26 $x(x+2)\frac{dy}{dx} = y$

$\Rightarrow \int \frac{1}{y} dy = \int \frac{1}{x(x+2)} dx$

$\frac{1}{x(x+2)} \equiv \frac{A}{x} + \frac{B}{x+2}$

$\Rightarrow 1 \equiv A(x+2) + Bx$

$x = 0 \Rightarrow 1 = 2A \Rightarrow A = \frac{1}{2}$

$x = -2 \Rightarrow 1 = -2B \Rightarrow B = -\frac{1}{2}$

So $\ln y = \int \left(\frac{\left(\frac{1}{2}\right)}{x} - \frac{\left(\frac{1}{2}\right)}{x+2}\right) dx$

$= \frac{1}{2}\ln|x| - \frac{1}{2}\ln|x+2| + c$

$\therefore y = \sqrt{\frac{kx}{x+2}} \quad \left(c = \frac{1}{2}\ln k\right)$

$x = 2, y = 2 \Rightarrow 2 = \sqrt{\frac{2k}{4}} \Rightarrow 4 \times 2 = k$

$\therefore y = \sqrt{\frac{8x}{x+2}}$ or $y^2 = \frac{8x}{x+2}$

27 a $A = \pi r^2 \Rightarrow \frac{dA}{dr} = 2\pi r$

$\frac{dr}{dt} = \frac{dr}{dA} \times \frac{dA}{dt} = \frac{1}{2\pi r} \times k \sin\left(\frac{t}{3\pi}\right)$

$\frac{dr}{dt} = \frac{k}{2\pi r}\sin\left(\frac{t}{3\pi}\right)$

27 b $\int 2\pi r \,\mathrm{d}r = \int k\sin\left(\frac{t}{3\pi}\right)\mathrm{d}t$

$\pi r^2 = -3\pi k\cos\left(\frac{t}{3\pi}\right) + c$

$r^2 = -3k\cos\left(\frac{t}{3\pi}\right) + c$

$r = 1, t = 0 \Rightarrow 1 = -3k + c \Rightarrow c = 3k + 1$

$r = 2, t = \pi^2 \Rightarrow 4 = -\frac{3k}{2} + 3k + 1$

So $r^2 = -6\cos\left(\frac{t}{3\pi}\right) + 6 + 1$

$r^2 = -6\cos\left(\frac{t}{3\pi}\right) + 7$

c $r = 1.5 \Rightarrow 2.25 = -6\cos\left(\frac{t}{3\pi}\right) + 7$

$6\cos\left(\frac{t}{3\pi}\right) = 4.75$

$\cos\left(\frac{t}{3\pi}\right) \approx 0.7917$

$\frac{t}{3\pi} = 0.6527$

$t = 6.19$ days

6 days, 5 hours

28 a $\frac{\pi}{2}$

b Area $= \int_0^{\frac{\pi}{2}} \sin 2t(6t)\,\mathrm{d}t$

$= 6\int_0^{\frac{\pi}{2}} t\sin 2t \,\mathrm{d}t$

Integrating by parts

$u = t \Rightarrow u' = 1$

$v' = \sin 2t \Rightarrow v = -\frac{\cos 2t}{2}$

$\int t\sin 2t = -\frac{t\cos 2t}{2} - \int -\frac{\cos 2t}{2}\,\mathrm{d}t$

$= -\frac{t\cos 2t}{2} + \frac{\sin 2t}{4}$

$6\int_0^{\frac{\pi}{2}} t\sin 2t = 6\left[\frac{\sin 2t}{4} - \frac{t\cos 2t}{2}\right]_0^{\frac{\pi}{2}}$

$= 6\left(\left(\frac{\pi}{4}\right) - (0)\right)$

$= \frac{3\pi}{2}$

29 a

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$$

$$\frac{dy}{d\theta} = 4\cos\theta, \quad \frac{dx}{d\theta} = -5\sin\theta$$

$$\frac{dy}{dx} = \frac{4\cos\theta}{-5\sin\theta} = -\frac{4}{5}\cot\theta$$

$$\theta = \frac{\pi}{4} \Rightarrow \frac{dy}{dx} = -\frac{4}{5}$$

b Point P is at $\theta = \frac{\pi}{4}$

$$x = 5\cos\frac{\pi}{4} = \frac{5}{\sqrt{2}}$$

$$y = 4\sin\frac{\pi}{4} = 2\sqrt{2}$$

$$\left(\frac{5}{\sqrt{2}}, 2\sqrt{2}\right)$$

Gradient of tangent = derivative at $P = -\frac{4}{5}$

Equation at P

$$y - 2\sqrt{2} = -\frac{4}{5}\left(x - \frac{5}{\sqrt{2}}\right)$$

$$y = -\frac{4}{5}x + 4\sqrt{2}$$

c P crosses x-axis at

$$0 = -\frac{4}{5}x + 4\sqrt{2}$$

$$x = 5\sqrt{2}$$

$$\text{Area} = \int_{\frac{5\sqrt{2}}{2}}^{5\sqrt{2}} -\frac{4}{5}x + 4\sqrt{2}\ dx - \int_{\frac{\pi}{4}}^{0} 4\sin\theta(-5\sin\theta)\,d\theta$$

$$= \left[-\frac{4}{10}x^2 + 4x\sqrt{2}\right]_{\frac{5\sqrt{2}}{2}}^{5\sqrt{2}} - 20\int_0^{\frac{\pi}{4}} \sin^2\theta\ d\theta$$

$$= ((20) - 15) - 20\int_0^{\frac{\pi}{4}} \sin^2\theta\ d\theta$$

$$= 5 - 20\int_0^{\frac{\pi}{4}} \sin^2\theta\ d\theta$$

Using trig identities

$$\int \sin^2\theta\ d\theta = \int \frac{1 - \cos 2\theta}{2} d\theta$$

$$= \int \frac{1}{2} d\theta - \frac{1}{2}\int \cos 2\theta\ d\theta$$

c Use the substitution $u = 2\theta$ to yield

$$= \frac{\theta}{2} - \frac{\sin 2\theta}{4}$$

$$\Rightarrow -20\int_0^{\frac{\pi}{4}} \sin^2\theta\ d\theta = [5\sin 2\theta - 10\theta]_0^{\frac{\pi}{4}}$$

$$= \frac{5(\pi - 2)}{2}$$

$$\text{Area } = 5 - \frac{5(\pi - 2)}{2} = 10 - \frac{5\pi}{2}$$

30 Point P is at $t = 1$

$y = 1$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

$$\frac{dy}{dt} = 2 - 3t^2$$

$$\frac{dx}{dt} = -2t$$

$$\frac{dy}{dx} = \frac{2 - 3t^2}{-2t} = \frac{1}{2} \quad (t = 1)$$

Gradient of normal is negative reciprocal of derivative $= -2$

Equation of P is therefore

$$y - 1 = -2(x)$$

$$y = -2x + 1$$

and crosses x-axis at $x = 0.5$

$$\text{Area} = \int_1^0 -2t(2t - t^3)\,dt - \int_0^{0.5} -2x + 1\ dx$$

$$= \int_1^0 -4t^2 + 2t^4 dt - \int_0^{0.5} -2x + 1\ dx$$

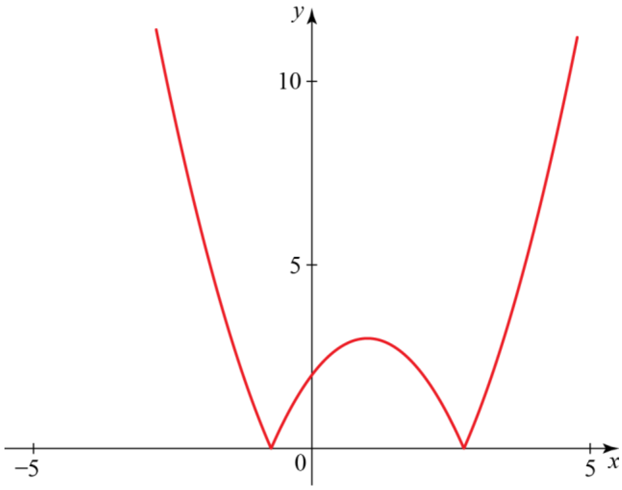
$$= \left[-\frac{4}{3}t^3 + \frac{2}{5}t^5\right]_1^0 - \left[-x^2 + x\right]_0^{0.5}$$

$$= \left[(0) - \left(-\frac{14}{15}\right)\right] - \left[\left(\frac{1}{4}\right) - 0\right]$$

$$= \frac{41}{60}$$

Challenge

a



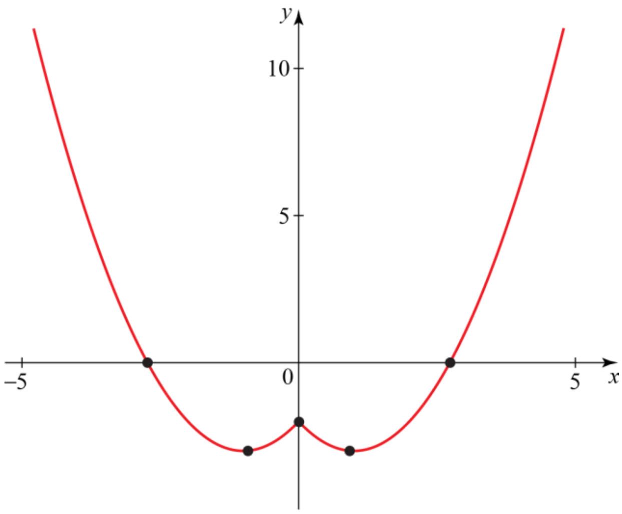
$$\int_{-3}^{3}|f(x)|\ dx = \int_{-3}^{3} f(x)\ dx + 2\times\left|\int_{-1}^{2} f(x)\ dx\right|$$

$$\int_{-3}^{3} f(x)\ dx = \left[\frac{x^3}{3}-\frac{x^2}{2}-2x\right]_{-3}^{3} = 6$$

$$\int_{-1}^{2} f(x)\ dx = \left[\frac{x^3}{3}-\frac{x^2}{2}-2x\right]_{-1}^{2} = \frac{9}{2}$$

$$\int_{-3}^{3}|f(x)|\ dx = 6+2\times\frac{9}{2} = 15$$

b



$$\int_{-3}^{3} f(|x|)\ dx = 2\times\int_{0}^{3} f(x)\ dx$$

$$\int_{0}^{3} f(x)\ dx = \left[\frac{x^3}{3}-\frac{x^2}{2}-2x\right]_{0}^{3} = -\frac{3}{2}$$

$$\int_{-3}^{3} f(|x|)\ dx = 2\times\left(-\frac{3}{2}\right) = -3$$