Sequences and series 3G

1 a $u_{n+1} = u_n + 3$, $u_1 = 1$

$n=1 \Rightarrow u_2 = u_1 + 3 = 1 + 3 = 4$
$n=2 \Rightarrow u_3 = u_2 + 3 = 4 + 3 = 7$
$n=3 \Rightarrow u_4 = u_3 + 3 = 7 + 3 = 10$

Terms are 1, 4, 7, 10, …

b $u_{n+1} = u_n - 5$, $u_1 = 9$

$n=1 \Rightarrow u_2 = u_1 - 5 = 9 - 5 = 4$
$n=2 \Rightarrow u_3 = u_2 - 5 = 4 - 5 = -1$
$n=3 \Rightarrow u_4 = u_3 - 5 = -1 - 5 = -6$

Terms are 9, 4, −1, −6, …

c $u_{n+1} = 2u_n$, $u_1 = 3$

$n=1 \Rightarrow u_2 = 2u_1 = 2 \times 3 = 6$
$n=2 \Rightarrow u_3 = 2u_2 = 2 \times 6 = 12$
$n=3 \Rightarrow u_4 = 2u_3 = 2 \times 12 = 24$

Terms are 3, 6, 12, 24, …

d $u_{n+1} = 2u_n + 1$, $u_1 = 2$

$n=1 \Rightarrow u_2 = 2u_1 + 1 = 2 \times 2 + 1 = 5$
$n=2 \Rightarrow u_3 = 2u_2 + 1 = 2 \times 5 + 1 = 11$
$n=3 \Rightarrow u_4 = 2u_3 + 1 = 2 \times 11 + 1 = 23$

Terms are 2, 5, 11, 23, …

e $u_{n+1} = \frac{u_n}{2}$, $u_1 = 10$

$n=1 \Rightarrow u_2 = \frac{u_1}{2} = \frac{10}{2} = 5$
$n=2 \Rightarrow u_3 = \frac{u_2}{2} = \frac{5}{2} = 2.5$
$n=3 \Rightarrow u_4 = \frac{u_3}{2} = \frac{2.5}{2} = 1.25$

Terms are 10, 5, 2.5, 1.25, …

f $u_{n+1} = (u_n)^2 - 1$, $u_1 = 2$

$n=1 \Rightarrow u_2 = (u_1)^2 - 1 = 2^2 - 1 = 4 - 1 = 3$
$n=2 \Rightarrow u_3 = (u_2)^2 - 1 = 3^2 - 1 = 9 - 1 = 8$
$n=3 \Rightarrow u_4 = (u_3)^2 - 1 = 8^2 - 1 = 64 - 1 = 63$

Terms are 2, 3, 8, 63, …

2 a $3 \underset{+2}{\rightarrow} 5 \underset{+2}{\rightarrow} 7 \underset{+2}{\rightarrow} 9 \ldots$

$u_{n+1} = u_n + 2$, $u_1 = 3$

b $20 \underset{-3}{\rightarrow} 17 \underset{-3}{\rightarrow} 14 \underset{-3}{\rightarrow} 11 \ldots$

$u_{n+1} = u_n - 3$, $u_1 = 20$

c $1 \underset{\times 2}{\rightarrow} 2 \underset{\times 2}{\rightarrow} 4 \underset{\times 2}{\rightarrow} 8 \ldots$

$u_{n+1} = 2 \times u_n$, $u_1 = 1$

d $100 \underset{\div 4}{\rightarrow} 25 \underset{\div 4}{\rightarrow} 6.25 \underset{\div 4}{\rightarrow} 1.5625 \ldots$

$u_{n+1} = \frac{u_n}{4}$, $u_1 = 100$

2 e $1 \underset{\times(-1)}{\rightarrow} -1 \underset{\times(-1)}{\rightarrow} 1 \underset{\times(-1)}{\rightarrow} -1 \ldots$

$u_{n+1} = (-1) \times u_n$, $u_1 = 1$

f $3 \underset{\times 2+1}{\rightarrow} 7 \underset{\times 2+1}{\rightarrow} 15 \underset{\times 2+1}{\rightarrow} 31 \ldots$

$u_{n+1} = 2u_n + 1$, $u_1 = 3$

g $0 \underset{0^2+1}{\rightarrow} 1 \underset{1^2+1}{\rightarrow} 2 \underset{2^2+1}{\rightarrow} 5 \underset{5^2+1}{\rightarrow} 26 \ldots$

$u_{n+1} = (u_n)^2 + 1$, $u_1 = 0$

h $26 \underset{+2\div 2}{\rightarrow} 14 \underset{+2\div 2}{\rightarrow} 8 \underset{+2\div 2}{\rightarrow} 5 \underset{+2\div 2}{\rightarrow} 3.5 \ldots$

$$u_{n+1} = \frac{u_n + 2}{2}, \quad u_1 = 26$$

3 a $u_n = 2n - 1$.

Substituting $n = 1, 2, 3$ and 4 gives

$$u_1 = 1 \underset{+2}{\rightarrow} u_2 = 3 \underset{+2}{\rightarrow} u_3 = 5 \underset{+2}{\rightarrow} u_4 = 7$$

Recurrence formula is
$u_{n+1} = u_n + 2$, $u_1 = 1$.

b $u_n = 3n + 2$. Substituting $n = 1, 2, 3$ and 4 gives

$$u_1 = 5 \underset{+3}{\rightarrow} u_2 = 8 \underset{+3}{\rightarrow} u_3 = 11 \underset{+3}{\rightarrow} u_4 = 14$$

Recurrence formula is
$u_{n+1} = u_n + 3$, $u_1 = 5$.

c $u_n = n + 2$. Substituting $n = 1, 2, 3$ and 4 gives

$$u_1 = 3 \underset{+1}{\rightarrow} u_2 = 4 \underset{+1}{\rightarrow} u_3 = 5 \underset{+1}{\rightarrow} u_4 = 6$$

Recurrence formula is
$u_{n+1} = u_n + 1$, $u_1 = 3$.

d $u_n = \frac{n+1}{2}$. Substituting $n = 1, 2, 3$ and 4 gives

$$u_1 = 1 \underset{+\frac{1}{2}}{\rightarrow} u_2 = \frac{3}{2} \underset{+\frac{1}{2}}{\rightarrow} u_3 = 2 \underset{+\frac{1}{2}}{\rightarrow} u_4 = \frac{5}{2}$$

Recurrence formula is
$u_{n+1} = u_n + \frac{1}{2}$, $u_1 = 1$.

e $u_n = n^2$. Substituting $n = 1, 2, 3$ and 4:

$$u_1 = 1 \underset{+3}{\rightarrow} u_2 = 4 \underset{+5}{\rightarrow} u_3 = 9 \underset{+7}{\rightarrow} u_4 = 16$$

Differences are
$2 \times 1 + 1, 2 \times 2 + 1, 2 \times 3 + 1$

$u_{n+1} = u_n + 2n + 1$, $u_1 = 1$.

f $u_n = 3^n - 1$
$u_1 = 3^1 - 1 = 2$
$u_2 = 3^2 - 1 = 8$
$u_3 = 3^3 - 1 = 26$
$u_4 = 3^4 - 1 = 80$
$u_{n+1} = 3u_n + 2$, $u_1 = 2$

4 a $u_{n+1} = ku_n + 2$,
$u_1 = 3$
$u_2 = ku_1 + 2$
$= 3k + 2$

b $u_3 = ku_2 + 2$
$= k(3k + 2) + 2$
$= 3k^2 + 2k + 2$

4 c $u_3 = 42$, so $3k^2 + 2k + 2 = 42$
$3k^2 + 2k - 40 = 0$
$(k + 4)(3k - 10) = 0$
So $k = -4$ or $k = \frac{10}{3}$

5 $u_{n+1} = pu_n + q$
$u_1 = 2$
$u_2 = 2p + q = -1$, so $q = -2p - 1$
$u_3 = p(2p + q) + q = 2p^2 + pq + q = 11$
$2p^2 + p(-2p - 1) - 2p - 1 = 11$
$2p^2 - 2p^2 - p - 2p - 1 = 11$
$-3p = 12$
$p = -4$
$q = -2(-4) - 1 = 7$
$p = -4$ and $q = 7$

6 a $x_{n+1} = x_n(p - 3x_n)$
$x_1 = 2$
$x_2 = 2(p - 3 \times 2) = 2p - 12$
$x_3 = (2p - 12)(p - 3(2p - 12))$
$= (2p - 12)(-5p + 36)$
$= -10p^2 + 132p - 432$

b $-10p^2 + 132p - 432 = -288$
$-10p^2 + 132p - 144 = 0$
$5p^2 - 66p + 72 = 0$
$(5p - 6)(p - 12) = 0$
$p = \frac{6}{5}$ or $p = 12$
As p is an integer, $p = 12$

c $x_4 = -288(12 - 3(-288)) = -252\,288$

7 a $a_1 = k$
$a_2 = 4k + 5$
$a_3 = 4(4k + 5) + 5 = 16k + 25$

b $a_4 = 4(16k + 25) + 5 = 64k + 105$
$\sum_{r=1}^{4} a_r = k + 4k + 5 + 16k + 25 + 64k + 105$
$= 85k + 135$
$= 5(17k + 27)$
Therefore, $\sum_{r=1}^{4} a_r$ is a multiple of 5.