Algebraic Methods 1G

1 $\dfrac{x^2+3x-2}{(x-1)(x-2)} \equiv \dfrac{x^2+3x-2}{x^2-3x+2}$

$$x^2-3x+2 \overline{)\, x^2+3x-2}$$
$$\underline{x^2-3x+2} \quad \Rightarrow 1$$
$$6x-4$$

Therefore $\dfrac{x^2+3x-2}{(x-1)(x-2)} \equiv 1+\dfrac{6x-4}{x^2-3x+2}$

$$\equiv 1+\frac{6x-4}{(x-1)(x-2)}$$

Let $\dfrac{6x-4}{(x-1)(x-2)} \equiv \dfrac{B}{x-1}+\dfrac{C}{x-2}$

$$\equiv \frac{B(x-2)+C(x-1)}{(x-1)(x-2)}$$

$$6x-4 \equiv B(x-2)+C(x-1)$$

Let $x = 2$:

$12 - 4 = 0 + C \times 1$

$C = 8$

Let $x = 1$:

$6 - 4 = B \times (-1) + 0$

$2 = -B$

$B = -2$

$$\frac{x^2+3x-2}{(x-1)(x-2)} \equiv 1+\frac{6x-4}{(x-1)(x-2)}$$

$$\equiv 1-\frac{2}{x-1}+\frac{8}{x-2}$$

So $A = 1$, $B = -2$ and $C = 8$.

2 $\dfrac{x^2-10}{(x-2)(x+1)} \equiv \dfrac{x^2-10}{x^2-x-2}$

$$x^2-x-2\overline{)\,x^2+0x-10}$$
$$\begin{array}{r} 1 \\ x^2-x-2 \overline{)\,x^2+0x-10} \\ \underline{x^2-x-2} \\ x-8 \end{array}$$

Therefore $\dfrac{x^2-10}{(x-2)(x+1)} \equiv 1+\dfrac{x-8}{x^2-x-2}$

$$\equiv 1+\frac{x-8}{(x-2)(x+1)}$$

Let $\dfrac{x-8}{(x-2)(x+1)} \equiv \dfrac{B}{x-2}+\dfrac{C}{x+1}$

$$\equiv \frac{B(x+1)+C(x-2)}{(x-2)(x+1)}$$

$$x-8 \equiv B(x+1)+C(x-2)$$

Let $x = -1$:

$-1-8 = 0 + C \times (-3)$

$-9 = -3C$

$C = 3$

Let $x = 2$:

$2-8 = B \times 3 + 0$

$-6 = 3B$

$B = -2$

$$\frac{x^2-10}{(x-2)(x+1)} \equiv 1+\frac{x-8}{(x-2)(x+1)}$$

$$\equiv 1-\frac{2}{x-2}+\frac{3}{x+1}$$

So $A = 1$, $B = -2$ and $C = 3$.

3 $\dfrac{x^3-x^2-x-3}{x(x-1)} \equiv \dfrac{x^3-x^2-x-3}{x^2-x}$

$$\begin{array}{r} x \\ x^2-x\overline{)\,x^3-x^2-x-3} \\ \underline{x^3-x^2} \\ -x-3 \end{array}$$

Therefore $\dfrac{x^3-x^2-x-3}{x(x-1)} \equiv x+\dfrac{-x-3}{x^2-x}$

$$\equiv x+\frac{-x-3}{x(x-1)}$$

Let $\dfrac{-x-3}{x(x-1)} \equiv \dfrac{C}{x}+\dfrac{D}{x-1}$

$$\equiv \frac{C(x-1)+Dx}{x(x-1)}$$

$$-x-3 \equiv C(x-1)+Dx$$

Let $x = 0$:

$0 - 3 = C \times (-1) + 0$

$-3 = -C$

$C = 3$

Let $x = 1$:

$-1 - 3 = 0 + D \times 1$

$D = -4$

$$\frac{x^3-x^2-x-3}{x(x-1)} \equiv x+\frac{-x-3}{x(x-1)}$$

$$\equiv x+\frac{3}{x}-\frac{4}{x-1}$$

So $A = 1$, $B = 0$, $C = 3$ and $D = -4$.

4 $x^2+2x-3\overline{)-3x^3-4x^2+19x+8}$

$$\begin{array}{r} -3x+2 \\ \hline -3x^3-4x^2+19x+8 \\ \underline{-3x^3-6x^2+9x} \\ 2x^2+10x+8 \\ \underline{2x^2+\ 4x-6} \\ 6x+14 \end{array}$$

Therefore $\dfrac{-3x^3-4x^2+19x+8}{x^2+2x-3} \equiv -3x+2+\dfrac{6x+14}{x^2+2x-3}$

$$\equiv -3x+2+\frac{6x+14}{(x-1)(x+3)}$$

Let $\dfrac{6x+14}{(x-1)(x+3)} \equiv \dfrac{C}{(x-1)}+\dfrac{D}{(x+3)}$

$$\equiv \frac{C(x+3)+D(x-1)}{(x-1)(x+3)}$$

$$6x+14 \equiv C(x+3)+D(x-1)$$

Let $x = 1$:

$$6+14 = C\times 4 + D\times 0$$

$$20 = 4C$$

$$5 = C$$

Let $x = -3$:

$$6\times(-3)+14 = C\times 0 + D\times(-4)$$

$$-4 = -4D$$

$$D = 1$$

$$\frac{-3x^3-4x^2+19x+8}{x^2+2x-3} \equiv 2-3x+\frac{6x+14}{(x-1)(x+3)}$$

$$\equiv 2-3x+\frac{5}{(x-1)}+\frac{1}{(x+3)}$$

So $A = 2,\ B = -3,\ C = 5$ and $D = 1$

5 $4x^2 - 25 \overline{)\,4x^2 + 25}$

$$\begin{array}{r} 1 \\ \hline 4x^2 - 25 \\ \hline 50 \end{array}$$

Therefore $\mathrm{p}(x) \equiv \dfrac{4x^2 + 25}{4x^2 - 25}$

$$\equiv 1 + \frac{50}{4x^2 - 25}$$

$$\equiv 1 + \frac{50}{(2x-5)(2x+5)}$$

Let $\dfrac{50}{(2x-5)(2x+5)} \equiv \dfrac{B}{2x-5} + \dfrac{C}{2x+5}$

$$\equiv \frac{B(2x+5) + C(2x-5)}{(2x-5)(2x+5)}$$

$$50 \equiv B(2x+5) + C(2x-5)$$

Let $x = \dfrac{5}{2}$:

$50 = B \times 10 + 0$

$50 = 10B$

$B = 5$

Let $x = -\dfrac{5}{2}$:

$50 = 0 + C \times (-10)$

$50 = -10C$

$C = -5$

$$\mathrm{p}(x) \equiv \frac{4x^2 + 25}{4x^2 - 25}$$

$$\equiv 1 + \frac{50}{(2x-5)(2x+5)}$$

$$\equiv 1 + \frac{5}{2x-5} - \frac{5}{2x+5}$$

So $A = 1$, $B = 5$ and $C = -5$.

6 $x^2+2x+1\overline{)2x^2+0x-1}$

$$\begin{array}{r} 2 \\ \hline 2x^2+0x-1 \\ \underline{2x^2+4x+2} \\ -4x-3 \end{array}$$

Therefore $\dfrac{2x^2-1}{x^2+2x+1} \equiv 2+\dfrac{-4x-3}{x^2+2x+1}$

$$\equiv 2+\frac{-4x-3}{(x+1)^2}$$

Let $\dfrac{-4x-3}{(x+1)^2} \equiv \dfrac{B}{x+1}+\dfrac{C}{(x+1)^2}$

$$\equiv \frac{B(x+1)+C}{(x+1)^2}$$

$$-4x-3 \equiv B(x+1)+C$$

Let $x = -1$:

$4 - 3 = 0 + C$

$C = 1$

Let $x = 0$:

$-3 = B \times 1 + C$

$-3 = B + 1$

$B = -4$

$$\frac{2x^2-1}{x^2+2x+1} \equiv 2+\frac{-4x-3}{(x+1)^2}$$

$$\equiv 2-\frac{4}{x+1}+\frac{1}{(x+1)^2}$$

So $A = 2$, $B = -4$ and $C = 1$.

7 a
$$x^2+3x-4\overline{)4x^2+17x-11}\quad 4$$
$$4x^2+12x-16$$
$$5x+5$$

Therefore $\dfrac{4x^2+17x-11}{x^2+3x-4} \equiv 4+\dfrac{5x+5}{x^2+3x-4}$

$$\equiv 4+\frac{5x+5}{(x+4)(x-1)}$$

Let $\dfrac{5x+5}{(x+4)(x-1)} \equiv \dfrac{A}{(x+4)}+\dfrac{B}{(x-1)}$

$$\equiv \frac{A(x-1)+B(x+4)}{(x+4)(x-1)}$$

$$5x+5 \equiv A(x-1)+B(x+4)$$

Let $x=1$:

$$5\times 1+5 = A\times 0+B\times 5$$
$$10 = 5B$$
$$B = 2$$

Let $x=-4$:

$$5\times(-4)+5 = A\times(-5)+B\times 0$$
$$-15 = -5A$$
$$A = 3$$

Hence $\dfrac{4x^2+17x-11}{x^2+3x-4} \equiv 4+\dfrac{5x+5}{(x+4)(x-1)}$

$$\equiv 4+\frac{3}{(x+4)}+\frac{2}{(x-1)}$$

7 b
$$x^3-4x^2+4x\overline{)x^4-4x^3+9x^2-17x+12}\quad x$$
$$\underline{x^4-4x^3+4x^2}$$
$$5x^2-17x+12$$

Therefore $\dfrac{x^4-4x^3+9x^2-17x+12}{x^3-4x^2+4x} \equiv x+\dfrac{5x^2-17x+12}{x^3-4x^2+4x}$

$$\equiv x+\frac{5x^2-17x+12}{x(x-2)^2}$$

Let $\dfrac{5x^2-17x+12}{x(x-2)^2} \equiv \dfrac{A}{x}+\dfrac{B}{x-2}+\dfrac{C}{(x-2)^2}$

$$\equiv \frac{A(x-2)^2+Bx(x-2)+Cx}{x(x-2)^2}$$

$$5x^2-17x+12 \equiv A(x-2)^2+Bx(x-2)+Cx$$

Let $x=0$:

$12=A\times(-2)^2$

$12=4A$

$A=3$

Let $x=2$:

$5\times(2)^2-17\times2+12=2C$

$-2=2C$

$C=-1$

Compare terms in x^2:

$5=A+B$

$5=3+B$

$B=2$

$$\frac{x^4-4x^3+9x^2-17x+12}{x^3-4x^2+4x} \equiv x+\frac{5x^2-17x+12}{x(x-2)^2}$$

$$\equiv x+\frac{3}{x}+\frac{2}{x-2}-\frac{1}{(x-2)^2}$$

8 $3x^2+x-10\overline{)6x^3-7x^2+0x+3}$

$$\begin{array}{r} 2x-3 \\ \hline 6x^3-7x^2+0x+3 \\ \underline{6x^3+2x^2-20x} \\ -9x^2+20x+3 \\ \underline{-9x^2-3x+30} \\ 23x-27 \end{array}$$

Therefore $\dfrac{6x^3-7x^2+3}{3x^2+x-10} \equiv 2x-3+\dfrac{23x-27}{3x^2+x-10}$

$$\equiv 2x-3+\frac{23x-27}{(3x-5)(x+2)}$$

Let $\dfrac{23x-27}{(3x-5)(x+2)} \equiv \dfrac{C}{3x-5}+\dfrac{D}{x+2}$

$$\equiv \frac{C(x+2)+D(3x-5)}{(3x-5)(x+2)}$$

$$23x-27 \equiv C(x+2)+D(3x-5)$$

Let $x=\dfrac{5}{3}$:

$$\frac{115}{3}-27 = C\times\frac{11}{3}+0$$

$$\frac{34}{3} = \frac{11}{3}C$$

$$C = \frac{34}{11}$$

Let $x=-2$:

$$-46-27 = 0+D\times(-11)$$

$$D = \frac{73}{11}$$

$$\frac{6x^3-7x^2+3}{3x^2+x-10} \equiv 2x-3+\frac{23x-27}{(3x-5)(x+2)}$$

$$\equiv 2x-3+\frac{34}{11(3x-5)}+\frac{73}{11(x+2)}$$

So $A=2$, $B=-3$, $C=\dfrac{34}{11}$ and $D=\dfrac{73}{11}$.

9 $4x^2-4x+1\overline{)8x^3+0x^2+0x+1}$

$$\begin{array}{r} 2x+2 \\ \hline 8x^3+0x^2+0x+1 \\ \underline{8x^3-8x^2+2x} \\ 8x^2-2x+1 \\ \underline{8x^2-8x+2} \\ 6x-1 \end{array}$$

Therefore $\dfrac{8x^3+1}{4x^2-4x+1} \equiv 2x+2+\dfrac{6x-1}{4x^2-4x+1}$

$$\equiv 2x+2+\frac{6x-1}{(2x-1)^2}$$

Let $\dfrac{6x-1}{(2x-1)^2} \equiv \dfrac{C}{2x-1}+\dfrac{D}{(2x-1)^2}$

$$\equiv \frac{C(2x-1)+D}{(2x-1)^2}$$

$$6x-1 \equiv C(2x-1)+D$$

Let $x=\dfrac{1}{2}$:

$3-1=0+D$

$D=2$

Let $x=0$:

$0-1=C\times(-1)+D$

$-1=-C+2$

$C=3$

$$\frac{8x^3+1}{4x^2-4x+1} \equiv 2x+2+\frac{6x-1}{(2x-1)^2}$$

$$\equiv 2x+2+\frac{3}{2x-1}+\frac{2}{(2x-1)^2}$$

So $A=2$, $B=2$, $C=3$ and $D=2$.

10
$$\begin{array}{r} x^2 - x + 5 \\ x^2 + x - 2 \overline{)\, x^4 + 0x^3 + 2x^2 - 3x + 8} \\ \underline{x^4 + x^3 - 2x^2} \\ -x^3 + 4x^2 - 3x \\ \underline{-x^3 - x^2 + 2x} \\ 5x^2 - 5x + 8 \\ \underline{5x^2 + 5x - 10} \\ -10x + 18 \end{array}$$

Therefore $\dfrac{x^4 + 2x^2 - 3x + 8}{x^2 + x - 2} \equiv x^2 - x + 5 + \dfrac{-10x + 18}{x^2 + x - 2}$

$$\equiv x^2 - x + 5 + \frac{-10x + 18}{(x+2)(x-1)}$$

Let $\dfrac{-10x + 18}{(x+2)(x-1)} \equiv \dfrac{D}{x+2} + \dfrac{E}{x-1}$

$$\equiv \frac{D(x-1) + E(x+2)}{(x+2)(x-1)}$$

$$-10x + 18 = D(x-1) + E(x+2)$$

Let $x = -2$:

$20 + 18 = D \times (-3) + 0$

$38 = -3D$

$D = -\dfrac{38}{3}$

Let $x = 1$:

$-10 + 18 = 0 + E \times 3$

$8 = 3E$

$E = \dfrac{8}{3}$

$$\frac{x^4 + 2x^2 - 3x + 8}{x^2 + x - 2} \equiv x^2 - x + 5 + \frac{-10x + 18}{(x+2)(x-1)}$$

$$\equiv x^2 - x + 5 - \frac{38}{3(x+2)} + \frac{8}{3(x-1)}$$

So $A = 1$, $B = -1$, $C = 5$, $D = -\dfrac{38}{3}$ and $E = \dfrac{8}{3}$.