Vectors 12B

1 a $\mathbf{a} = \begin{pmatrix} 1 \\ 2 \\ -4 \end{pmatrix}$ $\mathbf{b} = \begin{pmatrix} 4 \\ -3 \\ 5 \end{pmatrix}$

i $\mathbf{a} - \mathbf{b} = \begin{pmatrix} -3 \\ 5 \\ -9 \end{pmatrix}$

ii $-\mathbf{a} + 3\mathbf{b} = \begin{pmatrix} -1+12 \\ -2-9 \\ 4+15 \end{pmatrix} = \begin{pmatrix} 11 \\ -11 \\ 19 \end{pmatrix}$

b $\mathbf{a} - \mathbf{b}$ is parallel since
$-2(\mathbf{a} - \mathbf{b}) = 6\mathbf{i} - 10\mathbf{j} + 18\mathbf{k}$.

$-\mathbf{a} + 3\mathbf{b}$ is not parallel as it is not a multiple of $6\mathbf{i} - 10\mathbf{j} + 18\mathbf{k}$.

2 $3\mathbf{a} + 2\mathbf{b} = 3\begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} + 2\begin{pmatrix} -3 \\ -2 \\ 4 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 5 \end{pmatrix}$

$3\mathbf{i} + 2\mathbf{j} + 5\mathbf{k} = \frac{1}{2}(6\mathbf{i} + 4\mathbf{j} + 10\mathbf{k})$

So the vectors are parallel.

3 $\mathbf{a} = \begin{pmatrix} 1 \\ 2 \\ -4 \end{pmatrix}$ $\mathbf{b} = \begin{pmatrix} p \\ q \\ r \end{pmatrix}$

$\mathbf{a} + 2\mathbf{b} = (1 + 2p)\mathbf{i} + (2 + 2q)\mathbf{j} + (-4 + 2r)\mathbf{k}$

$(1 + 2p)\mathbf{i} + (2 + 2q)\mathbf{j} + (-4 + 2r)\mathbf{k} = 5\mathbf{i} + 4\mathbf{j}$

$1 + 2p = 5 \Rightarrow p = 2$
$2 + 2q = 4 \Rightarrow q = 1$
$-4 + 2r = 0 \Rightarrow r = 2$

4 a $|3\mathbf{i}+5\mathbf{j}+\mathbf{k}| = \sqrt{3^2+5^2+1^2}$
$= \sqrt{9+25+1} = \sqrt{35}$

b $|4\mathbf{i}-2\mathbf{k}| = \sqrt{4^2+0^2+(-2)^2}$
$= \sqrt{16+4} = \sqrt{20} = \sqrt{4}\sqrt{5} = 2\sqrt{5}$

c $|\mathbf{i}+\mathbf{j}-\mathbf{k}| = \sqrt{1^2+1^2+(-1)^2}$
$= \sqrt{1+1+1} = \sqrt{3}$

d $|5\mathbf{i}-9\mathbf{j}-8\mathbf{k}| = \sqrt{5^2+(-9)^2+(-8)^2}$
$= \sqrt{25+81+64} = \sqrt{170}$

e $|\mathbf{i}+5\mathbf{j}-7\mathbf{k}| = \sqrt{1^2+5^2+(-7)^2}$
$= \sqrt{1+25+49} = \sqrt{75}$
$= \sqrt{25}\sqrt{3} = 5\sqrt{3}$

5 a $\mathbf{p}+\mathbf{q} = \begin{pmatrix} 5 \\ 0 \\ 2 \end{pmatrix} + \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix} = \begin{pmatrix} 7 \\ 1 \\ -1 \end{pmatrix}$

b $\mathbf{q}-\mathbf{r} = \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix} - \begin{pmatrix} 7 \\ -4 \\ 2 \end{pmatrix} = \begin{pmatrix} -5 \\ 5 \\ -5 \end{pmatrix}$

c $\mathbf{p}+\mathbf{q}+\mathbf{r} = \begin{pmatrix} 5 \\ 0 \\ 2 \end{pmatrix} + \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix} + \begin{pmatrix} 7 \\ -4 \\ 2 \end{pmatrix} = \begin{pmatrix} 14 \\ -3 \\ 1 \end{pmatrix}$

d $3\mathbf{p}-\mathbf{r} = 3\begin{pmatrix} 5 \\ 0 \\ 2 \end{pmatrix} - \begin{pmatrix} 7 \\ -4 \\ 2 \end{pmatrix} = \begin{pmatrix} 15 \\ 0 \\ 6 \end{pmatrix} - \begin{pmatrix} 7 \\ -4 \\ 2 \end{pmatrix} = \begin{pmatrix} 8 \\ 4 \\ 4 \end{pmatrix}$

e $\mathbf{p}-2\mathbf{q}+\mathbf{r} = \begin{pmatrix} 5 \\ 0 \\ 2 \end{pmatrix} - 2\begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix} + \begin{pmatrix} 7 \\ -4 \\ 2 \end{pmatrix}$
$= \begin{pmatrix} 5 \\ 0 \\ 2 \end{pmatrix} - \begin{pmatrix} 4 \\ 2 \\ -6 \end{pmatrix} + \begin{pmatrix} 7 \\ -4 \\ 2 \end{pmatrix} = \begin{pmatrix} 8 \\ -6 \\ 10 \end{pmatrix}$

6 $\overrightarrow{AB} = \mathbf{b}-\mathbf{a}$, so $\mathbf{b} = \overrightarrow{AB}+\mathbf{a}$

$\mathbf{b} = \begin{pmatrix} 5 \\ 4 \\ -1 \end{pmatrix} + \begin{pmatrix} 2 \\ -7 \\ 3 \end{pmatrix} = \begin{pmatrix} 7 \\ -3 \\ 2 \end{pmatrix}$

Position vector of B is $7\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$

7 $|\mathbf{a}| = \sqrt{t^2 + 2^2 + 3^2} = 7$

$\sqrt{t^2 + 4 + 9} = 7$

$t^2 + 4 + 9 = 49$

$t^2 = 36$

$t = 6$ or $t = -6$

8 $|\mathbf{a}| = \sqrt{(5t)^2 + (2t)^2 + t^2} = 3\sqrt{10}$

$\sqrt{25t^2 + 4t^2 + t^2} = 3\sqrt{10}$

$\sqrt{30t^2} = 3\sqrt{10}$

$30t^2 = 9 \times 10$

$t^2 = 3$

$t = \sqrt{3}$ or $t = -\sqrt{3}$

9 a i Position vector of A is $2\mathbf{i} + \mathbf{j} + 4\mathbf{k}$
Position vector of B is $3\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}$
Position vector of C is $-\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$

ii $\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA}$

$= (-\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}) - (2\mathbf{i} + \mathbf{j} + 4\mathbf{k})$

$= -3\mathbf{i} + \mathbf{j} - 2\mathbf{k}$

b i $|\overrightarrow{AC}| = \sqrt{3^2 + 1^2 + 2^2} = \sqrt{14}$

ii $|\overrightarrow{OC}| = \sqrt{1^2 + 2^2 + 2^2} = \sqrt{9} = 3$

10 a $\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP}$

$= -\mathbf{i} + 3\mathbf{j} - 5\mathbf{k} - (3\mathbf{i} + 7\mathbf{k})$

$= -4\mathbf{i} + 3\mathbf{j} - 12\mathbf{k}$

b Distance between P and Q is
$|\overrightarrow{PQ}| = \sqrt{4^2 + 3^2 + 12^2} = \sqrt{169} = 13$

c Unit vector in the direction of $\overrightarrow{PQ}$ is
$\frac{1}{13}(-4\mathbf{i} + 3\mathbf{j} - 12\mathbf{k}) = -\frac{4}{13}\mathbf{i} + \frac{3}{13}\mathbf{j} - \frac{12}{13}\mathbf{k}$

11 a $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

$= -2\mathbf{i} + 3\mathbf{j} + \mathbf{k} - (4\mathbf{i} - \mathbf{j} - 2\mathbf{k})$

$= -6\mathbf{i} + 4\mathbf{j} + 3\mathbf{k}$

b Distance between A and B is
$|\overrightarrow{AB}| = \sqrt{6^2 + 4^2 + 3^2} = \sqrt{61}$

c Unit vector in the direction of $\overrightarrow{AB}$ is
$\frac{1}{\sqrt{61}}(-6\mathbf{i} + 4\mathbf{j} + 3\mathbf{k})$

$= -\frac{6}{\sqrt{61}}\mathbf{i} + \frac{4}{\sqrt{61}}\mathbf{j} + \frac{3}{\sqrt{61}}\mathbf{k}$

12 a $|\mathbf{p}| = \sqrt{3^2 + 4^2 + 2^2} = \sqrt{29}$

$\hat{\mathbf{p}} = \frac{1}{|\mathbf{p}|}\mathbf{p} = \frac{1}{\sqrt{29}}(3\mathbf{i} - 4\mathbf{j} - 2\mathbf{k})$

$= \frac{3}{\sqrt{29}}\mathbf{i} - \frac{4}{\sqrt{29}}\mathbf{j} - \frac{2}{\sqrt{29}}\mathbf{k}$

b $|\mathbf{q}| = \sqrt{2 + 4^2 + 7} = \sqrt{25} = 5$

$\hat{\mathbf{q}} = \frac{1}{|\mathbf{q}|}\mathbf{q} = \frac{1}{5}(\sqrt{2}\mathbf{i} - 4\mathbf{j} - \sqrt{7}\mathbf{k})$

$= \frac{\sqrt{2}}{5}\mathbf{i} - \frac{4}{5}\mathbf{j} - \frac{\sqrt{7}}{5}\mathbf{k}$

c $|\mathbf{r}| = \sqrt{5 + 8 + 3} = \sqrt{16} = 4$

$\hat{\mathbf{r}} = \frac{1}{|\mathbf{r}|}\mathbf{r} = \frac{1}{4}(\sqrt{5}\mathbf{i} - 2\sqrt{2}\mathbf{j} - \sqrt{3}\mathbf{k})$

$= \frac{\sqrt{5}}{4}\mathbf{i} - \frac{2\sqrt{2}}{4}\mathbf{j} - \frac{\sqrt{3}}{4}\mathbf{k}$

13 a $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

$= 8\mathbf{i} - 3\mathbf{j} + 3\mathbf{k} - (8\mathbf{i} - 7\mathbf{j} + 4\mathbf{k})$

$= 4\mathbf{j} - \mathbf{k}$

$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA}$

$= 12\mathbf{i} - 6\mathbf{j} + 3\mathbf{k} - (8\mathbf{i} - 7\mathbf{j} + 4\mathbf{k})$

$= 4\mathbf{i} + \mathbf{j} - \mathbf{k}$

$\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB}$

$= 12\mathbf{i} - 6\mathbf{j} + 3\mathbf{k} - (8\mathbf{i} - 3\mathbf{j} + 3\mathbf{k})$

$= 4\mathbf{i} - 3\mathbf{j}$

13 b $|\overrightarrow{AB}| = \sqrt{4^2+1} = \sqrt{17}$

$|\overrightarrow{AC}| = \sqrt{4^2+1+1} = \sqrt{18} = 3\sqrt{2}$

$|\overrightarrow{BC}| = \sqrt{4^2+3^2} = \sqrt{25} = 5$

c As the sides are all different lengths, the triangle is scalene.

14 a $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$
$= \mathbf{i} - 2\mathbf{j} + 5\mathbf{k} - (3\mathbf{i} + 4\mathbf{j} + 8\mathbf{k})$
$= -2\mathbf{i} - 6\mathbf{j} - 3\mathbf{k}$

$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA}$
$= 7\mathbf{i} - 5\mathbf{j} + 7\mathbf{k} - (3\mathbf{i} + 4\mathbf{j} + 8\mathbf{k})$
$= 4\mathbf{i} - 9\mathbf{j} - \mathbf{k}$

$\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB}$
$= 7\mathbf{i} - 5\mathbf{j} + 7\mathbf{k} - (\mathbf{i} - 2\mathbf{j} + 5\mathbf{k})$
$= 6\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$

b $|\overrightarrow{AB}| = \sqrt{2^2+6^2+3^2} = \sqrt{49} = 7$

$|\overrightarrow{AC}| = \sqrt{4^2+9^2+1} = \sqrt{98} = 7\sqrt{2}$

$|\overrightarrow{BC}| = \sqrt{6^2+3^2+2^2} = \sqrt{49} = 7$

c Triangle ABC is isosceles.

Since angle $ABC = 90°$, angle $BAC = 45°$

15 a $|-\mathbf{i} + 7\mathbf{j} + \mathbf{k}| = \sqrt{51}$

i $\cos\theta_x = \frac{-1}{\sqrt{51}} \Rightarrow \theta_x = 98.0°$

ii $\cos\theta_y = \frac{7}{\sqrt{51}} \Rightarrow \theta_y = 11.4°$

iii $\cos\theta_z = \frac{1}{\sqrt{51}} \Rightarrow \theta_z = 82.0°$

b $|3\mathbf{i} + 4\mathbf{j} + 7\mathbf{k}| = \sqrt{74}$

i $\cos\theta_x = \frac{3}{\sqrt{74}} \Rightarrow \theta_x = 69.6°$

ii $\cos\theta_y = \frac{4}{\sqrt{74}} \Rightarrow \theta_y = 62.3°$

iii $\cos\theta_z = \frac{7}{\sqrt{74}} \Rightarrow \theta_z = 35.5°$

c $|2\mathbf{i} - 3\mathbf{k}| = \sqrt{13}$

i $\cos\theta_x = \frac{2}{\sqrt{13}} \Rightarrow \theta_x = 56.3°$

ii $\cos\theta_y = \frac{0}{\sqrt{13}} \Rightarrow \theta_y = 90°$

iii $\cos\theta_z = \frac{-3}{\sqrt{13}} \Rightarrow \theta_z = -146.3°$

16 Let A be (2, 0, 0), B be (5, 0, 0) and C be (4, 2, 3).

$|\overrightarrow{AB}| = 3$

$|\overrightarrow{AC}| = \sqrt{2^2+2^2+3^2} = \sqrt{17} \approx 4.123$

$|\overrightarrow{BC}| = \sqrt{1^2+2^2+3^2} = \sqrt{14} \approx 3.742$

$\cos\angle ABC = \frac{9+14-17}{2\times3\times\sqrt{14}} = 0.2672...$

$\angle ABC = 74.49...°$

Area of triangle
$= \frac{1}{2}\times3\times\sqrt{14}\times\sin 74.49...°$
$= 5.41$

17 $|\overrightarrow{PQ}| = \sqrt{3^2+1+2^2} = \sqrt{14}$

$|\overrightarrow{QR}| = \sqrt{2^2+4^2+3^2} = \sqrt{29}$

$|\overrightarrow{PR}| = |\overrightarrow{PQ} + \overrightarrow{QR}| = \sqrt{1+3^2+5^2} = \sqrt{35}$

Using the cosine rule:

$35 = 29 + 14 - 2\sqrt{29\times14}\cos\angle PQR$

$\cos\angle PQR = \frac{4}{\sqrt{406}} = 0.1985...$

$\angle PQR = 78.5°$ (1 d.p.)

Challenge

The vector $\mathbf{a} = -2\mathbf{i} + 6\mathbf{j} - 3\mathbf{k}$ will be in the same vertical plane as the vector $\mathbf{b} = -2\mathbf{i} + 6\mathbf{j}$.

So the angle **a** makes with the *xy*-plane is the angle, θ, between **a** and **b**.

$$\cos\theta = \frac{|\mathbf{b}|}{|\mathbf{a}|} = \frac{\sqrt{40}}{\sqrt{49}} = 0.9035...$$
$$\theta = 25.4° \text{ (1 d.p.)}$$