Algebraic methods Mixed exercise 1

1 Assumption: $\sqrt{\frac{1}{2}}$ is a rational number.

Then $\sqrt{\frac{1}{2}} = \frac{a}{b}$ for some integers a and b.
There is a further assumption that this fraction is in its simplest terms: there are no common factors between a and b.

So $0.5 = \frac{a^2}{b^2}$ or $2a^2 = b^2$.
Therefore b^2 must be a multiple of 2.
We know that this means b must also be a multiple of 2.
Write $b = 2c$, which means $b^2 = (2c)^2 = 4c^2$.
Now $4c^2 = 2a^2$, or $2c^2 = a^2$.
Therefore a^2 must be a multiple of 2, which implies a is also a multiple of 2.
If a and b are both multiples of 2, this contradicts the statement that there are no common factors between a and b.

Therefore, $\sqrt{\frac{1}{2}}$ is an irrational number.

2 Assumption: There exists a rational number q where q^2 is irrational

So write $q = \frac{a}{b}$, where a and b are integers.

$q^2 = \frac{a^2}{b^2}$

As a and b are integers, a^2 and b^2 are integers.
So q^2 is rational.
This contradicts assumption that q^2 is irrational.
Therefore if q^2 is irrational then q is irrational.

3 **a**

$$\frac{x-4}{6}\times\frac{2x+8}{x^2-16} = \frac{x-4}{6}\times\frac{2(x+4)}{(x-4)(x+4)}$$
$$= \frac{1}{3}$$

b

$$\frac{x^2-3x-10}{3x^2-21}\times\frac{6x^2+24}{x^2+6x+8} = \frac{(x-5)(x+2)}{3(x^2-7)}\times\frac{6(x^2+4)}{(x+2)(x+4)}$$
$$= \frac{2(x^2+4)(x-5)}{(x^2-7)(x+4)}$$

3 c $$\frac{4x^2+12x+9}{x^2+6x} \div \frac{4x^2-9}{2x^2+9x-18} = \frac{4x^2+12x+9}{x^2+6x} \times \frac{2x^2+9x-18}{4x^2-9}$$
$$= \frac{(2x+3)^2}{x(x+6)} \times \frac{(2x-3)(x+6)}{(2x-3)(2x+3)}$$
$$= \frac{2x+3}{x}$$

4 a $$\frac{4x^2-8x}{x^2-3x-4} \times \frac{x^2+6x+5}{2x^2+10x} = \frac{4x(x-2)}{(x-4)(x+1)} \times \frac{(x+1)(x+5)}{2x(x+5)}$$
$$= \frac{2(x-2)}{x-4}$$
$$= \frac{2x-4}{x-4}$$

b $$6 = \ln\left(\left(4x^2-8x\right)\left(x^2+6x+5\right)\right) - \ln\left(\left(x^2-3x-4\right)\left(2x^2+10x\right)\right)$$
$$= \ln\left(\frac{\left(4x^2-8x\right)\left(x^2+6x+5\right)}{\left(x^2-3x-4\right)\left(2x^2+10x\right)}\right)$$
$$= \ln\left(\frac{4x\left(x-2\right)\left(x+1\right)\left(x+5\right)}{\left(x-4\right)\left(x+1\right)2x\left(x+5\right)}\right)$$
$$= \ln\left(\frac{2x-4}{x-4}\right)$$

$$\frac{2x-4}{x-4} = e^6$$
$$2x-4 = xe^6 - 4e^6$$
$$4e^6 - 4 = xe^6 - 2x$$
$$4(e^6-1) = x(e^6-2)$$
$$x = \frac{4(e^6-1)}{e^6-2}$$

5 a
$$\begin{aligned}g(x) &= \frac{4x^3-9x^2-9x}{32x+24} \div \frac{x^2-3x}{6x^2-13x-5} \\ &= \frac{4x^3-9x^2-9x}{32x+24} \times \frac{6x^2-13x-5}{x^2-3x} \\ &= \frac{x(4x+3)(x-3)}{8(4x+3)} \times \frac{(3x+1)(2x-5)}{x(x-3)} \\ &= \frac{(3x+1)(2x-5)}{8} \\ &= \frac{6x^2-13x-5}{8} \\ &= \frac{3}{4}x^2 - \frac{13}{8}x - \frac{5}{8}\end{aligned}$$

$a = \frac{3}{4}$, $b = -\frac{13}{8}$, $c = -\frac{5}{8}$

b
$$\begin{aligned}g'(x) &= \frac{3}{2}x - \frac{13}{8} \\ g'(-2) &= \frac{3}{2}(-2) - \frac{13}{8} \\ &= -\frac{37}{8}\end{aligned}$$

6
$$\begin{aligned}\frac{6x+1}{x-5} + \frac{5x+3}{x^2-3x-10} &= \frac{6x+1}{x-5} + \frac{5x+3}{(x-5)(x+2)} \\ &= \frac{(6x+1)(x+2)}{(x-5)(x+2)} + \frac{5x+3}{(x-5)(x+2)} \\ &= \frac{6x^2+13x+2}{(x-5)(x+2)} + \frac{5x+3}{(x-5)(x+2)} \\ &= \frac{6x^2+13x+2+5x+3}{(x-5)(x+2)} \\ &= \frac{6x^2+18x+5}{x^2-3x-10}\end{aligned}$$

7 $x+\frac{3}{x-1}-\frac{12}{x^2+2x-3}$

$$=\frac{x(x+3)(x-1)}{(x+3)(x-1)}+\frac{3(x+3)}{(x+3)(x-1)}-\frac{12}{(x+3)(x-1)}$$

$$=\frac{x(x+3)(x-1)+3x-3}{(x+3)(x-1)}$$

$$=\frac{(x-1)\left[x(x+3)+3\right]}{(x+3)(x-1)}$$

$$=\frac{x^2+3x+3}{x+3}$$

8 $\frac{x-3}{x(x-1)}\equiv\frac{A}{x}+\frac{B}{x-1}$

$$\equiv\frac{A(x-1)+Bx}{x(x-1)}$$

$$x-3\equiv A(x-1)+Bx$$

Let $x = 0$:

$0-3 = A\times(-1)+0$

$-3 = -A$

$A = 3$

Let $x = 1$:

$1-3 = 0+B\times 1$

$B = -2$

$$\frac{x-3}{x(x-1)}\equiv\frac{3}{x}-\frac{2}{x-1}$$

$A = 3$, $B = -2$

9 $\frac{-15x+21}{(x-2)(x+1)(x-5)} \equiv \frac{P}{x-2}+\frac{Q}{x+1}+\frac{R}{x-5}$

$$\equiv \frac{P(x+1)(x-5)+Q(x-2)(x-5)+R(x-2)(x+1)}{(x-2)(x+1)(x-5)}$$

$$-15x+21 \equiv P(x+1)(x-5)+Q(x-2)(x-5)+R(x-2)(x+1)$$

Let $x = 2$:

$-30 + 21 = P \times 3 \times (-3) + 0 + 0$

$-9 = -9P$

$P = 1$

Let $x = -1$:

$15 + 21 = 0 + Q \times (-3) \times (-6) + 0$

$36 = 18Q$

$Q = 2$

Let $x = 5$:

$-75 + 21 = 0 + 0 + R \times 3 \times 6$

$-54 = 18R$

$R = -3$

$P = 1$, $Q = 2$ and $R = -3$

10 $\frac{16x-1}{(3x+2)(2x-1)} \equiv \frac{D}{3x+2}+\frac{E}{2x-1}$

$$\equiv \frac{D(2x-1)+E(3x+2)}{(3x+2)(2x-1)}$$

$$16x-1 \equiv D(2x-1)+E(3x+2)$$

Let $x = -\frac{2}{3}$:

$-\frac{32}{3} - 1 = D \times \left(-\frac{7}{3}\right) + 0$

$-\frac{35}{3} = -\frac{7}{3}D$

$D = 5$

Let $x = \frac{1}{2}$:

$8 - 1 = 0 + E \times \left(\frac{7}{2}\right)$

$7 = \frac{7}{2}E$

$E = 2$

10 (continued)

$$\frac{16x-1}{(3x+2)(2x-1)} \equiv \frac{5}{3x+2} + \frac{2}{2x-1}$$

$D = 5$, $E = 2$

11 $$\frac{7x^2+2x-2}{x^2(x+1)} \equiv \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1}$$

$$\equiv \frac{Ax(x+1)+B(x+1)+Cx^2}{x^2(x+1)}$$

$$7x^2+2x-2 \equiv Ax(x+1)+B(x+1)+Cx^2$$

Let $x = 0$:

$0 + 0 - 2 = 0 + B \times 1 + 0$

$B = -2$

Let $x = -1$:

$7 - 2 - 2 = 0 + 0 + C \times (-1)^2$

$C = 3$

Compare terms in x^2:

$7 = A + C$

$7 = A + 3$

$A = 4$

$A = 4$, $B = -2$ and $C = 3$

12 $$\frac{21x^2-13}{(x+5)(3x-1)^2} \equiv \frac{D}{x+5} + \frac{E}{(3x-1)} + \frac{F}{(3x-1)^2}$$

$$\equiv \frac{D(3x-1)^2+E(x+5)(3x-1)+F(x+5)}{(x+5)(3x-1)^2}$$

$$21x^2-13 \equiv D(3x-1)^2+E(x+5)(3x-1)+F(x+5)$$

Let $x = -5$:

$525 - 13 = D \times (-16)^2 + 0 + 0$

$512 = 256D$

$D = 2$

Let $x = \frac{1}{3}$:

$\frac{7}{3} - 13 = 0 + 0 + F \times \frac{16}{3}$

$-\frac{32}{3} = \frac{16}{3}F$

$F = -2$

Compare terms in x^2:

12 (continued)

$21 = 9D + 3E$

$21 = 18 + 3E$

$E = 1$

$$\frac{21x^2-13}{(x+5)(3x-1)^2} \equiv \frac{2}{x+5} + \frac{1}{(3x-1)} - \frac{2}{(3x-1)^2}$$

$D = 2$, $E = 1$, $F = -2$

13

$$\begin{array}{r} x^2 - 4x + 3 \\ x-2 \overline{\smash{)}\, x^3 - 6x^2 + 11x + 2} \\ \underline{x^3 - 2x^2} \\ -4x^2 + 11x \\ \underline{-4x^2 + 8x} \\ 3x + 2 \\ \underline{3x - 6} \\ 8 \end{array}$$

$$x^3 - 6x^2 + 11x + 2 \equiv (x-2)(x^2-4x+3) + 8$$

$A = 1$, $B = -4$, $C = 3$ and $D = 8$

14

$$\begin{array}{r} 2x^2 - 4x + 6 \\ 2x+1 \overline{\smash{)}\, 4x^3 - 6x^2 + 8x - 5} \\ \underline{4x^3 + 2x^2} \\ -8x^2 + 8x \\ \underline{-8x^2 - 4x} \\ 12x - 5 \\ \underline{12x + 6} \\ -11 \end{array}$$

$$\frac{4x^3 - 6x^2 + 8x - 5}{2x+1} \equiv 2x^2 - 4x + 6 - \frac{11}{2x+1}$$

$A = 2$, $B = -4$, $C = 6$ and $D = -11$

15

$$\begin{array}{r} x^2+1 \\ x^2-1 \overline{)\, x^4+0x^2+2} \\ \underline{x^4-x^2} \\ x^2+\ 2 \\ \underline{x^2-\ 1} \\ 3 \end{array}$$

$$\frac{x^4+2}{x^2-1} \equiv x^2+1+\frac{3}{x^2-1}$$

So $A = 1$, $B = 0$, $C = 1$, and $D = 3$.

16

$$\begin{array}{r} x^2+2x+3 \\ x^2-2x+1 \overline{)\, x^4+0x^3+0x^2+0x+0} \\ \underline{x^4-2x^3+x^2} \\ 2x^3-x^2+0x \\ \underline{2x^3-4x^2+2x} \\ 3x^2-2x+0 \\ \underline{3x^2-6x+3} \\ 4x-3 \end{array}$$

$$\frac{x^4}{x^2-2x+1} \equiv x^2+2x+3+\frac{4x-3}{x^2-2x+1}$$

$$\equiv x^2+2x+3+\frac{4x-3}{(x-1)^2}$$

Let $\frac{4x-3}{(x-1)^2} \equiv \frac{D}{x-1}+\frac{E}{(x-1)^2}$

$$\equiv \frac{D(x-1)+E}{(x-1)^2}$$

$$4x-3 = D(x-1)+E$$

Let $x = 1$:
$4 - 3 = E$
$E = 1$

Let $x = 0$:
$-3 = D \times (-1) + E$
$-3 = -D + 1$
$D = 4$

$$\frac{x^4}{x^2-2x+1} \equiv x^2+2x+3+\frac{4x-3}{(x-1)^2}$$

$$\equiv x^2+2x+3+\frac{4}{x-1}+\frac{1}{(x-1)^2}$$

$A = 1$, $B = 2$, $C = 3$, $D = 4$ and $E = 1$

17 Using algebraic long division:

$$\begin{array}{r} 2 \\ x^2+2x-3\overline{)2x^2+2x-3} \\ \underline{2x^2+4x-6} \\ -2x+3 \end{array}$$

$$\frac{2x^2+2x-3}{x^2+2x-3} \equiv 2+\frac{-2x+3}{x^2+2x-3}$$

$$\equiv 2+\frac{-2x+3}{(x+3)(x-1)}$$

Let $\frac{-2x+3}{(x+3)(x-1)} \equiv \frac{B}{x+3}+\frac{C}{x-1}$

$$\equiv \frac{B(x-1)+C(x+3)}{x^2+2x-3}$$

$$-2x+3 \equiv B(x-1)+C(x+3)$$

Let $x = 1$:

$-2 + 3 = 0 + C \times 4$

$C = \frac{1}{4}$

Let $x = -3$:

$6 + 3 = B \times (-4) + 0$

$9 = -4B$

$B = -\frac{9}{4}$

$A = 2$, $B = -\frac{9}{4}$ and $C = \frac{1}{4}$.

18

$$\begin{array}{r} 1 \\ x^2-2x\overline{)x^2+0x+1} \\ \underline{x^2-2x} \\ 2x+1 \end{array}$$

$$\frac{x^2+1}{x(x-2)} \equiv 1+\frac{2x+1}{x(x-2)}$$

Let $\frac{2x+1}{x(x-2)} \equiv \frac{Q}{x}+\frac{R}{x-2}$

$$\equiv \frac{Q(x-2)+Rx}{x(x-2)}$$

$$2x+1 = Q(x-2)+Rx$$

18 (continued)

Let $x = 0$:

$$0 + 1 = Q \times (-2) + 0$$
$$1 = -2Q$$
$$Q = -\frac{1}{2}$$

Let $x = 2$:

$$4 + 1 = 0 + R \times 2$$
$$5 = 2R$$
$$R = \frac{5}{2}$$

$P = 1$, $Q = -\frac{1}{2}$ and $R = \frac{5}{2}$

19 a
$$\begin{aligned} f(-3) &= 2\times(-27)+9\times9+10\times(-3)+3 \\ &= -54+81-30+3 \\ &= 0 \end{aligned}$$

$f(-3)=0$ so $x=-3$ is a root of $f(x)$

OR

$$\begin{array}{r|l} & 2x^2 + 3x + 1 \\ x+3 & 2x^3 + 9x^2 + 10x + 3 \\ & \underline{2x^3 + 6x^2} \\ & 3x^2 + 10x \\ & \underline{3x^2 + 9x} \\ & x+3 \\ & \underline{x+3} \\ & 0 \end{array}$$

$(x+3)$ is a factor of $f(x)$, so by the factor theorem $x=-3$ is a root of $f(x)$

b
$$\begin{aligned} \frac{10}{f(x)} &\equiv \frac{10}{2x^3+9x^2+10x+3} \\ &\equiv \frac{10}{(x+3)(2x^2+3x+1)} \quad \text{[by part } \mathbf{a}\text{]} \\ &\equiv \frac{10}{(x+3)(2x+1)(x+1)} \\ &\equiv \frac{A}{(x+3)} + \frac{B}{(2x+1)} + \frac{C}{(x+1)} \\ &\equiv \frac{A(2x+1)(x+1)+B(x+3)(x+1)+C(x+3)(2x+1)}{(x+3)(2x+1)(x+1)} \end{aligned}$$

19 b (continued)

$$10 \equiv A(2x+1)(x+1) + B(x+3)(x+1) + C(x+3)(2x+1)$$

Let $x = -1$:

$$10 = A\times 0 + B\times 0 + C\times 2\times(-1)$$
$$10 = -2C$$
$$C = -5$$

Let $x = -3$:

$$10 = A\times(-5)\times(-2) + B\times 0 + C\times 0$$
$$10 = 10A$$
$$A = 1$$

Let $x = -\frac{1}{2}$:

$$10 = A\times 0 + B\times\left(\frac{5}{2}\right)\times\left(\frac{1}{2}\right) + C\times 0$$
$$10 = \frac{5}{4}B$$
$$B = 8$$

Hence $\frac{10}{\mathrm{f}(x)} \equiv \frac{1}{(x+3)} + \frac{8}{(2x+1)} - \frac{5}{(x+1)}$

Challenge

Assumption: *L* is not perpendicular to *OA*.
Draw the line through *O* which is perpendicular to *L*.
This line meets *L* at a point *B*, outside the circle.
Triangle *OBA* is right-angled at *B*, so *OA* is the hypotenuse of this triangle, so $OA > OB$.
This gives a contradiction, as *B* is outside the circle, so $OA < OB$.
Therefore *L* is perpendicular to *OA*.