Functions and graphs 2D

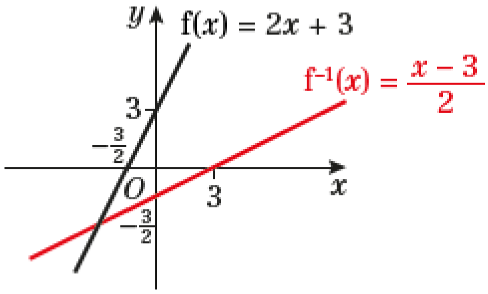
1 a i $y \in \mathbb{R}$

ii Let $y = \mathrm{f}(x)$

$$y = 2x + 3$$
$$x = \frac{y-3}{2}$$
$$\mathrm{f}^{-1}(x) = \frac{x-3}{2}$$

iii The domain of $\mathrm{f}^{-1}(x)$ is $x \in \mathbb{R}$
The range of $\mathrm{f}^{-1}(x)$ is $y \in \mathbb{R}$

iv



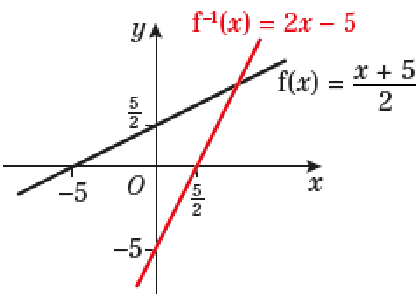
b i $y \in \mathbb{R}$

ii Let $y = \mathrm{f}(x)$

$$y = \frac{x+5}{2}$$
$$x = 2y - 5$$
$$\mathrm{f}^{-1}(x) = 2x - 5$$

iii The domain of $\mathrm{f}^{-1}(x)$ is $x \in \mathbb{R}$
The range of $\mathrm{f}^{-1}(x)$ is $y \in \mathbb{R}$

iv



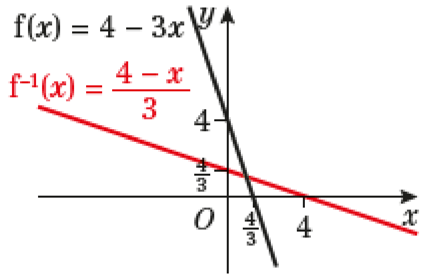
c i $y \in \mathbb{R}$

ii Let $y = \mathrm{f}(x)$

$$y = 4 - 3x$$
$$x = \frac{4-y}{3}$$
$$\mathrm{f}^{-1}(x) = \frac{4-x}{3}$$

iii The domain of $\mathrm{f}^{-1}(x)$ is $x \in \mathbb{R}$
The range of $\mathrm{f}^{-1}(x)$ is $y \in \mathbb{R}$

iv



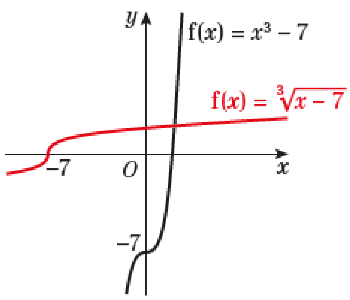
d i $y \in \mathbb{R}$

ii Let $y = \mathrm{f}(x)$

$$y = x^3 - 7$$
$$x = \sqrt[3]{y+7}$$
$$\mathrm{f}^{-1}(x) = \sqrt[3]{x+7}$$

iii The domain of $\mathrm{f}^{-1}(x)$ is $x \in \mathbb{R}$
The range of $\mathrm{f}^{-1}(x)$ is $y \in \mathbb{R}$

iv



2 a Range of f is $\mathrm{f}(x) \in \mathbb{R}$

Let $y = \mathrm{f}(x)$

$$y = 10 - x$$
$$x = 10 - y$$
$$\mathrm{f}^{-1}(x) = 10 - x, \ \{x \in \mathbb{R}\}$$

2 b Range of f is $f(x) \in \mathbb{R}$

Let $y = g(x)$

$y = \frac{x}{5}$

$x = 5y$

$g^{-1}(x) = 5x, \; \{x \in \mathbb{R}\}$

c Range of f is $f(x) \neq 0$

Let $y = h(x)$

$y = \frac{3}{x}$

$x = \frac{3}{y}$

$h^{-1}(x) = \frac{3}{x}, \; \{x \neq 0\}$

d Range of f is $f(x) \in \mathbb{R}$

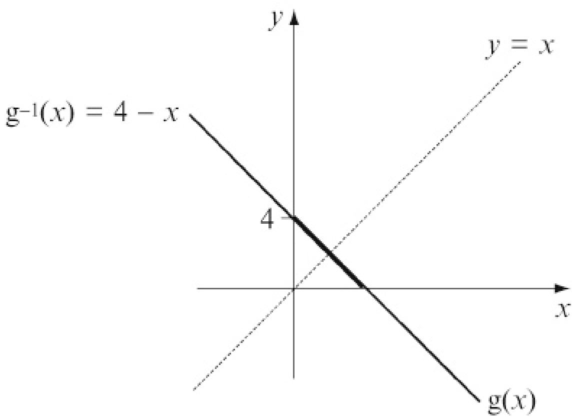
Let $y = k(x)$

$y = x - 8$

$x = y + 8$

$k^{-1}(x) = y + 8, \; \{x \in \mathbb{R}\}$

3



$g : x \mapsto 4 - x, \; \{x \in \mathbb{R}, \; x > 0\}$

g has range $\{g(x) \in \mathbb{R}, \; g(x) < 4\}$

The inverse function is $g^{-1}(x) = 4 - x$

Now $\{\text{Range g}\} = \{\text{Domain } g^{-1}\}$

and $\{\text{Domain g}\} = \{\text{Range g}\}$

Hence, $g^{-1}(x) = 4 - x, \; \{x \in \mathbb{R}, x < 4\}$

Although $g(x)$ and $g^{-1}(x)$ have identical equations, their domains and hence ranges are different, and so are not identical.

4 a i Maximum value of g when $x = \frac{1}{3}$

Hence $\left\{g(x) \in \mathbb{R}, \; 0 < g(x) \leq \frac{1}{3}\right\}$

ii $g^{-1}(x) = \frac{1}{x}$

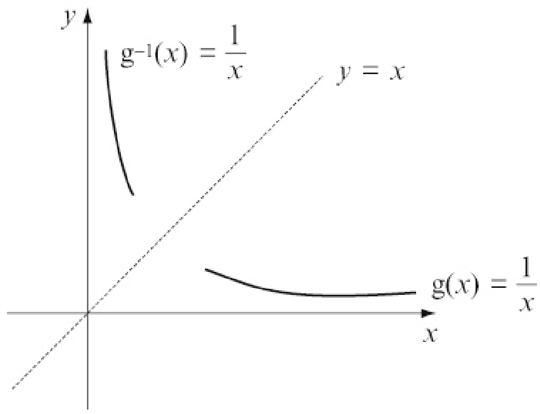
iii Domain g^{-1} = Range g

$\Rightarrow$ Domain $g^{-1} : \left\{x \in \mathbb{R}, 0 < x \leq \frac{1}{3}\right\}$

Range g^{-1} = Domain g

$\Rightarrow$ Range $g^{-1}(x) : \left\{g^{-1}(x) \in \mathbb{R}, \; g^{-1}(x) \geq 3\right\}$

iv



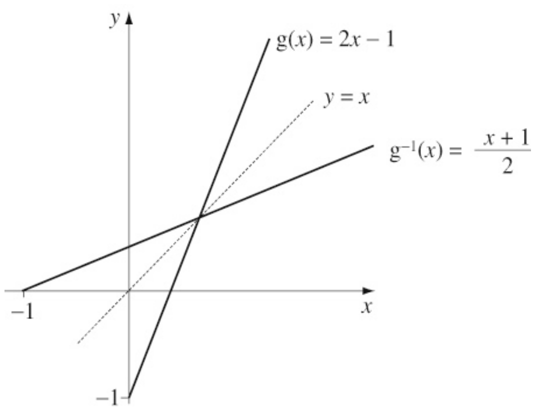
4 b i Minimum value of $g(x) = -1$
when $x = 0$
Hence $\{g(x) \in \mathbb{R},\ g(x) \geq -1\}$

ii Letting $y = 2x - 1 \Rightarrow x = \dfrac{y+1}{2}$

Hence $g^{-1}(x) = \dfrac{x+1}{2}$

iii Domain g^{-1} = Range g
$\Rightarrow$ Domain $g^{-1}: \{x \in \mathbb{R},\ x \geq -1\}$
Range g^{-1} = Domain g
$\Rightarrow$ Range $g^{-1}(x): \left\{ g^{-1}(x) \in \mathbb{R},\ g^{-1}(x) \geq 0 \right\}$

iv



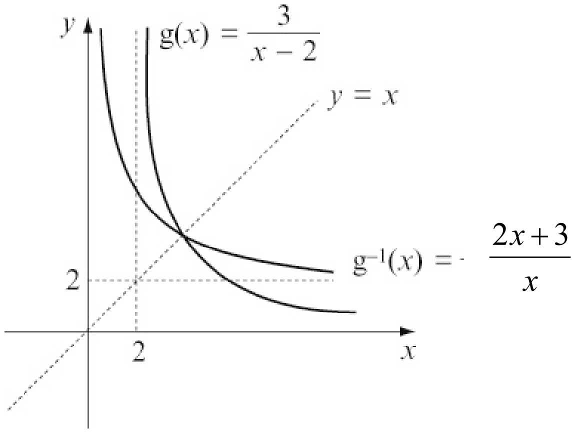
c i $g(x) \to +\infty$ as $x \to 2$
Hence $\{g(x) \in \mathbb{R},\ g(x) > 0\}$

ii Letting $y = \dfrac{3}{x-2} \Rightarrow x = \dfrac{2y+3}{y}$

Hence $g^{-1}(x) = \dfrac{2x+3}{x}$

iii Domain g^{-1} = Range g
$\Rightarrow$ Domain $g^{-1}: \{x \in \mathbb{R},\ x > 0\}$
Range g^{-1} = Domain g
$\Rightarrow$ Range $g^{-1}(x): \{g^{-1}(x) \in \mathbb{R},\ g^{-1}(x) > 2\}$

iv

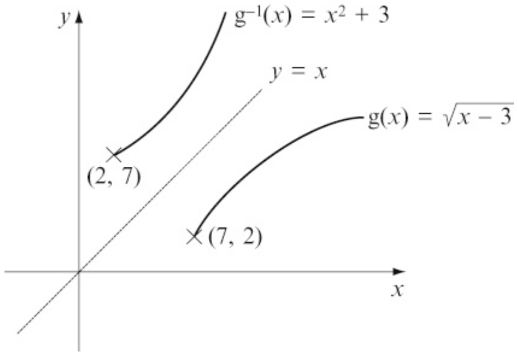


d i Minimum value of $g(x) = 2$
when $x = 7$
Hence $\{g(x) \in \mathbb{R},\ g(x) \geq 2\}$

ii Letting $y = \sqrt{x-3} \Rightarrow x = y^2 + 3$
Hence $g^{-1}(x) = x^2 + 3$

iii Domain g^{-1} = Range g
$\Rightarrow$ Domain $g^{-1}: \{x \in \mathbb{R},\ x \geq 2\}$
Range g^{-1} = Domain g
$\Rightarrow$ Range $g^{-1}(x): \{g^{-1}(x) \in \mathbb{R},\ g^{-1}(x) \geq 7\}$

iv



4 e i $2^2 + 2 = 6$

Hence $\{g(x) \in \mathbb{R},\ g(x) > 6\}$

ii Letting $y = x^2 + 2$

$y - 2 = x^2$

$x = \sqrt{y-2}$

Hence $g^{-1}(x) = \sqrt{x-2}$

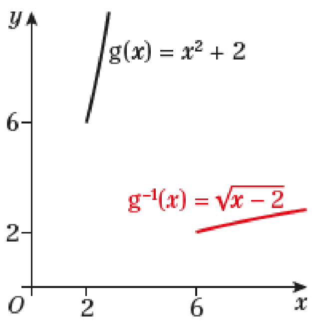
iii Domain g^{-1} = Range g

$\Rightarrow$ Domain $g^{-1}: \{x \in \mathbb{R}, x > 6\}$

Range g^{-1} = Domain g

$\Rightarrow$ Range $g^{-1}(x): \{g^{-1}(x) \in \mathbb{R},\ g^{-1}(x) > 2\}$

iv



f i Minimum value of $g(x) = 0$ when $x = 2$

Hence $\{g(x) \in \mathbb{R},\ g(x) \geq 0\}$

ii Letting $y = x^3 - 8 \Rightarrow x = \sqrt[3]{y+8}$

Hence $g^{-1}(x) = \sqrt[3]{x+8}$

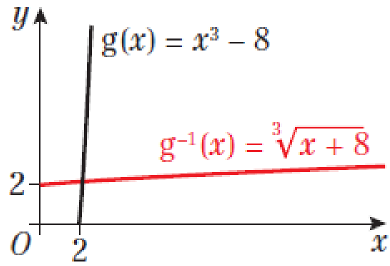
iii Domain g^{-1} = Range g

$\Rightarrow$ Domain $g^{-1}: \{x \in \mathbb{R}, x \geq 0\}$

Range g^{-1} = Domain g

$\Rightarrow$ Range $g^{-1}(x): \{g^{-1}(x) \in \mathbb{R},\ g^{-1}(x) \geq 2\}$

iv



5 $t(x) = x^2 - 6x + 5,\ \{x \in \mathbb{R},\ x \geq 5\}$

Let $y = x^2 - 6x + 5$

$y = (x-3)^2 - 9 + 5$ (completing the square)

$y = (x-3)^2 - 4$

This has a minimum point at $(3, -4)$

For the domain $x \geq 5$, $t(x)$ is a one-to-one function so we can find an inverse function.

Make y the subject:

$y = (x-3)^2 - 4$

$y + 4 = (x-3)^2$

$\sqrt{y+4} = x - 3$

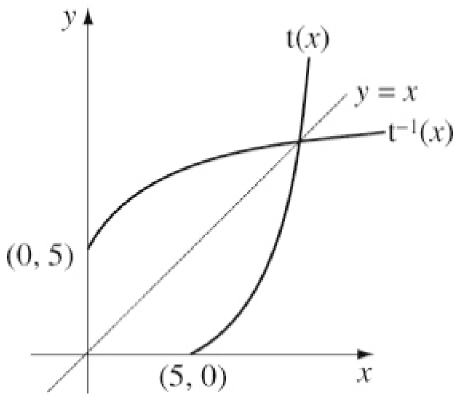
$\sqrt{y+4} + 3 = x$

5 (continued)

Domain t^{-1} = Range t

$\Rightarrow$ Domain $g^{-1}: \{x \in \mathbb{R}, x \geq 0\}$

Hence, $t^{-1}(x) = \sqrt{x+4} + 3, \ \{x \in \mathbb{R}, x \geq 0\}$



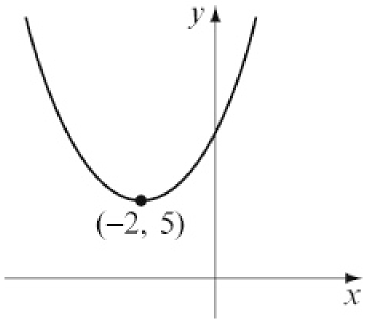
6 a $m(x) = x^2 + 4x + 9, \quad \{x \in \mathbb{R}, \ x > a\}$

Let $y = x^2 + 4x + 9$

$y = (x+2)^2 - 4 + 9$

$y = (x+2)^2 + 5$

This has a minimum value of $(-2, 5)$



For $m(x)$ to have an inverse it must be one-to-one. Hence the least value of a is -2

b Changing the subject of the formula:

$$y = (x+2)^2 + 5$$

$$y - 5 = (x+2)^2$$

$$\sqrt{y-5} = x + 2$$

$$\sqrt{y-5} - 2 = x$$

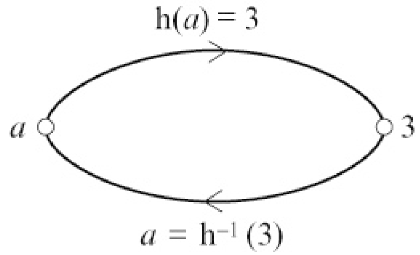
Hence $m^{-1}(x) = \sqrt{x-5} - 2$

c Domain of $m^{-1}(x)$: $\{x \in \mathbb{R}, \ x > 5\}$

7 a As $x \to 2, \ h(x) \to \dfrac{5}{0}$

and hence $h(x) \to \infty$

b To find $h^{-1}(3)$ we can find what element of the domain gets mapped to 3



Suppose $h(a) = 3$ for some a such that $a \neq 2$

Then $\dfrac{2a+1}{a-2} = 3$

$$2a + 1 = 3a - 6$$

$$7 = a$$

So $h^{-1}(3) = 7$

7 c Let $y = \frac{2x+1}{x-2}$ and find x as a function of y

$$y(x-2) = 2x+1$$
$$yx - 2y = 2x+1$$
$$yx - 2x = 2y+1$$
$$x(y-2) = 2y+1$$
$$x = \frac{2y+1}{y-2}$$

So $\mathrm{h}^{-1}(x) = \frac{2x+1}{x-2}, \quad \{x \in \mathbb{R},\ x \neq 2\}$

d If an element b is mapped to itself, then $\mathrm{h}(b) = b$

$$\frac{2b+1}{b-2} = b$$
$$2b+1 = b(b-2)$$
$$2b+1 = b^2 - 2b$$
$$0 = b^2 - 4b - 1$$
$$b = \frac{4 \pm \sqrt{16+4}}{2} = \frac{4 \pm \sqrt{20}}{2}$$
$$= \frac{4 \pm 2\sqrt{5}}{2} = 2 \pm \sqrt{5}$$

The elements $2+\sqrt{5}$ and $2-\sqrt{5}$ get mapped to themselves by the function.

8 a $\mathrm{nm}(x) = \mathrm{n}(2x+3)$

$$= \frac{2x+3-3}{2}$$
$$= x$$

b $\mathrm{mn}(x) = \mathrm{m}\left(\frac{x-3}{2}\right)$

$$= 2\left(\frac{x-3}{2}\right) + 3$$
$$= x$$

The functions m(x) and n(x) are the inverse of each other as mn(x) = nm(x) = x.

9 $\mathrm{st}(x) = \mathrm{s}\left(\frac{3-x}{x}\right)$

$$= \frac{3}{\left(\frac{3-x}{x}+1\right)}$$
$$= \frac{3}{\left(\frac{3-x+x}{x}\right)}$$
$$= x$$

$\mathrm{st}(x) = \mathrm{t}\left(\frac{3}{x+1}\right)$

$$= \frac{\left(3 - \frac{3}{x+1}\right)}{\left(\frac{3}{x+1}\right)}$$
$$= \frac{\left(\frac{3x+3-3}{x+1}\right)}{\left(\frac{3}{x+1}\right)}$$
$$= x$$

The functions s(x) and t(x) are the inverse of each other as st(x) = ts(x) = x

10 a Let $y = 2x^2 - 3$

The domain of $\mathrm{f}^{-1}(x)$ is the range of $\mathrm{f}(x)$.

$\mathrm{f}(x) = 2x^2 - 3,\ \{x \in \mathbb{R},\ x < 0\}$ has range $\mathrm{f}(x) > -3$

Letting $y = 2x^2 - 3 \Rightarrow x = \pm\sqrt{\frac{x+3}{2}}$

We need to consider the domain of $\mathrm{f}(x)$ to determine if *either*

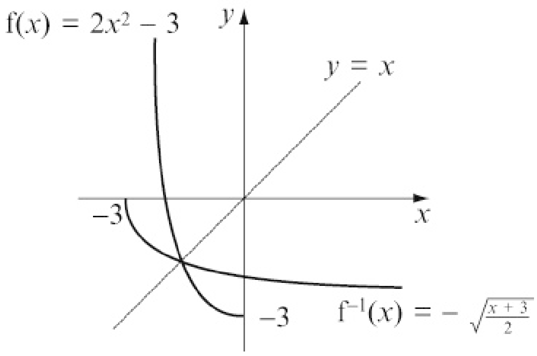
$$\mathrm{f}^{-1}(x) = +\sqrt{\frac{x+3}{2}} \text{ or } \mathrm{f}^{-1}(x) = -\sqrt{\frac{x+3}{2}}$$

$\mathrm{f}(x) = 2x^2 - 3$ has domain $\{x \in \mathbb{R},\ x < 0\}$

10 a (continued)

Hence $f^{-1}(x)$ must be the negative square root

$$f^{-1}(x) = -\sqrt{\frac{x+3}{2}}, \quad \{x \in \mathbb{R}, \ x > -3\}$$



b If $f(a) = f^{-1}(a)$ then a is negative (see graph).

Solve $f(a) = a$

$$2a^2 - 3 = a$$

$$2a^2 - a - 3 = 0$$

$$(2a-3)(a+1) = 0$$

$$a = \frac{3}{2}, -1$$

Therefore $a = -1$

11 a Range of $f(x)$ is $f(x) > -5$

b Let $y = f(x)$

$y = e^x - 5$

$e^x = y + 5$

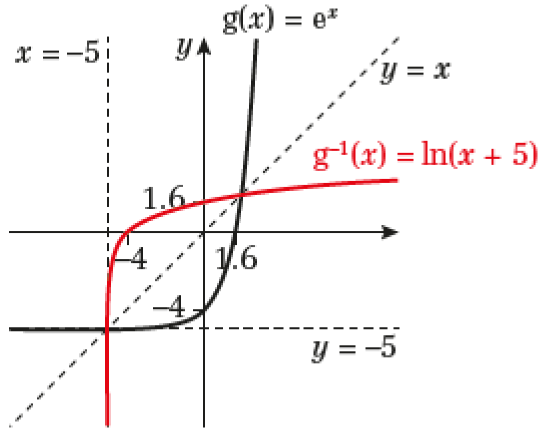
$x = \ln(y + 5)$

$f^{-1}(x) = \ln(x + 5)$

Range of $f(x)$ is $f(x) > -5$,

so domain of $f^{-1}(x)$ is $\{x \in \mathbb{R}, x > -5\}$

c



d Let $y = g(x)$

$y = \ln(x - 4)$

$e^y = x - 4$

$x = e^y + 4$

$g^{-1}(x) = e^x + 4$

Range of $g(x)$ is $g(x) \in \mathbb{R}$,

so domain of $g^{-1}(x)$ is $\{x \in \mathbb{R}\}$

e $g^{-1}(x) = 11$

$e^x + 4 = 11$

$e^x = 7$

$x = \ln 7$

$x = 1.95$

12 a
$$f(x) = \frac{3(x+2)}{x^2+x-20} - \frac{2}{x-4}$$
$$= \frac{3(x+2)}{(x+5)(x-4)} - \frac{2}{x-4}$$
$$= \frac{3(x+2)}{(x+5)(x-4)} - \frac{2(x+5)}{(x+5)(x-4)}$$
$$= \frac{3x+6-2x-10}{(x+5)(x-4)}$$
$$= \frac{x-4}{(x+5)(x-4)}$$
$$= \frac{1}{x+5}, \ x > 4$$

12 b The range of f is

$$\left\{ f(x) \in \mathbb{R},\ f(x) < \frac{1}{9} \right\}$$

c Let $y = f(x)$

$$y = \frac{1}{x+5}$$

$$yx + 5y = 1$$

$$yx = 1 - 5y$$

$$x = \frac{1-5y}{y}$$

$$x = \frac{1}{y} - 5$$

$$f^{-1}(x) = \frac{1}{x} - 5$$

The domain of $f^{-1}(x)$ is

$$\left\{ x \in \mathbb{R}, x < \frac{1}{9} \text{ and } x \neq 0 \right\}$$