Binomial expansion 4A

1 a i $(1+x)^{-4} = 1+(-4)x+\frac{(-4)(-5)}{2!}x^2+\frac{(-4)(-5)(-6)}{3!}x^3+\ldots$

$= 1-4x+10x^2-20x^3+\ldots$

ii $|x|<1$

b i $(1+x)^{-6} = 1+(-6)x+\frac{(-6)(-7)}{2!}x^2+\frac{(-6)(-7)(-8)}{3!}x^3+\ldots$

$= 1-6x+21x^2-56x^3+\ldots$

ii $|x|<1$

c i $(1+x)^{\frac{1}{2}} = 1+\left(\frac{1}{2}\right)x+\frac{\left(\frac{1}{2}\right)\left(\frac{1}{2}-1\right)}{2!}x^2+\frac{\left(\frac{1}{2}\right)\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{3!}x^3+\ldots$

$= 1+\left(\frac{1}{2}\right)x+\frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2}x^2+\frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{6}x^3+\ldots$

$= 1+\frac{1}{2}x-\frac{1}{8}x^2+\frac{1}{16}x^3+\ldots$

ii $|x|<1$

d i $(1+x)^{\frac{5}{3}} = 1+\left(\frac{5}{3}\right)x+\frac{\left(\frac{5}{3}\right)\left(\frac{5}{3}-1\right)}{2!}x^2+\frac{\left(\frac{5}{3}\right)\left(\frac{5}{3}-1\right)\left(\frac{5}{3}-2\right)}{3!}x^3+\ldots$

$(1+x)^{\frac{5}{3}} = 1+\left(\frac{5}{3}\right)x+\frac{\left(\frac{5}{3}\right)\left(\frac{2}{3}\right)}{2}x^2+\frac{\left(\frac{5}{3}\right)\left(\frac{2}{3}\right)\left(-\frac{1}{3}\right)}{6}x^3+\ldots$

$= 1+\frac{5}{3}x+\frac{5}{9}x^2-\frac{5}{81}x^3+\ldots$

ii $|x|<1$

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1 e i $(1+x)^{-\frac{1}{4}} = 1 + \left(-\frac{1}{4}\right)x + \frac{\left(-\frac{1}{4}\right)\left(-\frac{1}{4}-1\right)}{2!}x^2 + \frac{\left(-\frac{1}{4}\right)\left(-\frac{1}{4}-1\right)\left(-\frac{1}{4}-2\right)}{3!}x^3 + \ldots$

$= 1 + \left(-\frac{1}{4}\right)x + \frac{\left(-\frac{1}{4}\right)\left(-\frac{5}{4}\right)}{2}x^2 + \frac{\left(-\frac{1}{4}\right)\left(-\frac{5}{4}\right)\left(-\frac{9}{4}\right)}{6}x^3 + \ldots$

$= 1 - \frac{1}{4}x + \frac{5}{32}x^2 - \frac{15}{128}x^3 + \ldots$

ii $|x| < 1$

f i $(1+x)^{-\frac{3}{2}} = 1 + \left(-\frac{3}{2}\right)x + \frac{\left(-\frac{3}{2}\right)\left(-\frac{3}{2}-1\right)}{2!}x^2 + \frac{\left(-\frac{3}{2}\right)\left(-\frac{3}{2}-1\right)\left(-\frac{3}{2}-2\right)}{3!}x^3 + \ldots$

$= 1 + \left(-\frac{3}{2}\right)x + \frac{\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{2}x^2 + \frac{\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\left(-\frac{7}{2}\right)}{6}x^3 + \ldots$

$= 1 - \frac{3}{2}x + \frac{15}{8}x^2 - \frac{35}{16}x^3 + \ldots$

ii $|x| < 1$

2 a i $(1+3x)^{-3} = 1 + (-3)(3x) + \frac{(-3)(-4)}{2!}(3x)^2 + \frac{(-3)(-4)(-5)}{3!}(3x)^3 + \ldots$

$= 1 + (-3)(3x) + \frac{(-3)(-4)}{2}9x^2 + \frac{(-3)(-4)(-5)}{6}27x^3 + \ldots$

$= 1 - 9x + 54x^2 - 270x^3 + \ldots$

ii $|3x| < 1$

$|x| < \frac{1}{3}$

b i $\left(1+\frac{1}{2}x\right)^{-5} = 1 + (-5)\left(\frac{1}{2}x\right) + \frac{(-5)(-6)}{2!}\left(\frac{1}{2}x\right)^2 + \frac{(-5)(-6)(-7)}{3!}\left(\frac{1}{2}x\right)^3 + \ldots$

$= 1 + (-5)\left(\frac{1}{2}x\right) + \frac{(-5)(-6)}{2}\frac{1}{4}x^2 + \frac{(-5)(-6)(-7)}{6}\frac{1}{8}x^3 + \ldots$

$= 1 - \frac{5}{2}x + \frac{15}{4}x^2 - \frac{35}{8}x^3 + \ldots$

ii $\left|\frac{1}{2}x\right| < 1$

$|x| < 2$

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2 c i $(1+2x)^{\frac{3}{4}}=1+\left(\frac{3}{4}\right)(2x)+\frac{\left(\frac{3}{4}\right)\left(\frac{3}{4}-1\right)}{2!}(2x)^2+\frac{\left(\frac{3}{4}\right)\left(\frac{3}{4}-1\right)\left(\frac{3}{4}-2\right)}{3!}(2x)^3+\ldots$

$=1+\left(\frac{3}{4}\right)(2x)+\frac{\left(\frac{3}{4}\right)\left(-\frac{1}{4}\right)}{2}4x^2+\frac{\left(\frac{3}{4}\right)\left(-\frac{1}{4}\right)\left(-\frac{5}{4}\right)}{6}8x^3+\ldots$

$=1+\frac{3}{2}x-\frac{3}{8}x^2+\frac{5}{16}x^3+\ldots$

ii $|2x|<1$

$|x|<\frac{1}{2}$

d i $(1-5x)^{\frac{7}{3}}=1+\left(\frac{7}{3}\right)(-5x)+\frac{\left(\frac{7}{3}\right)\left(\frac{7}{3}-1\right)}{2!}(-5x)^2+\frac{\left(\frac{7}{3}\right)\left(\frac{7}{3}-1\right)\left(\frac{7}{3}-2\right)}{3!}(-5x)^3+\ldots$

$=1-\left(\frac{7}{3}\right)5x+\frac{\left(\frac{7}{3}\right)\left(\frac{4}{3}\right)}{2}25x^2-\frac{\left(\frac{7}{3}\right)\left(\frac{4}{3}\right)\left(\frac{1}{3}\right)}{6}125x^3+\ldots$

$=1-\frac{35}{3}x+\frac{350}{9}x^2-\frac{1750}{81}x^3+\ldots$

ii $|-5x|<1$

$|x|<\frac{1}{5}$

e i $(1+6x)^{-\frac{2}{3}}=1+\left(-\frac{2}{3}\right)(6x)+\frac{\left(-\frac{2}{3}\right)\left(-\frac{2}{3}-1\right)}{2!}(6x)^2+\frac{\left(-\frac{2}{3}\right)\left(-\frac{2}{3}-1\right)\left(-\frac{2}{3}-2\right)}{3!}(6x)^3+\ldots$

$=1+\left(-\frac{2}{3}\right)(6x)+\frac{\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)}{2}36x^2+\frac{\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)\left(-\frac{8}{3}\right)}{6}216x^3+\ldots$

$=1-4x+20x^2-\frac{320}{3}x^3+\ldots$

ii $|6x|<1$

$|x|<\frac{1}{6}$

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2 f i

$$\left(1-\frac{3}{4}x\right)^{-\frac{5}{3}} = 1+\left(-\frac{5}{3}\right)\left(-\frac{3}{4}x\right)+\frac{\left(-\frac{5}{3}\right)\left(-\frac{5}{3}-1\right)}{2!}\left(-\frac{3}{4}x\right)^2+\frac{\left(-\frac{5}{3}\right)\left(-\frac{5}{3}-1\right)\left(-\frac{5}{3}-2\right)}{3!}\left(-\frac{3}{4}x\right)^3+\ldots$$

$$=1+\left(-\frac{5}{3}\right)\left(-\frac{3}{4}x\right)+\frac{\left(-\frac{5}{3}\right)\left(-\frac{8}{3}\right)}{2}\frac{9}{16}x^2-\frac{\left(-\frac{5}{3}\right)\left(-\frac{8}{3}\right)\left(-\frac{11}{3}\right)}{6}\frac{27}{64}x^3+\ldots$$

$$=1+\frac{5}{4}x+\frac{5}{4}x^2+\frac{55}{48}x^3+\ldots$$

ii $\left|-\frac{3}{4}x\right|<1$

$|x|<\frac{4}{3}$

3 a i $\frac{1}{(1+x)^2}=(1+x)^{-2}=1+(-2)x+\frac{(-2)(-3)}{2!}x^2+\frac{(-2)(-3)(-4)}{3!}x^3+\ldots$

$=1-2x+3x^2-4x^3+\ldots$

ii $|x|<1$

b i $\frac{1}{(1+3x)^4}=(1+3x)^{-4}=1+(-4)(3x)+\frac{(-4)(-5)}{2!}(3x)^2+\frac{(-4)(-5)(-6)}{3!}(3x)^3+\ldots$

$=1+(-4)(3x)+\frac{(-4)(-5)}{2}9x^2+\frac{(-4)(-5)(-6)}{6}27x^3+\ldots$

$=1-12x+90x^2-540x^3+\ldots$

ii $|3x|<1$

$|x|<\frac{1}{3}$

3 c i $\sqrt{1-x}=(1-x)^{\frac{1}{2}}=1+\left(\frac{1}{2}\right)(-x)+\frac{\left(\frac{1}{2}\right)\left(\frac{1}{2}-1\right)}{2!}(-x)^2+\frac{\left(\frac{1}{2}\right)\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{3!}(-x)^3+\ldots$

$=1-\left(\frac{1}{2}\right)x+\frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2}x^2-\frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{6}x^3+\ldots$

$=1-\frac{1}{2}x-\frac{1}{8}x^2-\frac{1}{16}x^3+\ldots$

ii $|-x|<1$

$|x|<1$

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3 d i $\sqrt[3]{1-3x}=(1-3x)^{\frac{1}{3}}=1+\left(\frac{1}{3}\right)(-3x)+\frac{\left(\frac{1}{3}\right)\left(\frac{1}{3}-1\right)}{2!}(-3x)^2+\frac{\left(\frac{1}{3}\right)\left(\frac{1}{3}-1\right)\left(\frac{1}{3}-2\right)}{3!}(-3x)^3+\ldots$

$$=1-\left(\frac{1}{3}\right)3x+\frac{\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)}{2}9x^2-\frac{\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)}{6}27x^3+\ldots$$

$$=1-x-x^2-\frac{5}{3}x^3+\ldots$$

ii $|-3x|<1$

$|x|<\frac{1}{3}$

e i

$$\frac{1}{\sqrt{1+\frac{1}{2}x}}=\left(1+\frac{1}{2}x\right)^{-\frac{1}{2}}=1+\left(-\frac{1}{2}\right)\left(\frac{1}{2}x\right)+\frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)}{2!}\left(\frac{1}{2}x\right)^2+\frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)\left(-\frac{1}{2}-2\right)}{3!}\left(\frac{1}{2}x\right)^3+\ldots$$

$$=1+\left(-\frac{1}{2}\right)\left(\frac{1}{2}x\right)+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2}\frac{1}{4}x^2+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{6}\frac{1}{8}x^3+\ldots$$

$$=1-\frac{1}{4}x+\frac{3}{32}x^2-\frac{5}{128}x^3+\ldots$$

ii $\left|\frac{1}{2}x\right|<1$

$|x|<2$

3 f i

$$\frac{\sqrt[3]{1-2x}}{1-2x}=(1-2x)^{-\frac{2}{3}}=1+\left(-\frac{2}{3}\right)(-2x)+\frac{\left(-\frac{2}{3}\right)\left(-\frac{2}{3}-1\right)}{2!}(-2x)^2+\frac{\left(-\frac{2}{3}\right)\left(-\frac{2}{3}-1\right)\left(-\frac{2}{3}-2\right)}{3!}(-2x)^3+\ldots$$

$$=1-\left(-\frac{2}{3}\right)2x+\frac{\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)}{2}4x^2-\frac{\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)\left(-\frac{8}{3}\right)}{6}8x^3+\ldots$$

$$=1+\frac{4}{3}x+\frac{20}{9}x^2+\frac{320}{81}x^3+\ldots$$

ii $|-2x|<1$

$|x|<\frac{1}{2}$

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4 a $\dfrac{1+x}{1-2x}=(1+x)(1-2x)^{-1}$ Expand $(1-2x)^{-1}$ using binomial expansion

$$=(1+x)\left(1+(-1)(-2x)+\frac{(-1)(-2)(-2x)^2}{2!}+\frac{(-1)(-2)(-3)(-2x)^3}{3!}+\ldots\right)$$

$=(1+x)(1+2x+4x^2+8x^3+\ldots)$ Multiply out

$=1+2x+4x^2+8x^3+\ldots+x+2x^2+4x^3+8x^4+\ldots$ Add like terms

$=1+3x+6x^2+12x^3+\ldots$

b $(1-2x)^{-1}$ is only valid when $|-2x|<1\Rightarrow|x|<\dfrac{1}{2}$

So expansion of $\frac{1+x}{1-2x}$ is only valid when $|x|<\dfrac{1}{2}$

5 a $$\mathrm{f}(x)=(1+3x)^{\frac{1}{2}}=1+\left(\frac{1}{2}\right)(3x)+\frac{\left(\frac{1}{2}\right)\left(\frac{1}{2}-1\right)}{2!}(3x)^2+\frac{\left(\frac{1}{2}\right)\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{3!}(3x)^3+\ldots$$

$$=1+\left(\frac{1}{2}\right)(3x)+\frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2}9x^2+\frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{6}27x^3+\ldots$$

$$=1+\frac{3x}{2}-\frac{9x^2}{8}+\frac{27x^3}{16}+\ldots$$

b When $x=\dfrac{1}{100}$, $\mathrm{f}(x)=\sqrt{1+3\left(\dfrac{1}{100}\right)}=\sqrt{\dfrac{103}{100}}=\dfrac{\sqrt{103}}{10}$

c Using the expansion:

$$\mathrm{f}(0.01)\approx1+\frac{3(0.01)}{2}-\frac{9(0.01)^2}{8}+\frac{27(0.01)^3}{16}$$
$$=1.014889188\ldots$$

$$\text{Percentage error}=\frac{1.014889188-\frac{\sqrt{103}}{10}}{\frac{\sqrt{103}}{10}}\times100=0.0000031\%$$

6 a $$(1+ax)^{-\frac{1}{2}}=1+\left(-\frac{1}{2}\right)(ax)+\frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)}{2!}(ax)^2+\frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)\left(-\frac{1}{2}-2\right)}{3!}(ax)^3+\ldots$$

$$=1+\left(-\frac{1}{2}\right)(ax)+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2}a^2x^2+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{6}a^3x^3+\ldots$$

$$=1-\frac{ax}{2}+\frac{3a^2x^2}{8}-\frac{5a^3x^3}{16}+\ldots$$

$\dfrac{3a^2}{8}=24$

$a^2=64$

$a=\pm8$

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b When $a = 8$, $-\frac{5(8)^3}{16} = -160$

When $a = -8$, $-\frac{5(-8)^3}{16} = 160$

7 For small values of x, ignore terms in x^3 and higher.

$$\sqrt{\frac{1+x}{1-x}} = (1+x)^{\frac{1}{2}}(1-x)^{-\frac{1}{2}}$$

$$(1+x)^{\frac{1}{2}} = 1 + \left(\frac{1}{2}\right)x + \frac{\left(\frac{1}{2}\right)\left(\frac{1}{2}-1\right)}{2!}x^2$$

$$= 1 + \left(\frac{1}{2}\right)x + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2}x^2$$

$$= 1 + \frac{1}{2}x - \frac{1}{8}x^2$$

$$(1-x)^{-\frac{1}{2}} = 1 + \left(-\frac{1}{2}\right)(-x) + \frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)}{2!}(-x)^2$$

$$= 1 + \frac{1}{2}x + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2}x^2$$

$$= 1 + \frac{1}{2}x + \frac{3}{8}x^2$$

$$\sqrt{\frac{1+x}{1-x}} = (1+x)^{\frac{1}{2}}(1-x)^{-\frac{1}{2}} = \left(1 + \frac{1}{2}x - \frac{1}{8}x^2\right)\left(1 + \frac{1}{2}x + \frac{3}{8}x^2\right)$$

$$= 1 + \frac{1}{2}x + \frac{3}{8}x^2 + \frac{1}{2}x + \frac{1}{4}x^2 - \frac{1}{8}x^2$$

$$= 1 + x + \frac{1}{2}x^2$$

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8 a $\mathrm{h}(x)=\dfrac{6}{1+5x}-\dfrac{4}{1-3x}=6(1+5x)^{-1}-4(1-3x)^{-1}$

$(1+5x)^{-1}=1+(-1)(5x)+\dfrac{(-1)(-2)}{2!}(5x)^2+\ldots$

$=1+(-1)(5x)+\dfrac{(-1)(-2)}{2}25x^2+\ldots$

$=1-5x+25x^2+\ldots$

$(1-3x)^{-1}=1+(-1)(-3x)+\dfrac{(-1)(-2)}{2!}(-3x)^2+\ldots$

$=1+(-1)(-3x)+\dfrac{(-1)(-2)}{2}9x^2+\ldots$

$=1+3x+9x^2+\ldots$

$\mathrm{h}(x)=6(1+5x)^{-1}-4(1-3x)^{-1}$

$=6(1-5x+25x^2+\ldots)-4(1+3x+9x^2+\ldots)$

$=6-30x+150x^2-4-12x-36x^2+\ldots$

$=2-42x+114x^2+\ldots$

b $\mathrm{h}(0.01)=\dfrac{6}{1+5(0.01)}-\dfrac{4}{1-3(0.01)}=1.590\,574\,374$

$\mathrm{h}(0.01)\ =2-42(0.01)+114(0.01)^2=1.5914$

$\dfrac{1.5914-1.590574374}{1.590574374}\times 100=0.052\%$

c The expansion is only valid for $|x|<\dfrac{1}{5}$. $|0.5|$ is not less than $\dfrac{1}{5}$

9 a $(1-3x)^{\frac{3}{2}}=1+\left(\dfrac{3}{2}\right)(-3x)+\dfrac{\left(\frac{3}{2}\right)\left(\frac{3}{2}-1\right)}{2!}(-3x)^2+\dfrac{\left(\frac{3}{2}\right)\left(\frac{3}{2}-1\right)\left(\frac{3}{2}-2\right)}{3!}(-3x)^3+\ldots$

$=1+\left(\dfrac{3}{2}\right)(-3x)+\dfrac{\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)}{2}9x^2-\dfrac{\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{6}27x^3+\ldots$

$=1-\dfrac{9x}{2}+\dfrac{27x^2}{8}+\dfrac{27x^3}{16}+\ldots$

b When $x=\dfrac{1}{100}$, $\left(1-3\left(\dfrac{1}{100}\right)\right)^{\frac{3}{2}}=\left(\dfrac{97}{100}\right)^{\frac{3}{2}}=\left(\dfrac{\sqrt{97}}{10}\right)^3=\dfrac{97\sqrt{97}}{1000}$

c $\left(\sqrt{0.97}\right)^3=1-\dfrac{9(0.01)}{2}+\dfrac{27(0.01)^2}{8}+\dfrac{27(0.01)^3}{16}=0.955339...$

$\sqrt{0.97}=\sqrt[3]{0.955339...}=0.984886$

$\sqrt{97}=\sqrt{0.97\times 100}=0.984886\times 10=9.84886$

d To improve the accuracy of this approximation, use more terms from the binomial expansion of $(1-3x)^{\frac{3}{2}}$

Challenge

a $\left(1+\frac{1}{x}\right)^{-\frac{1}{2}}=1+\left(-\frac{1}{2}\right)\left(\frac{1}{x}\right)+\frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)}{2!}\left(\frac{1}{x}\right)^2+\ldots$

$=1+\left(-\frac{1}{2}\right)\left(\frac{1}{x}\right)+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2}\frac{1}{x^2}+\ldots$

$=1-\frac{1}{2x}+\frac{3}{8x^2}+\ldots$

b $\mathrm{h}(9)=\left(1+\frac{1}{9}\right)^{-\frac{1}{2}}=\left(\frac{10}{9}\right)^{-\frac{1}{2}}=\left(\frac{9}{10}\right)^{\frac{1}{2}}=\frac{3}{\sqrt{10}}=\frac{3\sqrt{10}}{10}$

c $\mathrm{h}(9)=\frac{3\sqrt{10}}{10}$

$\frac{10}{3}\mathrm{h}(9)=\sqrt{10}$

So $\sqrt{10}=\frac{10}{3}\left(1-\frac{1}{2(9)}+\frac{3}{8(9)^2}\right)$

$=\frac{10}{3}\left(1-\frac{1}{18}+\frac{3}{648}\right)$

$=3\frac{52}{324}=3.16$

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