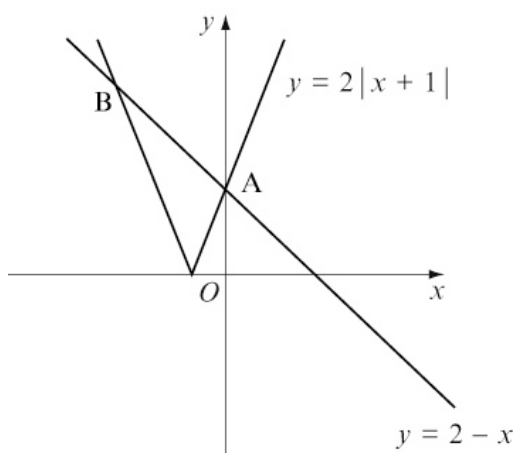
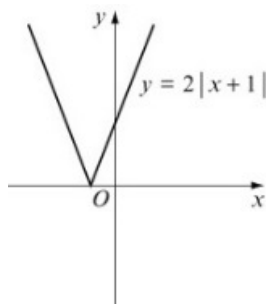
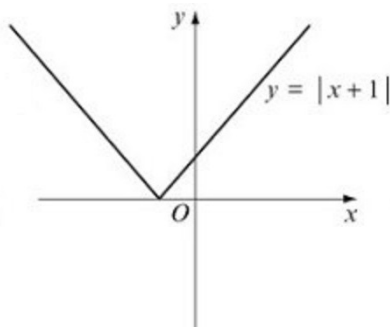
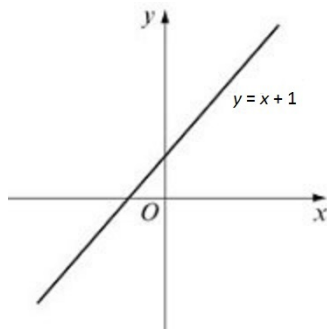


Functions and graphs Mixed exercise 2

1 a



b Intersection point A:

$$2(x+1) = 2-x$$

$$2x+2 = 2-x$$

$$3x = 0$$

$$x = 0$$

Intersection point B is on the reflected part of the modulus graph.

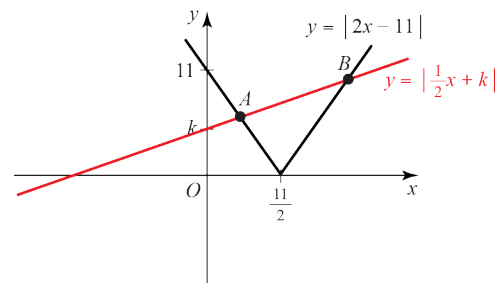
$$-2(x+1) = 2-x$$

$$-2x-2 = 2-x$$

$$-x = 4$$

$$x = -4$$

2



Minimum value of $y = |2x - 11|$ is

$$y = 0 \text{ at } x = \frac{11}{2}$$

For two distinct solutions to

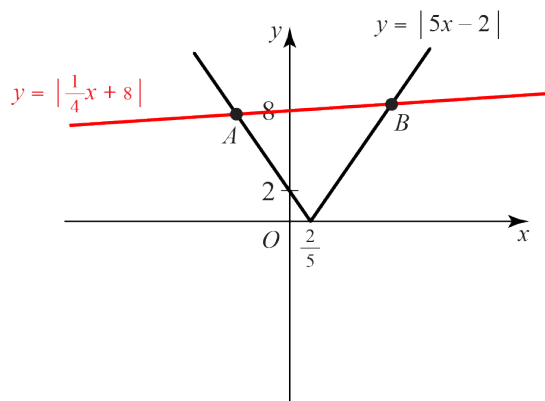
$$|2x - 11| = \frac{1}{2}x - k, \text{ we must have}$$

$$\frac{1}{2}x - k > 0 \text{ at } x = \frac{11}{2}$$

$$\frac{1}{2} \times \frac{11}{2} + k > 0$$

$$k > -\frac{11}{4}$$

3



At A:

$$\begin{aligned} -(5x - 2) &= -\frac{1}{4}x + 8 \\ -20x + 8 &= -x + 32 \\ -19x &= 24 \\ x &= -\frac{24}{19} \end{aligned}$$

At B:

$$\begin{aligned} 5x - 2 &= -\frac{1}{4}x + 8 \\ 20x - 8 &= -x + 32 \\ 21x &= 40 \\ x &= \frac{40}{21} \end{aligned}$$

So the solutions are

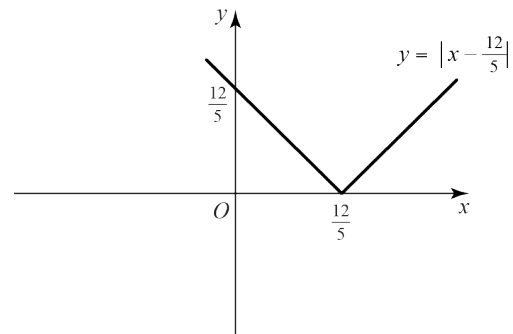
$$x = -\frac{24}{19} \text{ and } x = \frac{40}{21}$$

4 a $y = |12 - 5x| = 5 \left| -\left(x - \frac{12}{5}\right) \right|$

Start with $y = |x|$

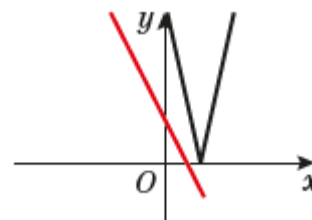
$y = \left|x - \frac{12}{5}\right|$ is a horizontal

translation of $+\frac{12}{5}$



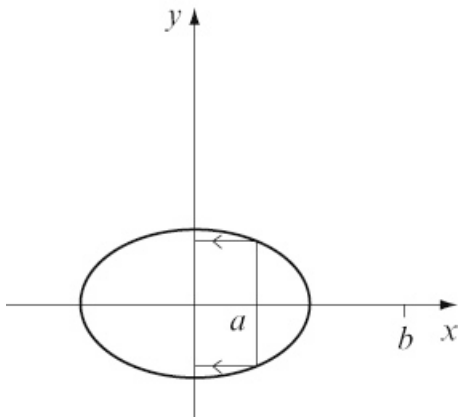
$y = 5 \left|x - \frac{12}{5}\right|$ is a vertical stretch,

scale factor 5



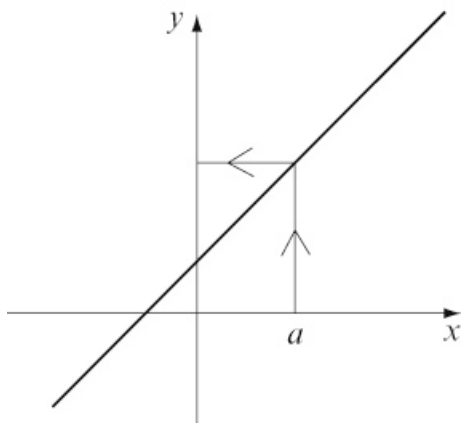
b The graphs do not intersect, so there are no solutions.

- 5 a i** One-to-many.
ii Not a function.

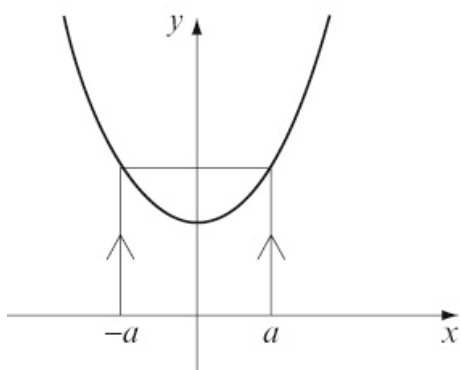


x value a gets mapped to two values of y .
 x value b gets mapped to no values.

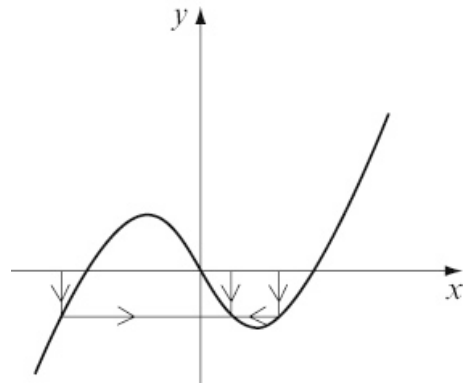
- b i** One-to-one.
ii Is a function.



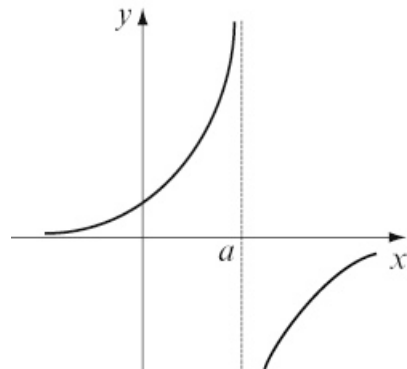
- c i** Many-to-one.
ii Is a function.



- d i** Many-to-one.
ii Is a function.

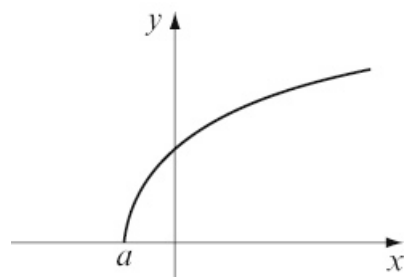


- 5 e i** One-to-one.
ii Not a function.



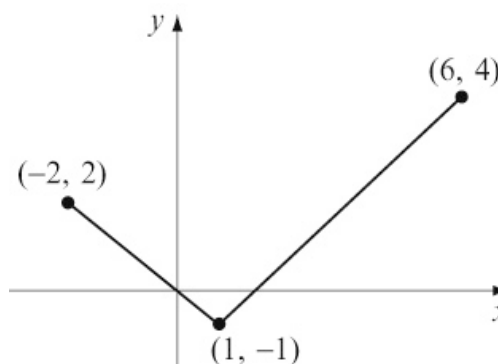
x value a doesn't get mapped to any value of y . It could be redefined as a function if the domain is said to exclude point a .

- f i** One-to-one.
ii Not a function for this domain.



x values less than a don't get mapped anywhere. Again, we could define the domain to be $x \geq a$ and then it would be a function.

6 a



For $x \leq 1$, $f(x) = -x$

This is a straight line of gradient -1 .

At point $x = 1$, its y -coordinate is -1 .

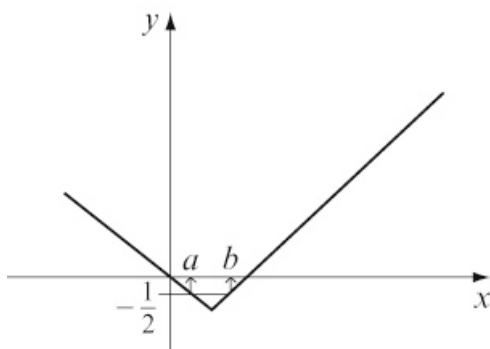
For $x > 1$, $f(x) = x - 2$

This is a straight line of gradient $+1$.

At point $x = 1$, its y -coordinate is also -1 .

Hence, the graph is said to be continuous.

- b There are two values x in the range $-2 \leq x \leq 6$ for which $f(x) = -\frac{1}{2}$



Point a is where

$$-x = -\frac{1}{2} \Rightarrow x = \frac{1}{2}$$

Point b is where

$$x - 2 = -\frac{1}{2} \Rightarrow x = 1\frac{1}{2}$$

Hence, the values of x for which

$$f(x) = -\frac{1}{2} \text{ are } x = \frac{1}{2} \text{ and } x = 1\frac{1}{2}$$

- 7 a $pq(x) = p(2x + 1)$

$$\begin{aligned} &= (2x + 1)^2 + 3(2x + 1) - 4 \\ &= 4x^2 + 4x + 1 + 6x + 3 - 4 \\ &= 4x^2 + 10x, \quad x \in \mathbb{R} \end{aligned}$$

- b $qq(x) = q(2x + 1)$

$$\begin{aligned} &= 2(2x + 1) + 1 \\ &= 4x + 3 \end{aligned}$$

$pq(x) = qq(x)$ gives

$$\begin{aligned} 4x^2 + 10x &= 4x + 3 \\ 4x^2 + 6x - 3 &= 0 \end{aligned}$$

Using the formula:

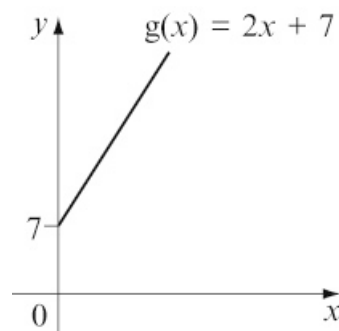
$$x = \frac{-6 \pm \sqrt{6^2 - 4 \times 4 \times (-3)}}{2 \times 4}$$

$$x = \frac{-6 \pm \sqrt{84}}{8}$$

$$x = \frac{-6 \pm 2\sqrt{21}}{8}$$

$$x = \frac{-3 \pm \sqrt{21}}{4}$$

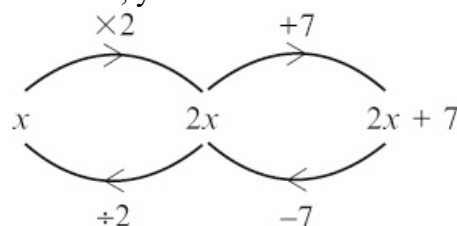
- 8 a $y = 2x + 7$ is a straight line with gradient 2 and y -intercept 7



For $x \geq 0$, the range is $g(x) \geq 7$

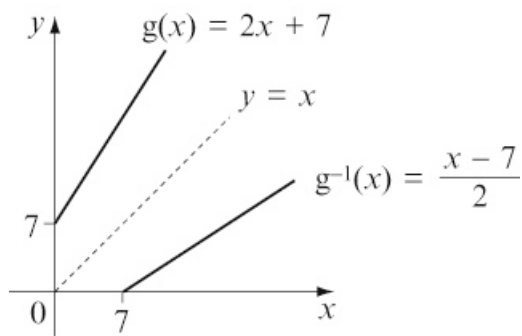
- b The range is $g^{-1}(x) \geq 0$.

To find the equation of the inverse function, you can use a flow chart.



$$g^{-1}(x) = \frac{x-7}{2} \text{ and has domain } x \geq 7$$

8 c



$g^{-1}(x)$ is the reflection of $g(x)$ in the line $y = x$.

9 a To find $f^{-1}(x)$, you can change the subject of the formula.

$$\begin{aligned} \text{Let } y &= \frac{2x+3}{x-1} \\ y(x-1) &= 2x+3 \\ yx - y &= 2x+3 \\ yx - 2x &= y+3 \\ x(y-2) &= y+3 \\ x &= \frac{y+3}{y-2} \end{aligned}$$

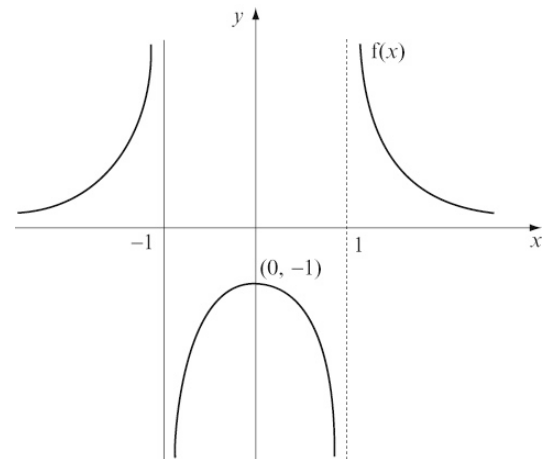
$$\text{Therefore } f^{-1}(x) = \frac{x+3}{x-2}, x \in \mathbb{R}, x > 2$$

b Domain $f(x) = \text{Range } f^{-1}(x)$
 $\therefore \text{Range } f^{-1}(x) = \{y \in \mathbb{R}, y > 1\}$

$$\begin{aligned} 10 \text{ a } f(x) &= \frac{x}{x^2-1} - \frac{1}{x+1} \\ &= \frac{x}{(x+1)(x-1)} - \frac{1}{(x+1)} \\ &= \frac{x}{(x+1)(x-1)} - \frac{x-1}{(x+1)(x-1)} \\ &= \frac{x-(x-1)}{(x+1)(x-1)} \\ &= \frac{1}{(x+1)(x-1)} \end{aligned}$$

b Consider the graph of

$$y = \frac{1}{(x-1)(x+1)} \text{ for } x \in \mathbb{R} :$$



For $x > 1$, $f(x) > 0$

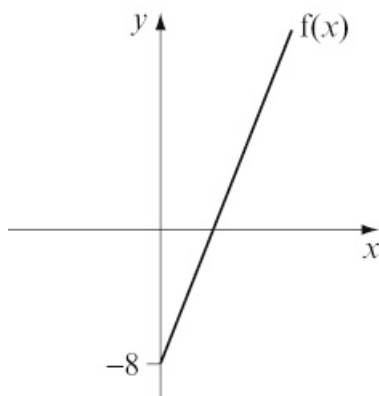
$$\begin{aligned} \text{c } gf(x) &= g\left(\frac{1}{(x-1)(x+1)}\right) \\ &= \frac{2}{\left(\frac{1}{(x-1)(x+1)}\right)} \\ &= 2 \times \frac{(x-1)(x+1)}{1} \\ &= 2(x-1)(x+1) \end{aligned}$$

$$\begin{aligned} gf(x) = 70 &\Rightarrow 2(x-1)(x+1) = 70 \\ (x-1)(x+1) &= 35 \\ x^2 - 1 &= 35 \\ x^2 &= 36 \\ x &= \pm 6 \end{aligned}$$

$$\begin{aligned}
 11 \text{ a } f(7) &= 4(7-2) \\
 &= 4 \times 5 \\
 &= 20 \\
 g(3) &= 3^3 + 1 \\
 &= 27 + 1 \\
 &= 28
 \end{aligned}$$

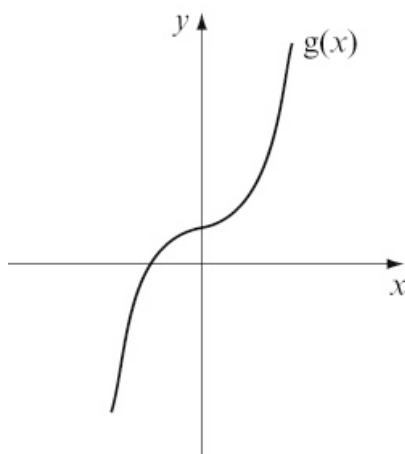
$$\begin{aligned}
 h(-2) &= 3^{-2} \\
 &= \frac{1}{3^2} \\
 &= \frac{1}{9}
 \end{aligned}$$

- b** $f(x) = 4(x-2) = 4x-8$
 This is a straight line with gradient 4 and intercept -8 .
 The domain tells us that $x \geq 0$, so the graph of $y = f(x)$ is:



The range of $f(x)$ is
 $f(x) \in \mathbb{R}, f(x) \geq -8$

$$g(x) = x^3 + 1$$



The range of $g(x)$ is $g(x) \in \mathbb{R}$

- c** Let $y = x^3 + 1$
 (change the subject of the formula)

$$y - 1 = x^3$$

$$\sqrt[3]{y-1} = x$$

$$\text{Hence } g^{-1}(x) = \sqrt[3]{x-1} \quad \{x \in \mathbb{R}\}$$

- d** $fg(x) = f(x^3 + 1)$
 $= 4(x^3 + 1 - 2)$
 $= 4(x^3 - 1), x \in \mathbb{R}, x \geq -1$

- e** First find $gh(x)$:

$$\begin{aligned}
 gh(x) &= g(3^x) \\
 &= (3^x)^3 + 1 \\
 &= 3^{3x} + 1
 \end{aligned}$$

$$gh(a) = 244$$

$$3^{3a} + 1 = 244$$

$$3^{3a} = 243$$

$$3^{3a} = 3^5$$

$$3a = 5$$

$$a = \frac{5}{3}$$

- f** First find $f^{-1}(x)$

$$\text{Let } y = 4(x-2)$$

(changing the subject of the formula)

$$\frac{y}{4} = x - 2$$

$$\frac{y}{4} + 2 = x$$

$$\text{Hence } f^{-1}(x) = \frac{x}{4} + 2$$

$$f^{-1}(x) = -\frac{1}{2}$$

$$\frac{x}{4} + 2 = -\frac{1}{2}$$

$$x = 4\left(-\frac{1}{2} - 2\right) = -10$$

- 12 a** f^{-1} exists when f is one-to-one.

$$\text{Now } f(x) = x^2 + 6x - 4$$

Completing the square:

$$f(x) = (x + 3)^2 - 13$$

The minimum value is

$$f(x) = -13 \text{ when } x + 3 = 0$$

$$\Rightarrow x = -3$$

Hence, f is one-to-one when $x > -3$

So least value of a is $a = -3$

- b** Let $y = f(x)$

$$y = x^2 + 6x - 4$$

$$y = (x + 3)^2 - 13$$

$$y + 13 = (x + 3)^2$$

$$x + 3 = \sqrt{y + 13}$$

$$x = \sqrt{y + 13} - 3$$

$$\text{So } f^{-1}: x \mapsto \sqrt{x + 13} - 3$$

For $a = 0$, Range $f(x)$ is $y > -4$

So Domain $f^{-1}(x)$ is $x > -4$

- 13 a** $f: x \mapsto 4x - 1$

Let $y = 4x - 1$ and change

the subject of the formula.

$$\Rightarrow y + 1 = 4x$$

$$\Rightarrow x = \frac{y + 1}{4}$$

$$\text{Hence } f^{-1}: x \mapsto \frac{x + 1}{4}, \quad x \in \mathbb{R}$$

- b** $gf(x) = g(4x - 1)$

$$= \frac{3}{2(4x - 1) - 1}$$

$$= \frac{3}{8x - 3}$$

$$\text{Hence } gf: x \mapsto \frac{3}{8x - 3}$$

$gf(x)$ is undefined when $8x - 3 = 0$

$$\text{That is, at } x = \frac{3}{8}$$

$$\therefore \text{Domain } gf(x) = \left\{ x \in \mathbb{R}, x \neq \frac{3}{8} \right\}$$

- c** If $2f(x) = g(x)$

$$2 \times (4x - 1) = \frac{3}{2x - 1}$$

$$8x - 2 = \frac{3}{2x - 1}$$

$$(8x - 2)(2x - 1) = 3$$

$$16x^2 - 12x + 2 = 3$$

$$16x^2 - 12x - 1 = 0$$

$$\text{Use } x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

with $a = 16$, $b = -12$ and $c = -1$.

$$\text{Then } x = \frac{12 \pm \sqrt{144 + 64}}{32}$$

$$= \frac{12 \pm \sqrt{208}}{32}$$

$$= 0.826, -0.076$$

Values of x are -0.076 and 0.826

- 14 a** Let $y = \frac{x}{x - 2}$

$$y(x - 2) = x$$

$$yx - 2y = x \quad (\text{rearrange})$$

$$yx - x = 2y$$

$$x(y - 1) = 2y$$

$$x = \frac{2y}{y - 1}$$

$$f^{-1}(x) = \frac{2x}{x - 1}, \quad x \neq 1$$

- b** The range of $f^{-1}(x)$ is the domain of $f(x)$:

$$\{f^{-1}(x) \in \mathbb{R}, f^{-1}(x) \neq 2\}$$

- c** $gf(1.5) = g\left(\frac{1.5}{1.5 - 2}\right)$

$$= g\left(\frac{1.5}{-0.5}\right)$$

$$= g(-3)$$

$$= \frac{3}{-3}$$

$$= -1$$

14 d If $g(x) = f(x) + 4$

$$\frac{3}{x} = \frac{x}{x-2} + 4$$

$$3(x-2) = x^2 + 4x(x-2)$$

$$3x - 6 = x^2 + 4x^2 - 8x$$

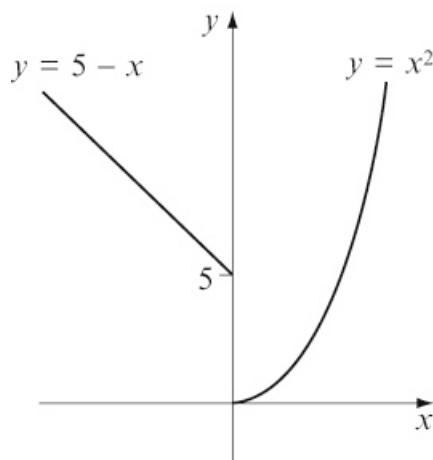
$$0 = 5x^2 - 11x + 6$$

$$0 = (5x-6)(x-1)$$

$$\Rightarrow x = \frac{6}{5}, 1$$

15 $y = 5 - x$ is a straight line with gradient -1 passing through 5 on the y axis.

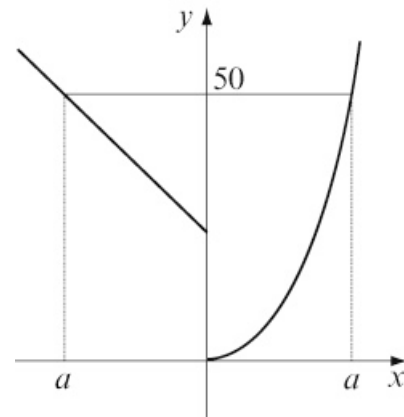
$y = x^2$ is a \cup -shaped quadratic passing through $(0, 0)$



$$\begin{aligned} \mathbf{a} \quad n(-3) &= 5 - (-3) \\ &= 5 + 3 \\ &= 8 \end{aligned}$$

$$\begin{aligned} n(3) &= 32 \\ &= 9 \end{aligned}$$

b From the diagram, you can see there are two values of x for which $n(x) = 50$



The negative value of x is where $5 - x = 50$

$$x = 5 - 50$$

$$x = -45$$

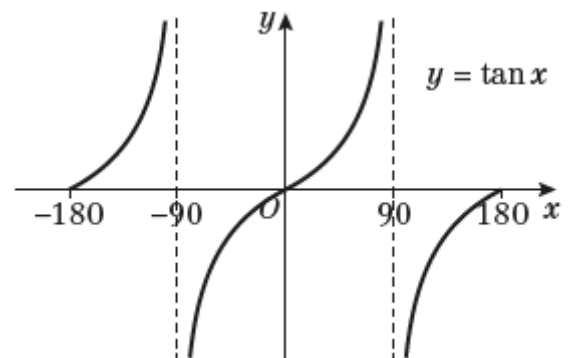
The positive value of x is where $x^2 = 50$

$$x = \sqrt{50}$$

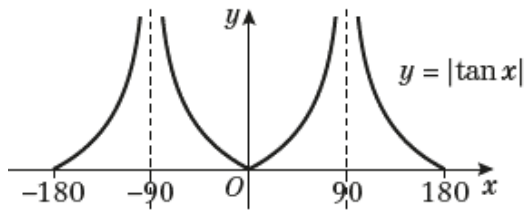
$$x = 5\sqrt{2}$$

The values of x such that $n(x) = 50$ are -45 and $+5\sqrt{2}$

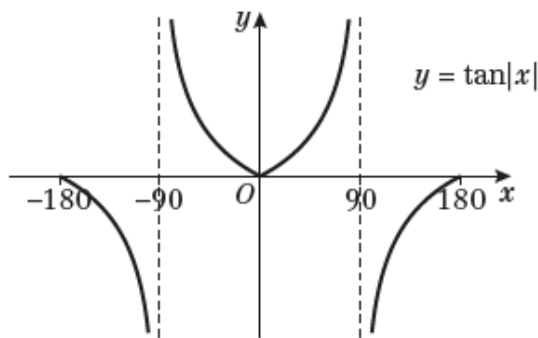
16 a



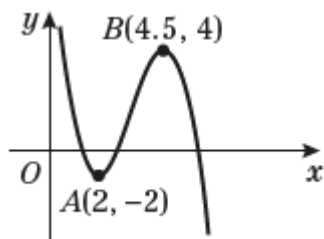
- 16 b** $y = |\tan(x)|$ reflects the negative parts of $\tan x$ in the x axis.



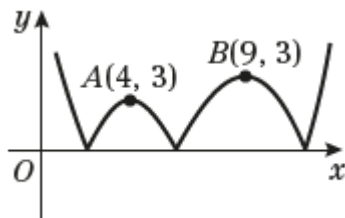
- c** $y = \tan(|x|)$ reflects $\tan x$ in the y -axis.



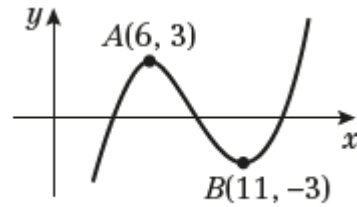
17 a



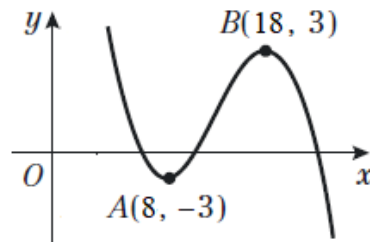
b



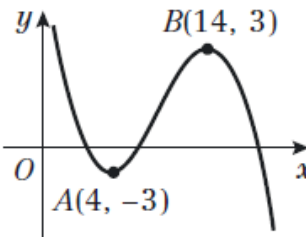
c



- d** $y = f(\frac{1}{2}x + 2)$ can be written as
 $y = f(\frac{1}{2}(x + 4))$
 $y = f(\frac{1}{2}x)$
 Horizontal stretch, scale factor 2.



- $y = f(\frac{1}{2}(x + 4))$
 Horizontal translation of -4

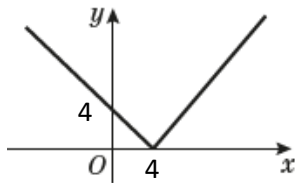


- 18 a** $g(x) \geq 0$

$$\begin{aligned} \text{b } gf(x) &= g(4 - x) \\ &= 3(4 - x)^2 \\ &= 3x^2 - 24x + 48 \end{aligned}$$

$$\begin{aligned} gf(x) &= 48 \\ 3x^2 - 24x + 48 &= 48 \\ 3x^2 - 24x &= 0 \\ 3x(x - 8) &= 0 \\ x &= 0 \text{ or } x = 8 \end{aligned}$$

18 c

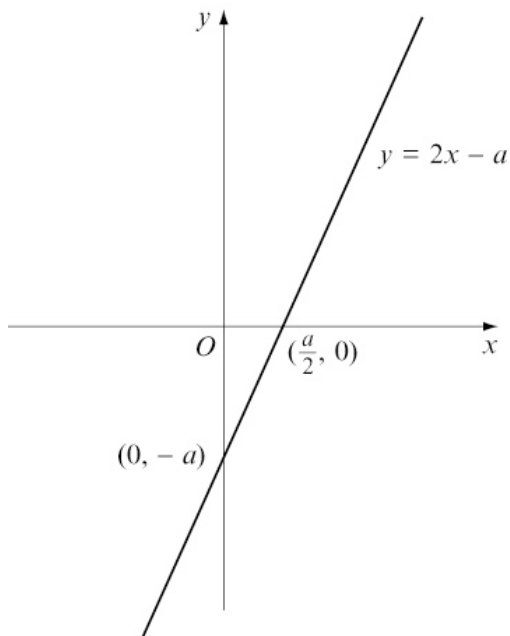


$$|f(x)| = 2 \text{ when } |4 - x| = 2, \text{ so}$$

$$4 - x = 2 \Rightarrow x = 2$$

$$\text{or } -(4 - x) = 2 \Rightarrow x = 6$$

19 a

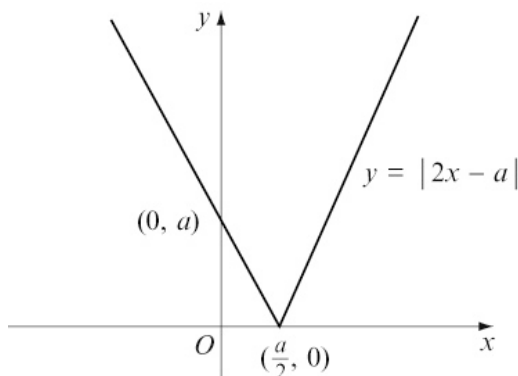


For $y = |2x - a|$:

$$\text{When } x = 0, y = |-a| = a \quad (0, a)$$

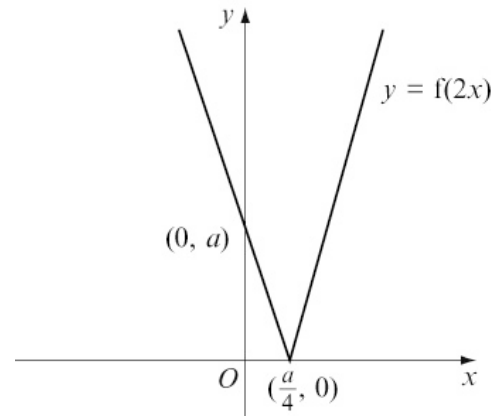
$$\text{When } y = 0, 2x - a = 0$$

$$\Rightarrow x = \frac{a}{2} \quad \left(\frac{a}{2}, 0\right)$$



19 b $y = f(2x)$

Horizontal stretch, scale factor $\frac{1}{2}$



c $|2x - a| = \frac{1}{2}x$

$$\text{Either } (2x - a) = \frac{1}{2}x$$

$$\Rightarrow a = \frac{3}{2}x$$

Given that $x = 4$,

$$a = \frac{3 \times 4}{2} = 6$$

Or

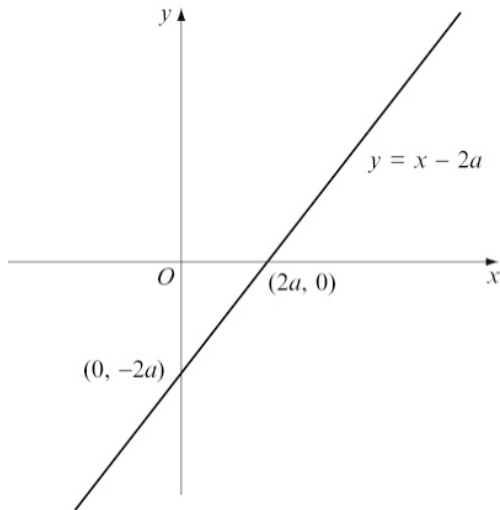
$$-(2x - a) = \frac{1}{2}x$$

$$\Rightarrow a = \frac{5}{2}x$$

Given that $x = 4$,

$$a = \frac{5 \times 4}{2} = 10$$

20 a

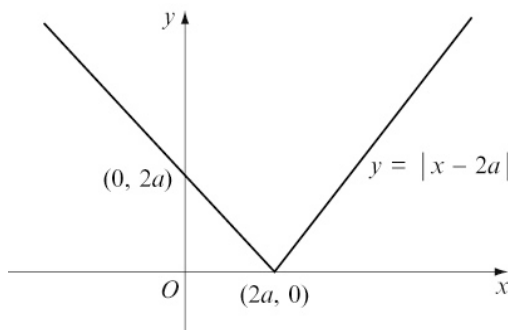


For $y = |x - 2a|$:

When $x = 0$, $y = |-2a| = 2a$ $(0, 2a)$

When $y = 0$, $x - 2a = 0$

$\Rightarrow x = 2a$ $(2a, 0)$



b $|x - 2a| = \frac{1}{3}x$

Either $(x - 2a) = \frac{1}{3}x$

$$\Rightarrow x - \frac{1}{3}x = 2a$$

$$\Rightarrow \frac{2}{3}x = 2a$$

$$\Rightarrow x = 3a$$

or $-(x - 2a) = \frac{1}{3}x$

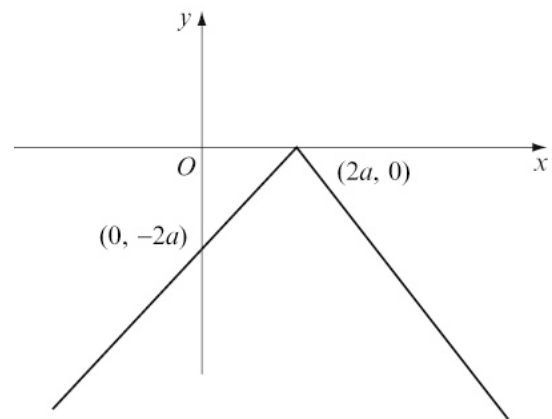
$$\Rightarrow -x + 2a = \frac{1}{3}x$$

$$\Rightarrow \frac{4}{3}x = 2a$$

$$\Rightarrow x = \frac{3}{2}a$$

c $y = -|x - 2a|$

Reflect $y = |x - 2a|$ in the x -axis



$y = a - |x - 2a|$ Vertical translation by $+a$

For $y = a - |x - 2a|$:

When $x = 0$,

$$\begin{aligned} y &= a - |-2a| \\ &= a - 2a \\ &= -a \end{aligned} \quad (0, -a)$$

When $y = 0$,

$$\begin{aligned} a - |x - 2a| &= 0 \\ |x - 2a| &= a \end{aligned}$$

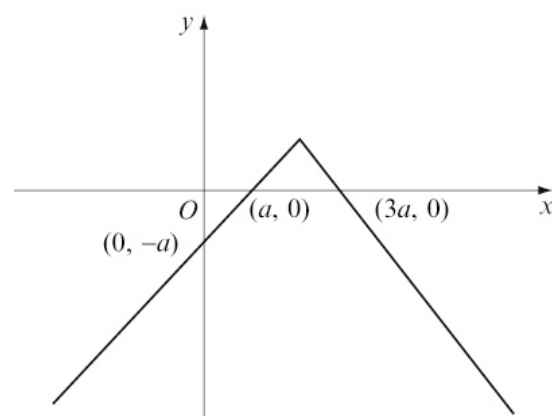
Either $x - 2a = a$

$$\Rightarrow x = 3a \quad (3a, 0)$$

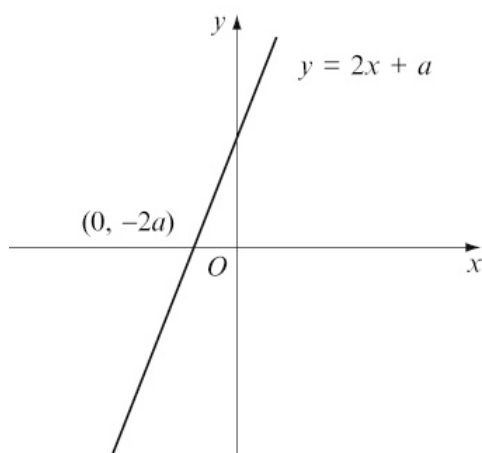
or $-(x - 2a) = a$

$$\Rightarrow -x + 2a = a$$

$$\Rightarrow x = a \quad (a, 0)$$



21 a & b

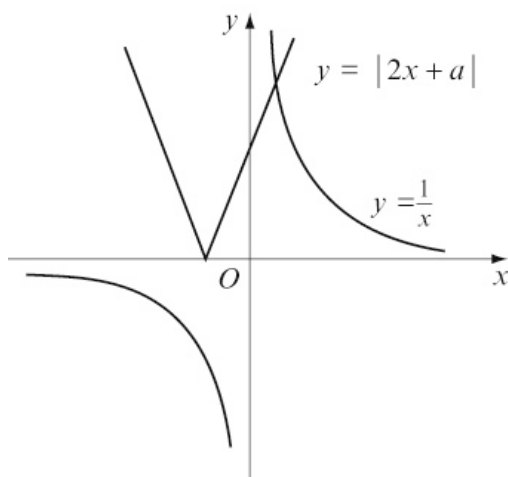


For $y = |2x + a|$:

When $x = 0$, $y = |a| = a$ $(0, a)$

When $y = 0$, $2x + a = 0$

$$\Rightarrow x = -\frac{a}{2} \quad \left(-\frac{a}{2}, 0\right)$$



c Intersection of graphs in b gives solutions to the equation:

$$|2x + a| = \frac{1}{x}$$

$$x|2x + a| = 1$$

$$x|2x + a| - 1 = 0$$

The graphs intersect once only, so $x|2x + a| - 1 = 0$ has only one solution.

d The intersection point is on the non-reflected part of the modulus graph, so here $|2x - a| = 2x - a$

$$x(2x + a) - 1 = 0$$

$$2x^2 + ax - 1 = 0$$

$$x = \frac{-a \pm \sqrt{a^2 + 8}}{4}$$

As shown on the graph,

x is positive at intersection,

$$\text{so } x = \frac{-a + \sqrt{a^2 + 8}}{4}$$

22 a $f(x) = x^2 - 7x + 5 \ln x + 8$

$$f'(x) = 2x - 7 + \frac{5}{x}$$

At stationary points, $f'(x) = 0$:

$$2x - 7 + \frac{5}{x} = 0$$

$$2x^2 - 7x + 5 = 0$$

$$(2x - 5)(x - 1) = 0$$

$$x = \frac{5}{2}, x = 1$$

Point A: $x = 1$,

$$f(x) = 1 - 7 + 5 \ln 1 + 8$$

$$= 2$$

A is $(1, 2)$

Point B: $x = \frac{5}{2}$,

$$f(x) = \frac{25}{4} - \frac{35}{2} + 5 \ln \frac{5}{2} + 8$$

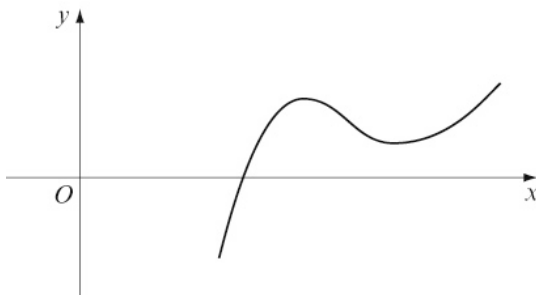
$$= 5 \ln \frac{5}{2} - \frac{13}{4}$$

$$B \text{ is } \left(\frac{5}{2}, 5 \ln \frac{5}{2} - \frac{13}{4}\right)$$

22 b $y = f(x - 2)$

Horizontal translation of +2.

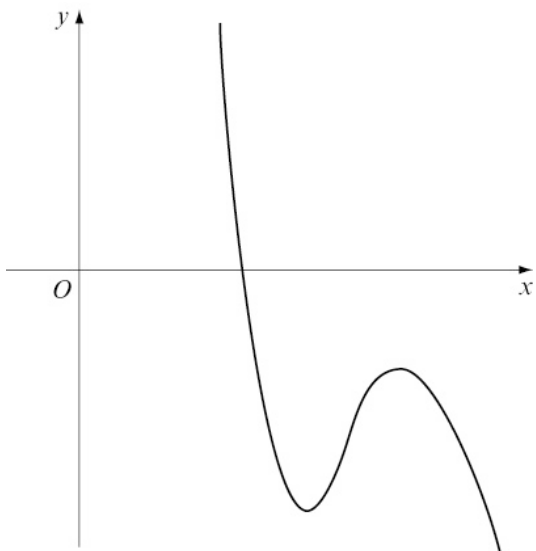
Graph looks like:



$y = -3f(x - 2)$

Reflection in the x-axis, and vertical stretch, scale factor 3.

Graph looks like:



c Using the transformations,

point (X, Y)

becomes $(X + 2, -3Y)$

$(1, 2) \rightarrow (3, -6)$

Minimum

$\left(\frac{5}{2}, 5\ln\frac{5}{2} - \frac{13}{4}\right) \rightarrow$

$\left(\frac{9}{2}, \frac{39}{4} - 15\ln\frac{5}{2}\right)$

Maximum

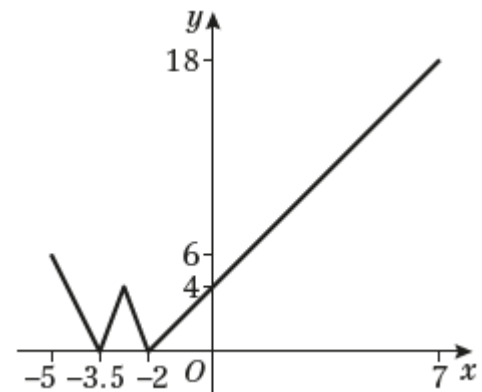
23 a The range of $f(x)$ is $-2 \leq f(x) \leq 18$

b $ff(-3) = f(-2)$

Using $f(x) = 2x + 4$

$f(-2) = 2 \times (-2) + 4 = 0$

c



- 23 d** Look at each section of $f(x)$ separately.

$$-5 \leq x \leq -3:$$

$$\text{Gradient} = \frac{-2-6}{-3-(-5)} = -4$$

$$\therefore f(x) - (-2) = -4(x - (-3)) \Rightarrow f(x) = -4x - 14$$

$$\text{So in this region, } f(x) = 2 \text{ when } x = -4$$

$$\therefore fg(x) = 2 \text{ has a corresponding solution if}$$

$$g(x) = -4 \Rightarrow g(x) + 4 = x^2 - 7x + 14 = 0$$

$$\text{Discriminant } (-7)^2 - 4(1)(14) = -7 < 0$$

So no solution

$$-3 \leq x \leq 7: \text{Gradient} = \frac{18-(-2)}{7-(-3)} = 2$$

$$\therefore f(x) - (-2) = 2(x - (-3)) \Rightarrow f(x) = 2x + 4$$

$$\text{So in this region, } f(x) = 2 \text{ when } x = -1$$

$$\therefore fg(x) = 2 \text{ has a corresponding solution if}$$

$$g(x) = -1 \Rightarrow g(x) + 1 = x^2 - 7x + 11 = 0$$

$$x = \frac{-(-7) \pm \sqrt{(-7)^2 - 4(1)(11)}}{2(1)} = \frac{7 \pm \sqrt{5}}{2}$$

$$\therefore x = \frac{7+\sqrt{5}}{2} \text{ or } x = \frac{7-\sqrt{5}}{2}$$

- d** For no solutions, $p(x) > 10$ at $x = -4$

$$\text{So } -\frac{1}{2}x + k > 10 \text{ at } x = -4$$

$$-\frac{1}{2}(-4) + k > 10$$

$$2 + k > 10$$

$$k > 8$$

- 25 a** Completing the square

$$3x^2 - 12x + 20 = 3(x^2 - 4x) + 20$$

$$= 3((x-2)^2 - 4) + 20$$

$$= 3(x-2)^2 - 12 + 20$$

$$= 3(x-2)^2 + 8$$

$$\mathbf{b} \quad g(x) = \frac{1}{3x^2 - 12x + 20} = \frac{1}{3(x-2)^2 + 8}$$

The maximum value of $g(x)$ is $\frac{1}{8}$ (when $x = 2$)

As x approaches infinity, $g(x)$ approaches 0

Therefore the range is $0 < g(x) \leq \frac{1}{8}$

- 24 a** The range of $p(x)$ is $p(x) \leq 10$

- b** $p(x)$ is many-to-one, therefore the inverse is one-to-many, which is not a function.

- c** At first point of intersection:

$$2(x+4) + 10 = -4$$

$$2x + 18 = -4$$

$$x = -11$$

At the other point of intersection:

$$-2(x+4) + 10 = -4$$

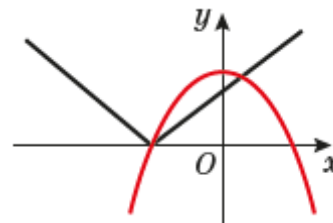
$$-2x + 2 = -4$$

$$x = 3$$

$$-11 < x < 3$$

Challenge

a



$$\mathbf{b} \quad y = (a+x)(a-x)$$

$$\text{When } y = 0, x = -a \text{ or } x = a$$

$$\text{When } x = 0, y = a^2$$

$$(-a, 0), (a, 0), (0, a^2)$$

$$\mathbf{c} \quad \text{When } x = 4, y = a^2 - x^2 = a^2 - 16$$

$$\text{and } y = x + a = 4 + a$$

$$a^2 - 16 = 4 + a$$

$$a^2 - a - 20 = 0$$

$$(a-5)(a+4) = 0$$

$$\text{As } a > 1, a = 5$$