Algebraic Methods 1D

Note that all questions in this exercise can be solved either by the method of substitution, or by equating coefficients. Questions **1b** and **1e** have been solved by equating coefficients. All others have been solved using substitution.

1 a $\dfrac{6x-2}{(x-2)(x+3)} \equiv \dfrac{A}{(x-2)} + \dfrac{B}{(x+3)}$

$$\equiv \frac{A(x+3)+B(x-2)}{(x-2)(x+3)}$$

$$6x-2 \equiv A(x+3)+B(x-2)$$

Let $x=2$:

$$6\times 2-2 = A(2+3)+B(2-2)$$

$$10 = 5A$$

$$A = 2$$

Let $x=-3$:

$$6\times(-3)-2 = A(-3+3)+B(-3-2)$$

$$-20 = B\times -5$$

$$B = 4$$

Hence $\dfrac{6x-2}{(x-2)(x+3)} \equiv \dfrac{2}{(x-2)} + \dfrac{4}{(x+3)}$

b $\dfrac{2x+11}{(x+1)(x+4)} \equiv \dfrac{A}{(x+1)} + \dfrac{B}{(x+4)}$

$$\equiv \frac{A(x+4)+B(x+1)}{(x+1)(x+4)}$$

$$2x+11 \equiv A(x+4)+B(x+1)$$

$$\equiv Ax+4A+Bx+B$$

$$\equiv (A+B)x+(4A+B)$$

Equate coefficients of x:

$$2 = A+B \qquad (1)$$

Equate constant terms:

$$11 = 4A+B \qquad (2)$$

$(2)-(1)$:

$$9 = 3A$$

$$A = 3$$

Substitute $A=3$ in (1): $2 = 3+B$

$$B = -1$$

Hence $\dfrac{2x+11}{(x+1)(x+4)} \equiv \dfrac{3}{(x+1)} - \dfrac{1}{(x+4)}$

1 c $\dfrac{-7x-12}{2x(x-4)} \equiv \dfrac{A}{2x} + \dfrac{B}{(x-4)}$

$$\equiv \frac{A(x-4)+B\times 2x}{2x(x-4)}$$

$$-7x-12 \equiv A(x-4)+2Bx$$

Let $x=4$:

$$-7\times 4-12 = A(4-4)+2B\times 4$$

$$-40 = 8B$$

$$B = -5$$

Let $x=0$:

$$-7\times 0-12 = A(0-4)+2B\times 0$$

$$-12 = -4A$$

$$A = 3$$

Hence $\dfrac{-7x-12}{2x(x-4)} \equiv \dfrac{3}{2x} - \dfrac{5}{(x-4)}$

d $\dfrac{2x-13}{(2x+1)(x-3)} \equiv \dfrac{A}{(2x+1)} + \dfrac{B}{(x-3)}$

$$\equiv \frac{A(x-3)+B(2x+1)}{(2x+1)(x-3)}$$

$$2x-13 \equiv A(x-3)+B(2x+1)$$

Let $x=3$:

$$2\times 3-13 = A(3-3)+B(2\times 3+1)$$

$$-7 = B\times 7$$

$$B = -1$$

Let $x=-\dfrac{1}{2}$:

$$2\times\left(\frac{1}{2}\right)-13 = A\left(-\frac{1}{2}-3\right)+B\left(2\times\left(-\frac{1}{2}\right)+1\right)$$

$$-14 = A\times -3\frac{1}{2}$$

$$A = 4$$

Hence $\dfrac{2x-13}{(2x+1)(x-3)} \equiv \dfrac{4}{(2x+1)} - \dfrac{1}{(x-3)}$

1 e First factorise the denominator:

$$\frac{6x+6}{x^2+9} \equiv \frac{6x+6}{(x+3)(x-3)}$$

Then $\dfrac{6x+6}{(x+3)(x-3)} \equiv \dfrac{A}{(x+3)} + \dfrac{B}{(x-3)}$

$$\equiv \frac{A(x-3)+B(x+3)}{(x+3)(x-3)}$$

$$6x+6 \equiv A(x-3)+B(x+3)$$

$$\equiv Ax-3A+Bx+3B$$

$$\equiv (A+B)x+(3B-3A)$$

Equate coefficients of x:

$$6 = A+B \qquad (1)$$

Equate constant terms:

$$6 = 3B-3A \qquad (2)$$

$(2)+3\times(1)$:

$$24 = 6B$$

$$B = 4$$

Substitute $B=4$ in (1): $6 = A+4$

$$A = 2$$

Hence $\dfrac{6x+6}{x^2-9} \equiv \dfrac{2}{(x+3)} + \dfrac{4}{(x-3)}$

f First factorise the denominator:

$$\frac{7-3x}{x^2-3x-4} \equiv \frac{7-3x}{(x-4)(x+1)}$$

Then $\dfrac{7-3x}{(x-4)(x+1)} \equiv \dfrac{A}{(x-4)} + \dfrac{B}{(x+1)}$

$$\equiv \frac{A(x+1)+B(x-4)}{(x-4)(x+1)}$$

$$7-3x \equiv A(x+1)+B(x-4)$$

Let $x=-1$:

$$7-3\times(-1) = A(-1+1)+B(-1-4)$$

$$10 = B\times -5$$

$$B = -2$$

1 f (continued)

Let $x = 4$:

$$7 - 3 \times 4 = A(4+1) + B(4-4)$$
$$-5 = A \times 5$$
$$A = -1$$

Hence $\dfrac{7-3x}{x^2-3x-4} \equiv -\dfrac{1}{(x-4)} - \dfrac{2}{(x+1)}$

g First factorise the denominator:

$$\frac{8-x}{x^2+4x} \equiv \frac{8-x}{x(x+4)}$$

Then $\dfrac{8-x}{x(x+4)} \equiv \dfrac{A}{x} + \dfrac{B}{(x+4)}$

$$\equiv \frac{A(x+4)+Bx}{x(x+4)}$$
$$8 - x \equiv A(x+4) + Bx$$

Let $x = 0$:

$$8 - 0 = A(0+4) + B \times 0$$
$$8 = 4A$$
$$A = 2$$

Let $x = -4$:

$$8 - (-4) = A(-4+4) + B \times (-4)$$
$$12 = -4B$$
$$B = -3$$

Hence $\dfrac{8-x}{x^2+4x} \equiv \dfrac{2}{x} - \dfrac{3}{(x+4)}$

1 h First factorise the denominator:

$$\frac{2x-14}{x^2+2x-15} \equiv \frac{2x-14}{(x+5)(x-3)}$$

Then $\frac{2x-14}{(x+5)(x-3)} \equiv \frac{A}{(x+5)} + \frac{B}{(x-3)}$

$$\equiv \frac{A(x-3)+B(x+5)}{(x+5)(x-3)}$$

$$2x-14 \equiv A(x-3)+B(x+5)$$

Let $x=3$:

$$2\times 3-14 = A(3-3)+B(3+5)$$
$$-8 = B\times 8$$
$$B=-1$$

Let $x=-5$:

$$2\times(-5)-14 = A(-5-3)+B(-5+5)$$
$$-24 = A\times(-8)$$
$$A=3$$

Hence $\frac{2x-14}{x^2+2x-15} \equiv \frac{3}{(x+5)} - \frac{1}{(x-3)}$

2 $\frac{-2x-5}{(4+x)(2-x)} \equiv \frac{A}{4+x} + \frac{B}{2-x}$

$$\equiv \frac{A(2-x)+B(4+x)}{(4+x)(2-x)}$$

$$-2x-5 \equiv A(2-x)+B(4+x)$$

Let $x=2$:

$$-2\times 2-5 = A(2-2)+B(4+2)$$
$$-9 = B\times 6$$
$$B=\frac{-3}{2}$$

Let $x=-4$:

$$-2\times(-4)-5 = A(2-(-4)+B(4+(-4))$$
$$3 = A\times 6$$
$$\frac{1}{2} = A$$

Hence $\frac{-2x-5}{(4+x)(2-x)} = \frac{1}{2(4+x)} - \frac{3}{2(2-x)}$

3 $\dfrac{A}{(x-4)(x+8)} \equiv \dfrac{2}{x-4} + \dfrac{B}{x+8}$

$$\equiv \frac{2(x+8)+B(x-4)}{(x-4)(x+8)}$$

$$A \equiv 2(x+8)+B(x-4)$$

Let $x = 4$:

$$A = 2(4+8)+B(4-4)$$
$$= 24$$

Let $x = -8$:

$$24 = 2(-8+8)+B(-8-4)$$
$$= -12B$$
$$\Rightarrow B = -2$$

$A = 24,\ B = -2$

4 $\dfrac{2x^2-12x-26}{(x+1)(x-2)(x+5)} \equiv \dfrac{A}{x+1} + \dfrac{B}{x-2} + \dfrac{C}{x+5}$

$$\equiv \frac{A(x-2)(x+5)+B(x+1)(x+5)+C(x+1)(x-2)}{(x+1)(x-2)(x+5)}$$

$$2x^2-12x-26 \equiv A(x-2)(x+5)+B(x+1)(x+5)+C(x+1)(x-2)$$

Let $x = -1$:

$$2+12-26 = A\times(-3)\times 4+0+0$$
$$-12 = -12A$$
$$A = 1$$

Let $x = 2$:

$$8-24-26 = 0+B\times 3\times 7+0$$
$$-42 = 21B$$
$$B = -2$$

Let $x = -5$:

$$50+60-26 = 0+0+C\times(-4)\times(-7)$$
$$84 = 28C$$
$$C = 3$$

$A = 1,\ B = -2,\ C = 3$

5 $\dfrac{-10x^2-8x+2}{x(2x+1)(3x-2)} \equiv \dfrac{D}{x}+\dfrac{E}{2x+1}+\dfrac{F}{3x-2}$

$$\equiv \frac{D(2x+1)(3x-2)+Ex(3x-2)+Fx(2x+1)}{x(2x+1)(3x-2)}$$

$$-10x^2-8x+2 \equiv D(2x+1)(3x-2)+Ex(3x-2)+Fx(2x+1)$$

Let $x=0$:

$$2 = D\times 1\times(-2)+0+0$$
$$= -2D$$
$$\Rightarrow D=-1$$

Let $x=-\dfrac{1}{2}$:

$$-\frac{5}{2}+4+2 = 0+E\times\left(-\frac{1}{2}\right)\times\left(-\frac{7}{2}\right)+0$$
$$\frac{7}{2}=\frac{7}{4}E$$
$$\Rightarrow E=2$$

Let $x=\dfrac{2}{3}$:

$$-\frac{40}{9}-\frac{16}{3}+2 = 0+0+F\times\left(\frac{2}{3}\right)\times\left(\frac{7}{3}\right)$$
$$-\frac{70}{9}=\frac{14}{9}F$$
$$F=-5$$

$D=-1, E=2, F=-5$

6 Let $\dfrac{-5x^2-19x-32}{(x+1)(x+2)(x-5)} \equiv \dfrac{A}{(x+1)} + \dfrac{B}{(x+2)} + \dfrac{C}{(x-5)}$

$$\equiv \frac{A(x+2)(x-5)+B(x+1)(x-5)+C(x+1)(x+2)}{(x+1)(x+2)(x-5)}$$

$$-5x^2-19x-32 \equiv A(x+2)(x-5)+B(x+1)(x-5)+C(x+1)(x+2)$$

Let $x=-1$:

$$-5+19-32 = A\times 1\times(-6)+B\times 0+C\times 0$$

$$-18 = -6A$$

$$A = 3$$

Let $x=5$:

$$-125-95-32 = A\times 0+B\times 0+C\times 6\times 7$$

$$-252 = 42C$$

$$C = -6$$

Let $x=-2$:

$$-20+38-32 = A\times 0+B\times(-1)\times(-7)+C\times 0$$

$$-14 = 7B$$

$$B = -2$$

Hence $\dfrac{-5x^2-19x-32}{(x+1)(x+2)(x-5)} \equiv \dfrac{3}{(x+1)} - \dfrac{2}{(x+2)} - \dfrac{6}{(x-5)}$

7 a First factorise the denominator:

$$\frac{6x^2+7x-3}{x^3-x} = \frac{6x^2+7x-3}{x(x+1)(x-1)}$$

Then $\dfrac{6x^2+7x-3}{x(x+1)(x-1)} \equiv \dfrac{A}{x} + \dfrac{B}{x+1} + \dfrac{C}{x-1}$

$$\equiv \frac{A(x+1)(x-1)+Bx(x-1)+Cx(x+1)}{x(x+1)(x-1)}$$

$$6x^2+7x-3 \equiv A(x+1)(x-1)+Bx(x-1)+Cx(x+1)$$

7 a (continued)

Let $x = 0$:

$-3 = A \times 1 \times (-1) + 0 + 0$

$= -A$

$\Rightarrow A = 3$

Let $x = -1$:

$6 - 7 - 3 = 0 + B \times (-1) \times (-2) + 0$

$-4 = 2B$

$\Rightarrow B = -2$

Let $x = 1$:

$6 + 7 - 3 = 0 + 0 + C \times 1 \times 2$

$10 = 2C$

$\Rightarrow C = 5$

So $\dfrac{6x^2+7x-3}{x(x+1)(x-1)} \equiv \dfrac{3}{x} - \dfrac{2}{x+1} + \dfrac{5}{x-1}$

b First factorise the denominator:

$$\frac{8x+9}{10x^2+3x-4} = \frac{8x+9}{(5x+4)(2x-1)}$$

Then $\dfrac{8x+9}{(5x+4)(2x-1)} \equiv \dfrac{A}{5x+4} + \dfrac{B}{2x-1}$

$$\equiv \frac{A(2x-1)}{(5x+4)(2x-1)} + \frac{B(5x+4)}{(5x+4)(2x-1)}$$

$$8x+9 \equiv A(2x-1) + B(5x+4)$$

Let $x = -\dfrac{4}{5}$:

$$-\frac{32}{5} + 9 = A \times \left(-\frac{13}{5}\right) + 0$$

$$\frac{13}{5} = -\frac{13}{5}A$$

$$\Rightarrow A = -1$$

Let $x = \dfrac{1}{2}$:

$$4 + 9 = 0 + B \times \left(\frac{13}{2}\right)$$

$$13 = \frac{13}{2}B$$

$$\Rightarrow B = 2$$

So $\dfrac{8x+9}{(5x+4)(2x-1)} \equiv -\dfrac{1}{5x+4} + \dfrac{2}{2x-1}$

Challenge

Evaluating the denominator at $x = 2$:

$$2^3 - 4(2^2) + 2 + 6 = 0$$

By the factor theorem, $(x-2)$ is a factor of $x^3 - 4x^2 + x + 6$

So we can write $x^3 - 4x^2 + x + 6 = (x-2) \times \text{p}(x)$ for some quadratic polynomial p.

We can find p using long division:

$$\begin{array}{r} x^2 - 2x - 3 \\ x-2 \overline{)\, x^3 - 4x^2 + x + 6} \\ \underline{x^3 - 2x^2} \\ -2x^2 + x \\ \underline{-2x^2 + 4x} \\ -3x + 6 \\ \underline{-3x + 6} \\ 0 \end{array}$$

$$\begin{aligned} \frac{5x^2 - 15x - 8}{x^3 - 4x^2 + x + 6} &\equiv \frac{5x^2 - 15x - 8}{(x-2)(x^2 - 2x - 3)} \\ &\equiv \frac{5x^2 - 15x - 8}{(x-2)(x+1)(x-3)} \\ &\equiv \frac{A}{x-2} + \frac{B}{x+1} + \frac{C}{x-3} \\ &\equiv \frac{A(x+1)(x-3) + B(x-2)(x-3) + C(x-2)(x+1)}{(x-2)(x+1)(x-3)} \end{aligned}$$

$$5x^2 - 15x - 8 \equiv A(x+1)(x-3) + B(x-2)(x-3) + C(x-2)(x+1)$$

Let $x = 2$:

$20 - 30 - 8 = A \times 3 \times (-1) + 0 + 0$

$-18 = -3A$

$\Rightarrow A = 6$

Let $x = -1$:

$5 + 15 - 8 = 0 + B \times (-3) \times (-4) + 0$

$12 = 12B$

$\Rightarrow B = 1$

Let $x = 3$:

$45 - 45 - 8 = 0 + 0 + C \times 1 \times 4$

$-8 = 4C$

$C = -2$

So $\dfrac{5x^2 - 15x - 8}{x^3 - 4x^2 + x + 6} \equiv \dfrac{6}{x-2} + \dfrac{1}{x+1} - \dfrac{2}{x-3}$