Vectors, Mixed exercise 12

1 Coordinates of M are $(3, 5, 4)$

Distance from M to C

$= \sqrt{(5-3)^2 + (8-5)^2 + (7-4)^2}$

$= \sqrt{4+9+9} = \sqrt{22}$

2 Distance from P to Q

$= \sqrt{((a-2)-2)^2 + (6-3)^2 + (7-a)^2}$

$= \sqrt{a^2 - 8a + 16 + 9 + 49 - 14a + a^2}$

$= \sqrt{2a^2 - 22a + 74} = \sqrt{14}$

$2a^2 - 22a + 74 = 14$

$a^2 - 11a + 30 = 0$

$(a-5)(a-6) = 0$

$a = 5$ or $a = 6$

3 $|\overrightarrow{AB}| = \sqrt{3^2 + t^2 + 5^2} = \sqrt{t^2 + 34}$

$\sqrt{t^2 + 34} = 5\sqrt{2}$

$t^2 + 34 = 50$

$t^2 = 16$

$t = 4$ (since $t > 0$)

So $\overrightarrow{AB} = -3\mathbf{i} + 4\mathbf{j} + 5\mathbf{k}$

$6\mathbf{i} - 8\mathbf{j} - \frac{5}{2}t\mathbf{k} = 6\mathbf{i} - 8\mathbf{j} - 10\mathbf{k}$

$= -2\overrightarrow{AB}$

So $\overrightarrow{AB}$ is parallel to $6\mathbf{i} - 8\mathbf{j} - \frac{5}{2}t\mathbf{k}$

4 a Let O be the fixed origin.

$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = -3\mathbf{i} - 8\mathbf{j} + 3\mathbf{k}$

$\overrightarrow{PR} = \overrightarrow{OR} - \overrightarrow{OP} = -3\mathbf{i} - 9\mathbf{j} + 8\mathbf{k}$

$\overrightarrow{QR} = \overrightarrow{OR} - \overrightarrow{OQ} = -\mathbf{j} + 5\mathbf{k}$

b $|\overrightarrow{PQ}| = \sqrt{9+64+9} = \sqrt{82}$

$|\overrightarrow{PR}| = \sqrt{9+81+64} = \sqrt{154}$

$|\overrightarrow{QR}| = \sqrt{1+25} = \sqrt{26}$

$\cos\angle QPR = \frac{82+154-26}{2\times\sqrt{82}\times\sqrt{154}} = 0.9343...$

$\angle QPR = 20.87...°$

Area of triangle

$= \frac{1}{2}\times\sqrt{82}\times\sqrt{154}\sin 20.87...°$

$= 20.0$ (1 d.p.)

5 a $\overrightarrow{DE} = \overrightarrow{OE} - \overrightarrow{OD} = 4\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}$

$\overrightarrow{EF} = \overrightarrow{OF} - \overrightarrow{OE} = -3\mathbf{i} - 4\mathbf{j} + 4\mathbf{k}$

$\overrightarrow{FD} = \overrightarrow{OD} - \overrightarrow{OF} = -\mathbf{i} + \mathbf{j} - 8\mathbf{k}$

b $|\overrightarrow{DE}| = \sqrt{16+9+16} = \sqrt{41}$

$|\overrightarrow{EF}| = \sqrt{9+16+16} = \sqrt{41}$

$|\overrightarrow{FD}| = \sqrt{1+1+64} = \sqrt{66}$

c Two sides are equal in length so the triangle is isosceles.

6 a $\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = 9\mathbf{i} - 4\mathbf{j}$

$\overrightarrow{PR} = \overrightarrow{OR} - \overrightarrow{OP} = 7\mathbf{i} + \mathbf{j} - 3\mathbf{k}$

$\overrightarrow{QR} = \overrightarrow{OR} - \overrightarrow{OQ} = -2\mathbf{i} + 5\mathbf{j} - 3\mathbf{k}$

b $|\overrightarrow{PQ}| = \sqrt{81+16} = \sqrt{97}$

$|\overrightarrow{PR}| = \sqrt{49+1+9} = \sqrt{59}$

$|\overrightarrow{QR}| = \sqrt{4+25+9} = \sqrt{38}$

c $\angle QRP = 90°$ so PQ is the hypotenuse.

$\sin\angle PQR = \frac{|\overrightarrow{PR}|}{|\overrightarrow{PQ}|} = \sqrt{\frac{59}{97}} = 0.7799...$

$\angle PQR = 51.3°$

7 $\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC} = -2\mathbf{j} + \mathbf{k}$

$|\overrightarrow{AB}| = \sqrt{1+1} = \sqrt{2}$

$|\overrightarrow{BC}| = \sqrt{1+9+1} = \sqrt{11}$

$|\overrightarrow{AC}| = \sqrt{4+1} = \sqrt{5}$

$\cos\angle ABC = \frac{2+11-5}{2\times\sqrt{2}\times\sqrt{11}} = 0.8528...$

$\angle ABC = 31.5°$

8 $\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC} \Rightarrow \overrightarrow{BC} = \overrightarrow{AC} - \overrightarrow{AB}$

So $\overrightarrow{BC} = \begin{pmatrix} 9 \\ 10 \\ -6 \end{pmatrix}$

$|\overrightarrow{AB}| = \sqrt{36+4+121} = \sqrt{161}$

$|\overrightarrow{AC}| = \sqrt{225+64+25} = \sqrt{314}$

$|\overrightarrow{BC}| = \sqrt{81+100+36} = \sqrt{217}$

$\cos\angle ABC = \frac{161+217-314}{2\times\sqrt{161}\times\sqrt{217}} = 0.1712...$

$\angle ABC = 80.14...°$

Area of triangle *ABC*

$= \frac{1}{2}\times\sqrt{161}\times\sqrt{217}\times\sin\angle ABC$

Area of parallelogram *ABCD*

$= \sqrt{161}\times\sqrt{217}\times\sin\angle ABC$

$= \sqrt{161}\times\sqrt{217}\times\sin 80.14...°$

$= 184$ (3 s.f.)

9 a $|\overrightarrow{AB}| = \sqrt{4+25+9} = \sqrt{38}$

$|\overrightarrow{AC}| = \sqrt{4+25+9} = \sqrt{38}$

So *ABC* is an isosceles triangle.
Therefore *DBC* is an isosceles triangle.

So $\overrightarrow{AB}$ is parallel to $\overrightarrow{CD}$ and $\overrightarrow{AC}$ is parallel to $\overrightarrow{BD}$.

Let *O* be the fixed origin.

$\overrightarrow{OD} = \overrightarrow{OC} + \overrightarrow{CD}$

$= \overrightarrow{OC} + \overrightarrow{AB}$

$= \overrightarrow{OC} + \overrightarrow{OB} - \overrightarrow{OA}$

$= \begin{pmatrix} 4 \\ -2 \\ -5 \end{pmatrix} + \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix} - \begin{pmatrix} 2 \\ 3 \\ -2 \end{pmatrix} = \begin{pmatrix} 2 \\ -7 \\ -2 \end{pmatrix}$

Coordinates of *D* are (2, −7, −2).

b *ABCD* is a parallelogram with four sides of equal length. It is a rhombus.

c $|\overrightarrow{BC}| = \sqrt{16+36} = \sqrt{52}$

$\cos\angle BAC = \frac{38+38-52}{2\times\sqrt{38}\times\sqrt{38}} = 0.3157...$

$\angle BAC = 71.59...°$

Area of triangle *ABC*

$= \frac{1}{2}\times\sqrt{38}\times\sqrt{38}\times\sin\angle ABC$

Area of parallelogram *ABCD*

$= \sqrt{38}\times\sqrt{38}\times\sin\angle ABC$

$= \sqrt{38}\times\sqrt{38}\times\sin 71.59...°$

$= 36.1$ (3 s.f.)

10 $\overrightarrow{OP} = \frac{1}{2}\overrightarrow{OC} = \frac{1}{2}\mathbf{c}$

$\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$

$\overrightarrow{OQ} = \overrightarrow{OA} + \frac{1}{2}\overrightarrow{AB} = \mathbf{a} + \frac{1}{2}(\mathbf{b} - \mathbf{a})$

$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = \frac{1}{2}(\mathbf{a} + \mathbf{b} - \mathbf{c})$

$\overrightarrow{OR} = \frac{1}{2}\overrightarrow{OA} = \frac{1}{2}\mathbf{a}$

$\overrightarrow{BC} = \mathbf{c} - \mathbf{b}$

$\overrightarrow{OS} = \overrightarrow{OB} + \frac{1}{2}\overrightarrow{BC} = \mathbf{b} + \frac{1}{2}(\mathbf{c} - \mathbf{b})$

$\overrightarrow{RS} = \overrightarrow{OS} - \overrightarrow{OR} = \frac{1}{2}(-\mathbf{a} + \mathbf{b} + \mathbf{c})$

$\overrightarrow{OT} = \frac{1}{2}\overrightarrow{OB} = \frac{1}{2}\mathbf{b}$

$\overrightarrow{AC} = \mathbf{c} - \mathbf{a}$

$\overrightarrow{OU} = \overrightarrow{OA} + \frac{1}{2}\overrightarrow{AC} = \mathbf{a} + \frac{1}{2}(\mathbf{c} - \mathbf{a})$

$\overrightarrow{TU} = \overrightarrow{OU} - \overrightarrow{OT} = \frac{1}{2}(\mathbf{a} - \mathbf{b} + \mathbf{c})$

Suppose there is a point of intersection, X, of PQ, RS and TU.

$\overrightarrow{PX} = r\overrightarrow{PQ} = \frac{r}{2}(\mathbf{a} + \mathbf{b} - \mathbf{c})$

$\overrightarrow{RX} = s\overrightarrow{RS} = \frac{s}{2}(-\mathbf{a} + \mathbf{b} + \mathbf{c})$

$\overrightarrow{TX} = t\overrightarrow{TU} = \frac{t}{2}(\mathbf{a} - \mathbf{b} + \mathbf{c})$

for scalars r, s and t.

But $\overrightarrow{RX} = \overrightarrow{RO} + \overrightarrow{OP} + \overrightarrow{PX}$

$= -\frac{1}{2}\mathbf{a} + \frac{1}{2}\mathbf{c} + \frac{r}{2}(\mathbf{a} + \mathbf{b} - \mathbf{c})$

so $\frac{s}{2}(-\mathbf{a} + \mathbf{b} + \mathbf{c}) = -\frac{1}{2}\mathbf{a} + \frac{1}{2}\mathbf{c} + \frac{r}{2}(\mathbf{a} + \mathbf{b} - \mathbf{c})$

$s(-\mathbf{a} + \mathbf{b} + \mathbf{c}) = (r-1)\mathbf{a} + r\mathbf{b} + (1-r)\mathbf{c}$

Comparing coefficients of **b** and **c**:
$s = r$ and $s = 1 - r$

Hence $r = s = \frac{1}{2}$

Also $\overrightarrow{TX} = \overrightarrow{TO} + \overrightarrow{OP} + \overrightarrow{PX}$

$= -\frac{1}{2}\mathbf{b} + \frac{1}{2}\mathbf{c} + \frac{1}{4}(\mathbf{a} + \mathbf{b} - \mathbf{c})$

so $\frac{t}{2}(\mathbf{a} - \mathbf{b} + \mathbf{c}) = -\frac{1}{2}\mathbf{b} + \frac{1}{2}\mathbf{c} + \frac{1}{4}(\mathbf{a} + \mathbf{b} - \mathbf{c})$

$t(\mathbf{a} - \mathbf{b} + \mathbf{c}) = \frac{1}{2}(\mathbf{a} - \mathbf{b} + \mathbf{c})$

Hence $t = \frac{1}{2}$

So the point X is the midpoint of all three line segments PQ, RS and TU. Therefore the line segments do meet at a point and bisect each other.

11 Total force on particle

$= \mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3$

$= ((b+1)\mathbf{i} + (4-b)\mathbf{j} + (7-b)\mathbf{k})$ N

$|\mathbf{F}| = \sqrt{(b+1)^2 + (4-b)^2 + (7-b)^2}$

$= \sqrt{b^2 + 2b + 1 + 16 - 8b + b^2 + 49 - 14b + b^2}$

$= \sqrt{3b^2 - 20b + 66}$

$|\mathbf{F}| = m|\mathbf{a}|$

$\Rightarrow \sqrt{3b^2 - 20b + 66} = 2 \times 3.5 = 7$

$3b^2 - 20b + 66 = 49$

$3b^2 - 20b + 17 = 0$

$(b-1)(3b-17) = 0$

$b = 1$ or $b = \frac{17}{3}$

12 a Air resistance acts in opposition to the motion of the BASE jumper. The motion downwards will be greater than the motion in the other directions.

b Gravitational force downwards
$= 50 \times 9.8 = 490$ N

Total force on BASE jumper
$= \mathbf{W} + \mathbf{F} - 490\mathbf{k}$

$= (16\mathbf{i} + 13\mathbf{j} - 40\mathbf{k})$ N

12 c $|16\mathbf{i}+13\mathbf{j}-40\mathbf{k}|=\sqrt{256+169+1600}$

$=\sqrt{2025}=45\text{ N}$

Acceleration $=\dfrac{45}{50}=\dfrac{9}{10}\text{ m s}^{-2}$

Using $s=ut+\dfrac{1}{2}at^2$:

$180=0+\dfrac{1}{2}\times\dfrac{9}{10}t^2$

$t^2=400$

$t=20$

The descent took 20 seconds.

Challenge

For example, if **a** = (1, 0, 0), **b** = (0, 1, 0) and **c** = (1, 1, 0) then $p = q = r = 1$ gives

a + **b** + **c** = 2**a** + 2**b**

So $s = 2 \neq p$, $t = 2 \neq q$ and $u = 0 \neq r$, and the result does not hold.

The statement is also untrue if any of the scalars p, q and r is zero. For example, with **a**, **b** and **c** as above, if $p = 0$ and $q = r = 1$, then $s = 1 \neq p$, $t = 2 \neq q$ and $u = 0 \neq r$.