Sequences and series 3B

1 a 3 + 7 + 11 + 14 + . . . (for 20 terms)

Substitute $a = 3$, $d = 4$, $n = 20$ into

$$S_n = \frac{n}{2}\left(2a + (n-1)d\right) = \frac{20}{2}\left(6 + 19\times 4\right)$$
$$= 10\times 82 = 820$$

b 2 + 6 + 10 + 14 + . . . (for 15 terms)

Substitute $a = 2$, $d = 4$, $n = 15$ into

$$S_n = \frac{n}{2}\left(2a + (n-1)d\right) = \frac{15}{2}\left(4 + 14\times 4\right)$$
$$= \frac{15}{2}\times 60 = 450$$

c 30 + 27 + 24 + 21 + . . . (40 terms)

Substitute $a = 30$, $d = -3$, $n = 40$ into

$$S_n = \frac{n}{2}\left(2a + (n-1)d\right)$$
$$= \frac{40}{2}\left(60 + 39\times(-3)\right)$$
$$= 20\times(-57) = -1140$$

d 5 + 1 + −3 + −7 + . . . (14 terms)

Substitute $a = 5$, $d = -4$, $n = 14$ into

$$S_n = \frac{n}{2}\left(2a + (n-1)d\right)$$
$$= \frac{14}{2}\left(10 + 13\times(-4)\right)$$
$$= 7\times(-42) = -294$$

e 5 + 7 + 9 + . . . + 75

Here, $a = 5$, $d = 2$ and $L = 75$.

Use $L = a + (n - 1)d$ to find n:

$$75 = 5 + (n-1)\times 2$$
$$70 = (n-1)\times 2$$
$$35 = n - 1$$
$$n = 36\ (36\ \text{terms})$$

Substitute $a = 5$, $d = 2$, $n = 36$ and $L = 75$ into

$$S_n = \frac{n}{2}(a + L) = \frac{36}{2}(5 + 75)$$
$$= 18\times 80 = 1440$$

f 4 + 7 + 10 + . . . + 91

Here, $a = 4$, $d = 3$ and $L = 91$.

Use $L = a + (n − 1)d$ to find n:

$$91 = 4 + (n-1)\times 3$$
$$87 = (n-1)\times 3$$
$$29 = (n-1)$$
$$n = 30\ (30\ \text{terms})$$

Substitute $a = 4$, $d = 3$, $L = 91$ and $n = 30$ into

$$S_n = \frac{n}{2}(a + L) = \frac{30}{2}(4 + 91)$$
$$= 15\times 95 = 1425$$

1 g $34 + 29 + 24 + 19 + \ldots + -111$

Here, $a = 34$, $d = -5$ and $L = -111$.

Use $L = a + (n - 1)d$ to find n:

$$\begin{aligned}-111 &= 34 + (n-1)\times(-5)\\ -145 &= (n-1)\times(-5)\\ 29 &= (n-1)\\ n &= 30 \ (30 \text{ terms})\end{aligned}$$

Substitute $a = 34$, $d = -5$, $L = -111$ and $n = 30$ into

$$\begin{aligned}S_n &= \frac{n}{2}(a+L) = \frac{30}{2}(34+(-111))\\ &= 15\times(-77) = -1155\end{aligned}$$

h $(x + 1) + (2x + 1) + (3x + 1) + \ldots + (21x + 1)$

Here, $a = x + 1$, $d = x$ and $L = 21x + 1$.

Use $L = a + (n - 1)d$ to find n:

$$\begin{aligned}21x+1 &= x+1+(n-1)\times x\\ 20x &= (n-1)\times x\\ 20 &= (n-1)\\ n &= 21\,(21 \text{ terms})\end{aligned}$$

Substitute $a = x + 1$, $d = x$, $L = 21x + 1$ and $n = 21$ into

$$\begin{aligned}S_n &= \frac{n}{2}(a+L) = \frac{21}{2}(x+1+21x+1)\\ &= \frac{21}{2}\times(22x+2) = 21(11x+1)\\ &= 231x+21\end{aligned}$$

2 a $5 + 8 + 11 + 14 + \ldots = 670$

Substitute $a = 5$, $d = 3$, $S_n = 670$ into

$$\begin{aligned}S_n &= \frac{n}{2}(2a+(n-1)d)\\ 670 &= \frac{n}{2}(10+(n-1)\times 3)\\ 670 &= \frac{n}{2}(3n+7)\\ 1340 &= n(3n+7)\\ 0 &= 3n^2+7n-1340\\ 0 &= (n-20)(3n+67)\\ n &= 20 \text{ or } -\frac{67}{3}\end{aligned}$$

Number of terms is 20.

b $3 + 8 + 13 + 18 + \ldots = 1575$

Substitute $a = 3$, $d = 5$, $S_n = 1575$ into

$$\begin{aligned}S_n &= \frac{n}{2}(2a+(n-1)d)\\ 1575 &= \frac{n}{2}(6+(n-1)\times 5)\\ 1575 &= \frac{n}{2}(5n+1)\\ 3150 &= n(5n+1)\\ 0 &= 5n^2+n-3150\\ 0 &= (5n+126)(n-25)\\ n &= -\frac{126}{5}, 25\end{aligned}$$

Number of terms is 25.

2 c $64 + 62 + 60 + ... = 0$

Substitute $a = 64$, $d = -2$ and $S_n = 0$ into

$$S_n = \frac{n}{2}\left(2a + (n-1)d\right)$$
$$0 = \frac{n}{2}\left(128 + (n-1)\times(-2)\right)$$
$$0 = \frac{n}{2}(130 - 2n)$$
$$0 = n(65 - n)$$
$$n = 0 \text{ or } 65$$

Number of terms is 65.

d $34 + 30 + 26 + 22 + ... = 112$

Substitute $a = 34$, $d = -4$ and $S_n = 112$ into

$$S_n = \frac{n}{2}\left(2a + (n-1)d\right)$$
$$112 = \frac{n}{2}\left(68 + (n-1)\times(-4)\right)$$
$$112 = \frac{n}{2}(72 - 4n)$$
$$112 = n(36 - 2n)$$
$$2n^2 - 36n + 112 = 0$$
$$n^2 - 18n + 56 = 0$$
$$(n-4)(n-14) = 0$$
$$n = 4 \text{ or } 14$$

Number of terms is 4 or 14

3 $S = \underbrace{2 + 4 + 6 + 8 + \ldots}_{50 \text{ terms}}$

This is an arithmetic series with $a = 2$, $d = 2$ and $n = 50$.

Use $S_n = \frac{n}{2}\left(2a + (n-1)d\right)$

So $S = \frac{50}{2}(4 + 49\times 2)$
$= 25 \times 102 = 2550$

4 $7 + 12 + 17 + 22 + 27 + ... > 1000$

Using $S_n = \frac{n}{2}(2a + (n-1)d)$

$$1000 = \frac{n}{2}(2\times 7 + (n-1)5)$$
$$2000 = n(14 + 5n - 5)$$
$$2000 = n(5n + 9)$$
$$5n^2 + 9n - 2000 = 0$$
$$n = \frac{-9 \pm \sqrt{9^2 - 4\times 5\times(-2000)}}{2\times 5}$$
$$n = \frac{-9 \pm \sqrt{40081}}{10}$$
$$n = 19.12... \text{ or } n = -20.92...$$

So 20 terms are needed.

5 Let common difference = d.

Substitute $a = 4$, $n = 20$, and $S_{20} = -15$ into

$$S_n = \frac{n}{2}\left(2a + (n-1)d\right)$$
$$-15 = \frac{20}{2}\left(8 + (20-1)d\right)$$
$$-15 = 10(8 + 19d)$$
$$-1.5 = 8 + 19d$$
$$19d = -9.5$$
$$d = -0.5$$

The common difference is −0.5.

Use nth term = $a + (n - 1)d$ to find

$$\begin{aligned}20\text{th term} &= a + 19d \\ &= 4 + 19 \times (-0.5) \\ &= 4 - 9.5 = -5.5\end{aligned}$$

20th term is −5.5.

6 Let the first term be a and the common difference d.

Sum of first three terms is 12, so

$$a + (a + d) + (a + 2d) = 12$$
$$3a + 3d = 12$$

$$a + d = 4 \qquad \mathbf{(1)}$$

20th term is −32, so

$$a + 19d = -32 \qquad \mathbf{(2)}$$

Equation (**2**) − Equation (**1**):

$$18d = -36$$
$$d = -2$$

Substitute $d = -2$ into Equation (**1**):

$$a + (-2) = 4$$
$$a = 6$$

Therefore, first term is 6 and common difference is −2.

7
$$S_{50} = 1 + 2 + 3 + \ldots + 48 + 49 + 50 \qquad \mathbf{(1)}$$
$$S_{50} = 50 + 49 + 48 + \ldots + 3 + 2 + 1 \qquad \mathbf{(2)}$$

Adding (**1**) and (**2**):

$$2 \times S_{50} = 50 \times 51$$
$$S_{50} = \frac{50 \times 51}{2}$$
$$= 1275$$

8 Sum required $= \underbrace{1 + 2 + 3 + \ldots + 2n}$

Arithmetic series with $a = 1$, $d = 1$ and $n = 2n$.

Use $S_n = \frac{n}{2}\left(2a + (n-1)d\right)$

$= \frac{2n}{2}\left(2 \times 1 + (2n-1) \times 1\right)$

$= \frac{\cancel{2}n}{\cancel{2}}(2n+1)$

$= n(2n+1)$

9 Required sum $= \underbrace{1 + 3 + 5 + 7 + \ldots}_{n \text{ terms}}$

This is an arithmetic series with $a = 1$, $d = 2$ and $n = n$.

Use $S_n = \frac{n}{2}\left(2a + (n-1)d\right)$

$= \frac{n}{2}\left(2 \times 1 + (n-1) \times 2\right)$

$= \frac{n}{2}(2 + 2n - 2)$

$= \frac{n \times \cancel{2}n}{\cancel{2}}$

$= n \times n$

$= n^2$

10 a $u_5 = 33$, so $a + 4d = 33$ (**1**)
$u_{10} = 68$, so $a + 9d = 68$ (**2**)
(**2**) − (**1**) gives:
$5d = 35$
$d = 7$
$a = 5$

$2225 = \frac{n}{2}(2 \times 5 + (n-1)7)$

$4450 = n(7n+3)$

$7n^2 + 3n - 4450 = 0$

10 b $n = \frac{-3 \pm \sqrt{3^2 - 4 \times 7 \times (-4450)}}{2 \times 7}$

$n = \frac{-3 \pm \sqrt{124\,609}}{14}$

$n = \frac{-3 \pm 353}{14}$

$n = 25$ or -25.42
So $n = 25$

11 a $u_n = a + (n-1)d$
$303 = k + 1 + (n-1)(k+2)$
$303 = k + 1 + nk + 2n - k - 2$
$303 = nk + 2n - 1$
$304 = n(k+2)$

$n = \frac{304}{k+2}$

b $S_n = \frac{\left(\frac{304}{k+2}\right)}{2}(k + 1 + 303)$

$S_n = \frac{152}{k+2}(k + 304)$

$S_n = \frac{152k + 46\,208}{k+2}$

c $2568 = \frac{152k + 46\,208}{k+2}$

$2568(k+2) = 152k + 46\,208$

$2416k = 41\,072$

$k = 17$

12 a $S_n = \frac{33}{2}(3 + 99)$

$= 1683$

b i $4p + (n-1)4p = 400$
$4pn = 400$

$n = \frac{100}{p}$

12 b ii $S_n = \frac{\left(\frac{100}{p}\right)}{2}(4p+400)$

$S_n = \frac{50}{p}(4p+400)$

$S_n = 200 + \frac{20\,000}{p}$

c $u_{80} = 3p + 2 + (80 - 1)(2p + 1)$
$= 3p + 2 + 158p + 79$
$= 161p + 81$

13 a $u_n = a + (n - 1)d$
$= 6 + (n - 1)5$
$= 6 + 5n - 5$
$= 5n + 1$

b $u_{10} = 5 \times 10 + 1 = 51$

$S_{10} = \frac{10}{2}(6+51)$

$= 5 \times 57$

$= 285$

c $S_k = \frac{k}{2}(2\times 6 + (k-1)5)$

$= \frac{k}{2}(12 + 5k - 5)$

$= \frac{k}{2}(5k+7)$

$\frac{k}{2}(5k+7) \leqslant 1029$

$5k^2 + 7k \leqslant 2058$

$5k^2 + 7k - 2058 \leqslant 0$

$(5k-98)(k+21) \leqslant 0$

d As $k > 0$, $5k - 98 = 0$, $k = 19.6$
So $k = 19$

Challenge

$u_n = \ln 9 + (n - 1)\ln 3$
$a = \ln 9$, $d = \ln 3$

$S_n = \frac{n}{2}(2\ln 9 + (n-1)\ln 3)$

$= \frac{n}{2}(\ln 81 - \ln 3 + n\ln 3)$

$= \frac{n}{2}(\ln 27 + n\ln 3)$

$= \frac{n}{2}(\ln 3^3 + \ln 3^n)$

$= \frac{n}{2}(\ln 3^{n+3})$

$= \frac{1}{2}(\ln 3^{n^2+3n})$

Therefore, $a = \frac{1}{2}$