Binomial expansion Mixed exercise 4

1 a i $(1-4x)^3$ Use binomial expansion with $n = 3$ and $x = -4x$

$=1+(3)(-4x)+\frac{(3)(2)(-4x)^2}{2!}+\frac{(3)(2)(1)(-4x)^3}{3!}$ As $n=3$, expansion is finite

and exact

$=1-12x+48x^2-64x^3$

ii Valid for all x

b i $\sqrt{16+x}$ Write in index form

$=(16+x)^{\frac{1}{2}}$ Take out a factor of 16

$=\left(16\left(1+\frac{x}{16}\right)\right)^{\frac{1}{2}}$

$=16^{\frac{1}{2}}\left(1+\frac{x}{16}\right)^{\frac{1}{2}}$ Use binomial expansion with $n=\frac{1}{2}$ and $x=\frac{x}{16}$

$=4\left(1+\frac{1}{2}\left(\frac{x}{16}\right)+\frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2!}\left(\frac{x}{16}\right)^2+\frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{3!}\left(\frac{x}{16}\right)^3+\ldots\right)$

$=4\left(1+\frac{x}{32}-\frac{x^2}{2048}+\frac{x^3}{65\,536}+\ldots\right)$ Multiply by 4

$=4+\frac{x}{8}-\frac{x^2}{512}+\frac{x^3}{16\,384}+\ldots$

ii Valid for $\left|\frac{x}{16}\right|<1\Rightarrow|x|<16$

c i $\frac{1}{1-2x}$ Write in index form

$=(1-2x)^{-1}$ Use binomial expansion with $n=-1$ and $x=-2x$

$=1+(-1)(-2x)+\frac{(-1)(-2)(-2x)^2}{2!}+\frac{(-1)(-2)(-3)(-2x)^3}{3!}+\ldots$

$=1+2x+4x^2+8x^3+\ldots$

ii Valid for $|2x|<1\Rightarrow|x|<\frac{1}{2}$

1 d i $\dfrac{4}{2+3x}$ Write in index form

$= 4(2+3x)^{-1}$ Take out a factor of 2

$= 4\left(2\left(1+\dfrac{3x}{2}\right)\right)^{-1}$

$= 4\times 2^{-1}\times\left(1+\dfrac{3x}{2}\right)^{-1}$ Use binomial expansion with $n=-1$ and $x=\dfrac{3x}{2}$

$= 2\left(1+(-1)\left(\dfrac{3x}{2}\right)+\dfrac{(-1)(-2)}{2!}\left(\dfrac{3x}{2}\right)^2+\dfrac{(-1)(-2)(-3)}{3!}\left(\dfrac{3x}{2}\right)^3+\ldots\right)$

$= 2\left(1-\dfrac{3x}{2}+\dfrac{9x^2}{4}-\dfrac{27x^3}{8}+\ldots\right)$ Multiply by 2

$= 2-3x+\dfrac{9x^2}{2}-\dfrac{27x^3}{4}+\ldots$

ii Valid for $\left|\dfrac{3x}{2}\right|<1 \Rightarrow |x|<\dfrac{2}{3}$

e i $\dfrac{4}{\sqrt{4-x}}=4(\sqrt{4-x})^{-1}$ Write in index form

$= 4(4-x)^{-\frac{1}{2}}$ Take out a factor of 4

$= 4\left(4\left(1-\dfrac{x}{4}\right)\right)^{-\frac{1}{2}}$

$= 4\times 4^{-\frac{1}{2}}\left(1-\dfrac{x}{4}\right)^{-\frac{1}{2}}$ Use binomial expansion with $n=-\dfrac{1}{2}$ and $x=-\dfrac{x}{4}$

$= 4^{\frac{1}{2}}\left(1+\left(-\dfrac{1}{2}\right)\left(-\dfrac{x}{4}\right)+\dfrac{(-\frac{1}{2})(-\frac{3}{2})}{2!}\left(-\dfrac{x}{4}\right)^2+\dfrac{(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})}{3!}\left(-\dfrac{x}{4}\right)^3+\ldots\right)$

$= 2\left(1+\dfrac{x}{8}+\dfrac{3}{128}x^2-\dfrac{5}{1024}x^3+\ldots\right)$ Multiply by 2

$= 2+\dfrac{x}{4}+\dfrac{3}{64}x^2+\dfrac{5}{512}x^3+\ldots$

ii Valid $\left|-\dfrac{x}{4}\right|<1 \Rightarrow |x|<4$

1 f i $\frac{1+x}{1+3x}=(1+x)(1+3x)^{-1}$ Write $\frac{1}{1+3x}$ in index form then expand

$=(1+x)\left(1+(-1)(3x)+\frac{(-1)(-2)(3x)^2}{2!}+\frac{(-1)(-2)(-3)(3x)^3}{3!}+\ldots\right)$

$=(1+x)(1-3x+9x^2-27x^3+\ldots)$ Multiply out

$=1-3x+9x^2-27x^3+x-3x^2+9x^3+\ldots$ Collect like terms

$=1-2x+6x^2-18x^3+\ldots$

ii Valid for $|3x|<1\Rightarrow|x|<\frac{1}{3}$

g i $\left(\frac{1+x}{1-x}\right)^2=\frac{(1+x)^2}{(1-x)^2}$ Write in index form

$=(1+x)^2(1-x)^{-2}$ Expand $(1-x)^{-2}$ using binomial expansion

$=(1+2x+x^2)\left(1+(-2)(-x)+\frac{(-2)(-3)(-x)^2}{2!}+\frac{(-2)(-3)(-4)(-x)^3}{3!}+\ldots\right)$

$=(1+2x+x^2)(1+2x+3x^2+4x^3+\ldots)$ Multiply out brackets

$=1+2x+3x^2+4x^3+2x+4x^2+6x^3+x^2+2x^3+\ldots$ Collect like terms

$=1+4x+8x^2+12x^3+\ldots$

ii Valid for $|x|<1$

1 h i Let $\frac{x-3}{(1-x)(1-2x)} \equiv \frac{A}{(1-x)} + \frac{B}{(1-2x)}$ Put in partial fraction form

$\equiv \frac{A(1-2x)+B(1-x)}{(1-x)(1-2x)}$ Add fractions.

Set the numerators equal:

$x - 3 = A(1 - 2x) + B(1 - x)$

Substitute $x = 1$:

$1 - 3 = A \times -1 + B \times 0$

$\Rightarrow -2 = -1A$

$\Rightarrow A = 2$

Substitute $x = \frac{1}{2}$: $\frac{1}{2} - 3 = A \times 0 + B \times \frac{1}{2}$

$\Rightarrow -2\frac{1}{2} = \frac{1}{2}B$

$\Rightarrow B = -5$

Hence $\frac{x-3}{(1-x)(1-2x)} \equiv \frac{2}{(1-x)} - \frac{5}{(1-2x)}$

$$\frac{2}{(1-x)} = 2(1-x)^{-1}$$
$$= 2\left(1 + (-1)(-x) + \frac{(-1)(-2)(-x)^2}{2!} + \frac{(-1)(-2)(-3)(-x)^3}{3!} + \ldots\right)$$
$$= 2(1 + x + x^2 + x^3 + \ldots)$$
$$= 2 + 2x + 2x^2 + 2x^3 + \ldots$$

$$\frac{5}{(1-2x)} = 5(1-2x)^{-1}$$
$$= 5\left(1 + (-1)(-2x) + \frac{(-1)(-2)(-2x)^2}{2!} + \frac{(-1)(-2)(-3)(-2x)^3}{3!} + \ldots\right)$$
$$= 5(1 + 2x + 4x^2 + 8x^3 + \ldots)$$
$$= 5 + 10x + 20x^2 + 40x^3 + \ldots$$

Hence $\frac{x-3}{(1-x)(1-2x)} \equiv \frac{2}{(1-x)} - \frac{5}{(1-2x)}$

$= (2 + 2x + 2x^2 + 2x^3 + \ldots) - (5 + 10x + 20x^2 + 40x^3 + \ldots)$

$= -3 - 8x - 18x^2 - 38x^3 + \ldots$

1 h ii $\frac{2}{1-x}$ is valid for $|x| < 1$

$\frac{5}{1-2x}$ is valid for $|2x|<1 \Rightarrow |x|<\frac{1}{2}$

Both are valid when $|x|<\frac{1}{2}$

2 $$\left(1-\frac{1}{2}x\right)^{\frac{1}{2}} = 1+\left(\frac{1}{2}\right)\left(-\frac{1}{2}x\right)+\frac{\left(\frac{1}{2}\right)\left(\frac{1}{2}-1\right)}{2!}\left(-\frac{1}{2}x\right)^2+\frac{\left(\frac{1}{2}\right)\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{3!}\left(-\frac{1}{2}x\right)^3+\ldots$$

$$=1+\left(\frac{1}{2}\right)\left(-\frac{1}{2}x\right)+\frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2}\frac{1}{4}x^2-\frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{6}\frac{1}{8}x^3+\ldots$$

$$=1-\frac{1}{4}x-\frac{1}{32}x^2-\frac{1}{128}x^3+\ldots$$

3 a Using binomial expansion

$$(1+x)^{\frac{1}{2}} = 1+\left(\frac{1}{2}\right)(x)+\frac{(\frac{1}{2})(-\frac{1}{2})(x)^2}{2!}+\frac{(\frac{1}{2})(-\frac{1}{2})(-\frac{3}{2})(x)^3}{3!}+\ldots$$

$$=1+\frac{1}{2}x-\frac{1}{8}x^2+\frac{1}{16}x^3+\ldots$$

Expansion is valid if $|x| < 1$.

b Substituting $x=\frac{1}{4}$ in both sides of the expansion gives

$$\left(1+\frac{1}{4}\right)^{\frac{1}{2}} \approx 1+\frac{1}{2}\times\frac{1}{4}-\frac{1}{8}\times\left(\frac{1}{4}\right)^2+\frac{1}{16}\times\left(\frac{1}{4}\right)^3$$

$$\left(\frac{5}{4}\right)^{\frac{1}{2}} \approx 1+\frac{1}{8}-\frac{1}{128}+\frac{1}{1024}$$

$$\sqrt{\frac{5}{4}} \approx \frac{1145}{1024}$$

$\frac{\sqrt{5}}{2} \approx \frac{1145}{1024}$ Multiply both sides by 2

$$\sqrt{5} \approx \frac{1145}{512}$$

4 a $(1+9x)^{\frac{2}{3}} = 1+\left(\frac{2}{3}\right)(9x)+\frac{\left(\frac{2}{3}\right)\left(\frac{2}{3}-1\right)}{2!}(9x)^2+\frac{\left(\frac{2}{3}\right)\left(\frac{2}{3}-1\right)\left(\frac{2}{3}-2\right)}{3!}(9x)^3+\ldots$

$= 1+\left(\frac{2}{3}\right)(9x)+\frac{\left(\frac{2}{3}\right)\left(-\frac{1}{3}\right)}{2}81x^2+\frac{\left(\frac{2}{3}\right)\left(-\frac{1}{3}\right)\left(-\frac{4}{3}\right)}{6}729x^3+\ldots$

$= 1+6x-9x^2+36x^3+\ldots$

Equating coefficients:
$c = -9$ and $d = 36$

b $1 + 9x = 1.45$
$x = 0.05$

$(1.45)^{\frac{2}{3}} \approx 1+6(0.05)-9(0.05)^2+36(0.05)^3$
$= 1.282$

c $(1.45)^{\frac{2}{3}} = 1.28108713$

The approximation is correct to 2 decimal places.

5 a The x^2 term of $(1+ax)^{\frac{1}{2}} = \frac{\left(\frac{1}{2}\right)\left(\frac{1}{2}-1\right)}{2!}(ax)^2$

$-\frac{1}{8}a^2 = -2$

$a^2 = 16$

$a = \pm 4$

b The x^3 term of $(1+ax)^{\frac{1}{2}} = \frac{\left(\frac{1}{2}\right)\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{3!}(ax)^3$

When $a = 4$:

$\frac{\left(\frac{1}{2}\right)\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{3!}(ax)^3 = \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{6}(4x)^3$

$= 4x^3$

When $a = -4$:

$\frac{\left(\frac{1}{2}\right)\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{3!}(ax)^3 = \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{6}(-4x)^3$

$= -4x^3$

The coefficient of the x^3 term is 4 or −4

6 **a** $(1+3x)^{-1}$ Use binomial expansion with $n=-1$ and $x=3x$

$$=1+(-1)(3x)+\frac{(-1)(-2)(3x)^2}{2!}+\frac{(-1)(-2)(-3)(3x)^3}{3!}+\dots$$

$$=1-3x+9x^2-27x^3+\dots$$

b $\frac{1+x}{1+3x}=(1+x)(1+3x)^{-1}$ Use expansion from part **a**

$=(1+x)(1-3x+9x^2-27x^3+\dots)$ Multiply out

$=1-3x+9x^2-27x^3+x-3x^2+9x^3+\dots$ Collect like terms

$=1-2x+6x^2-18x^3+\dots$ Ignore terms greater than x^3

Hence $\frac{1+x}{1+3x}\approx 1-2x+6x^2-18x^3$

c Substitute $x=0.01$ into both sides of the above

$$\frac{1+0.01}{1+3\times 0.01}\approx 1-2\times 0.01+6\times 0.01^2-18\times 0.01^3$$

$$\frac{1.01}{1.03}\approx 1-0.02+0.0006-0.000018, \quad \left(\frac{1.01}{1.03}=\frac{101}{103}\right)$$

$\frac{101}{103}\approx 0.980582$ Round to 5 d.p.

$\frac{101}{103}\approx 0.98058$ (5 d.p.)

7 **a** Using binomial expansion

$$(1+ax)^n=1+n(ax)+\frac{n(n-1)(ax)^2}{2!}+\frac{n(n-1)(n-2)(ax)^3}{3!}+\dots$$

If coefficient of x is -6 then $na=-6$ **(1)**

If coefficient of x^2 is 27 then $\frac{n(n-1)a^2}{2}=27$ **(2)**

From (**1**), $a=\frac{-6}{n}$. Substitute in (**2**):

$$\frac{n(n-1)}{2}\left(\frac{-6}{n}\right)^2=27$$

$$\frac{n(n-1)}{2}\times\frac{36}{n^2}=27$$

$$\frac{(n-1)18}{n}=27$$

$(n-1)18=27n$

$18n-18=27n$

$-18=9n$

$n=-2$

Substitute $n=-2$ back in (**1**): $-2a=-6\Rightarrow a=3$

7 **b** Coefficient of x^3 is

$$\frac{n(n-1)(n-2)a^3}{3!}=\frac{(-2)\times(-3)\times(-4)\times 3^3}{3\times 2\times 1}=-108$$

c $(1+3x)^{-2}$ is valid if $|3x|<1 \Rightarrow |x|<\frac{1}{3}$

8

$$\frac{3}{\sqrt{4+x}}=3(\sqrt{4+x})^{-1} \quad \text{Write in index form}$$

$$=3(4+x)^{-\frac{1}{2}} \quad \text{Take out a factor of 4}$$

$$=3\left(4\left(1+\frac{x}{4}\right)\right)^{-\frac{1}{2}}$$

$$=3\times 4^{-\frac{1}{2}}\times\left(1+\frac{x}{4}\right)^{-\frac{1}{2}} \quad 4^{-\frac{1}{2}}=\frac{1}{4^{\frac{1}{2}}}=\frac{1}{2}$$

$$=\frac{3}{2}\times\left(1+\left(-\frac{1}{2}\right)\left(\frac{x}{4}\right)+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(\frac{x}{4}\right)^2}{2!}+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\left(\frac{x}{4}\right)^3}{3!}+\dots\right)$$

$$=\frac{3}{2}\left(1-\frac{x}{8}+\frac{3}{128}x^2+\dots\right) \quad \text{Multiply by } \frac{3}{2}$$

$$=\frac{3}{2}-\frac{3}{16}x+\frac{9}{256}x^2+\dots$$

$$\approx\frac{3}{2}-\frac{3}{16}x+\frac{9}{256}x^2 \quad \text{if terms higher than } x^2 \text{ are ignored.}$$

9 a

$$\frac{1}{\sqrt{4-x}}=(4-x)^{-\frac{1}{2}}=\left(4\left(1-\frac{1}{4}x\right)\right)^{-\frac{1}{2}}=\frac{1}{2}\left(1-\frac{1}{4}x\right)^{-\frac{1}{2}}$$

$$=\frac{1}{2}\left(1+\left(-\frac{1}{2}\right)\left(-\frac{1}{4}x\right)+\frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)}{2!}\left(-\frac{1}{4}x\right)^2+\dots\right)$$

$$=\frac{1}{2}\left(1+\left(-\frac{1}{2}\right)\left(-\frac{1}{4}x\right)+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2}\frac{1}{16}x^2+\dots\right)$$

$$=\frac{1}{2}\left(1+\frac{1}{8}x+\frac{3}{128}x^2+\dots\right)$$

$$=\frac{1}{2}+\frac{1}{16}x+\frac{3}{256}x^2+\dots$$

9 b $\frac{1+2x}{\sqrt{4-x}} = (1+2x)\left(\frac{1}{2}+\frac{1}{16}x+\frac{3}{256}x^2+\ldots\right)$

$= \frac{1}{2}+\frac{1}{16}x+\frac{3}{256}x^2+x+\frac{1}{8}x^2+\ldots$

$= \frac{1}{2}+\frac{17}{16}x+\frac{35}{256}x^2+\ldots$

10 a $(2+3x)^{-1}$ Take out factor of 2

$= \left(2\left(1+\frac{3x}{2}\right)\right)^{-1}$

$= 2^{-1}\left(1+\frac{3x}{2}\right)^{-1}$ Use binomial expansion with $n=-1$ and $x=\frac{3x}{2}$

$= \frac{1}{2}\left(1+(-1)\left(\frac{3x}{2}\right)+\frac{(-1)(-2)}{2!}\left(\frac{3x}{2}\right)^2+\frac{(-1)(-2)(-3)}{3!}\left(\frac{3x}{2}\right)^3+\ldots\right)$

$= \frac{1}{2}\left(1-\frac{3}{2}x+\frac{9}{4}x^2-\frac{27}{8}x^3+\ldots\right)$ Multiply by $\frac{1}{2}$

$= \frac{1}{2}-\frac{3}{4}x+\frac{9}{8}x^2-\frac{27}{16}x^3+\ldots$

Valid for $\left|\frac{3x}{2}\right|<1 \Rightarrow |x|<\frac{2}{3}$

b $\frac{1+x}{2+3x}$ Put in index form

$= (1+x)(2+3x)^{-1}$ Use expansion from part **a**

$= (1+x)\left(\frac{1}{2}-\frac{3}{4}x+\frac{9}{8}x^2-\frac{27}{16}x^3+\ldots\right)$ Multiply out

$= \frac{1}{2}-\frac{3}{4}x+\frac{9}{8}x^2-\frac{27}{16}x^3+\frac{1}{2}x-\frac{3}{4}x^2+\frac{9}{8}x^3+\ldots$ Collect like terms

$= \frac{1}{2}-\frac{1}{4}x+\frac{3}{8}x^2-\frac{9}{16}x^3+\ldots$

Valid for $\left|\frac{3x}{2}\right|<1 \Rightarrow |x|<\frac{2}{3}$

11 a $(4+x)^{-\frac{1}{2}} = \left(4\left(1+\frac{x}{4}\right)\right)^{-\frac{1}{2}}$ Take out factor of 4

$= 4^{-\frac{1}{2}}\left(1+\frac{x}{4}\right)^{-\frac{1}{2}}$ $\quad 4^{-\frac{1}{2}} = \frac{1}{4^{\frac{1}{2}}} = \frac{1}{2}$

$= \frac{1}{2}\left(1+\frac{x}{4}\right)^{-\frac{1}{2}}$ Use binomial expansion with $n = -\frac{1}{2}$ and $x = \frac{x}{4}$

$$= \frac{1}{2}\left(1+\left(-\frac{1}{2}\right)\left(\frac{x}{4}\right)+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(\frac{x}{4}\right)^2}{2!}+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\left(\frac{x}{4}\right)^3}{3!}+\ldots\right)$$

$$= \frac{1}{2}\left(1-\frac{1}{8}x+\frac{3}{128}x^2-\frac{5}{1024}x^3+\ldots\right)$$

$$= \frac{1}{2}-\frac{1}{16}x+\frac{3}{256}x^2-\frac{5}{2048}x^3+\ldots$$

Valid for $\left|\frac{x}{4}\right| < 1 \Rightarrow |x| < 4$

b i $(4+x)^{-\frac{1}{2}} \approx \frac{1}{2}-\frac{x}{16}+\frac{3x^2}{256}-\frac{5x^3}{2048}$

When $x = -2$

$$\frac{1}{\sqrt{4+(-2)}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

So $\frac{\sqrt{2}}{2} \approx \frac{1}{2}-\frac{(-2)}{16}+\frac{3(-2)^2}{256}-\frac{5(-2)^3}{2048}$

$\approx \frac{1}{2}+\frac{2}{16}+\frac{12}{256}+\frac{40}{2048}$

$\approx \frac{177}{256}$

So $\sqrt{2} \approx 2\times\frac{177}{256} = 1.3828$ (4 d.p.)

b ii $(4+x)^{-\frac{1}{2}} \approx \frac{1}{2} - \frac{x}{16} + \frac{3x^2}{256} - \frac{5x^3}{2048}$

When $x = \frac{1}{2}$

$$\frac{1}{\sqrt{4+\frac{1}{2}}} = \frac{1}{\sqrt{\frac{9}{2}}} = \frac{\sqrt{2}}{3}$$

So $\frac{\sqrt{2}}{3} \approx \frac{1}{2} - \frac{\left(\frac{1}{2}\right)}{16} + \frac{3\left(\frac{1}{2}\right)^2}{256} - \frac{5\left(\frac{1}{2}\right)^3}{2048}$

$$\approx \frac{1}{2} - \frac{1}{32} + \frac{3}{1024} - \frac{5}{16384}$$

$$\approx \frac{7723}{16384}$$

So $\sqrt{2} \approx 3 \times \frac{7723}{16384} = 1.4141$ (4 d.p.)

c $x = \frac{1}{2}$ because it is closer to 0.

12 $(3+4x)^{-3} = \left(3\left(1+\frac{4}{3}x\right)\right)^{-3} = \frac{1}{27}\left(1+\frac{4}{3}x\right)^{-3}$

$$= \frac{1}{27}\left(1+(-3)\left(\frac{4}{3}x\right) + \frac{(-3)(-4)}{2!}\left(\frac{4}{3}x\right)^2 + \ldots\right)$$

$$= \frac{1}{27}\left(1-4x+\frac{32}{3}x^2+\ldots\right)$$

$$= \frac{1}{27} - \frac{4}{27}x + \frac{32}{81}x^2 + \ldots$$

13 a $\frac{39x+12}{(x+1)(x+4)(x-8)} \equiv \frac{A}{x+1}+\frac{B}{x+4}+\frac{C}{x-8}$

$\equiv \frac{A(x+4)(x-8)+B(x+1)(x-8)+C(x+1)(x+4)}{(x+1)(x+4)(x-8)}$

$39x+12 \equiv A(x+4)(x-8)+B(x+1)(x-8)+C(x+1)(x+4)$

Let $x=-1$:

$-39+12 = A\times 3\times(-9)+0+0$

$-27=-27A$

$A=1$

Let $x=-4$:

$-156+12=0+B\times(-3)\times(-12)+0$

$-144=36B$

$B=-4$

Let $x=8$:

$312+12=0+0+C\times 9\times 12$

$324=108C$

$C=3$

$A = 1$, $B = -4$ and $C = 3$

13 b $\frac{39x+12}{(x+1)(x+4)(x-8)} \equiv \frac{1}{x+1} - \frac{4}{x+4} + \frac{3}{x-8}$

$\frac{1}{x+1} - \frac{4}{x+4} + \frac{3}{x-8} = (1+x)^{-1} - 4(4+x)^{-1} + 3(-8+x)^{-1}$

$= (1+x)^{-1} - 4\left(4\left(1+\frac{1}{4}x\right)\right)^{-1} + 3\left(-8\left(1-\frac{1}{8}x\right)\right)^{-1}$

$= (1+x)^{-1} - \left(1+\frac{1}{4}x\right)^{-1} - \frac{3}{8}\left(1-\frac{1}{8}x\right)^{-1}$

$(1+x)^{-1} = 1 + (-1)x + \frac{(-1)(-2)}{2!}x^2 + \ldots$

$= 1 - x + x^2 + \ldots$

$\left(1+\frac{1}{4}x\right)^{-1} = 1 + (-1)\left(\frac{1}{4}x\right) + \frac{(-1)(-2)}{2!}\left(\frac{1}{4}x\right)^2 + \ldots$

$= 1 - \frac{1}{4}x + \frac{1}{16}x^2 + \ldots$

$\frac{3}{8}\left(1-\frac{1}{8}x\right)^{-1} = \frac{3}{8}\left(1 + (-1)\left(-\frac{1}{8}x\right) + \frac{(-1)(-2)}{2!}\left(-\frac{1}{8}x\right)^2 + \ldots\right)$

$= \frac{3}{8} + \frac{3}{64}x + \frac{3}{512}x^2 + \ldots$

$\frac{39x+12}{(x+1)(x+4)(x-8)} = \left(1 - x + x^2 + \ldots - \left(1 - \frac{1}{4}x + \frac{1}{16}x^2 + \ldots\right) - \left(\frac{3}{8} + \frac{3}{64}x + \frac{3}{512}x^2 + \ldots\right)\right)$

$= -\frac{3}{8} - \frac{51}{64}x + \frac{477}{512}x^2 + \ldots$

14 a $\frac{12x+5}{(1+4x)^2} \equiv \frac{A}{1+4x} + \frac{B}{(1+4x)^2}$

$\equiv \frac{A(1+4x)+B}{(1+4x)^2}$

$12x + 5 \equiv A(1+4x) + B$

Let $x = -\frac{1}{4}$:

$-3 + 5 = 0 + B$

$B = 2$

Let $x = 0$:

$5 = A \times 1 + B$

$5 = A + 2$

$A = 3$

$A = 3, B = 2$

14 b $\dfrac{12x+5}{(1+4x)^2} \equiv \dfrac{3}{1+4x} + \dfrac{2}{(1+4x)^2}$

$= 3(1+4x)^{-1} + 2(1+4x)^{-2}$

$3(1+4x)^{-1} = 3\left(1 + (-1)(4x) + \dfrac{(-1)(-2)}{2!}(4x)^2 + \ldots\right)$

$= 3\left(1 - 4x + 16x^2 + \ldots\right)$

$= 3 - 12x + 48x^2 + \ldots$

$2(1+4x)^{-2} = 2\left(1 + (-2)(4x) + \dfrac{(-2)(-3)}{2!}(4x)^2 + \ldots\right)$

$= 2\left(1 - 8x + 48x^2 + \ldots\right)$

$= 2 - 16x + 96x^2 + \ldots$

$\dfrac{12x+5}{(1+4x)^2} = 3 - 12x + 48x^2 + 2 - 16x + 96x^2 + \ldots$

$= 5 - 28x + 144x^2 + \ldots$

15 a $\dfrac{9x^2+26x+20}{(1+x)(2+x)} \equiv A + \dfrac{B}{1+x} + \dfrac{C}{2+x}$

$$\begin{array}{r} 9 \\ x^2+3x+2 \overline{\smash{)}\, 9x^2+26x+20} \\ \underline{9x^2+27x+18} \\ -x+2 \end{array}$$

$A = 9$

$\dfrac{9x^2+26x+20}{(1+x)(2+x)} \equiv 9 + \dfrac{-x+2}{(1+x)(2+x)}$

$\dfrac{-x+2}{(1+x)(2+x)} \equiv \dfrac{B}{1+x} + \dfrac{C}{2+x}$

$= \dfrac{B(2+x)+C(1+x)}{(1+x)(2+x)}$

$-x+2 \equiv B(2+x)+C(1+x)$

Let $x=-1$:

$1+2 = B\times 1 + 0$:

$B = 3$

Let $x=-2$:

$2+2 = 0 + C\times(-1)$:

$C = -4$

15 a (continued)

$$\frac{9x^2+26x+20}{(1+x)(2+x)} \equiv 9+\frac{3}{1+x}-\frac{4}{2+x}$$
$$=9+3(1+x)^{-1}-4(2+x)^{-1}$$
$$=9+3(1+x)^{-1}-4\left(2\left(1+\frac{1}{2}x\right)\right)^{-1}$$
$$=9+3(1+x)^{-1}-2\left(1+\frac{1}{2}x\right)^{-1}$$

$$3(1+x)^{-1}=3\left(1+(-1)x+\frac{(-1)(-2)}{2!}x^2+\frac{(-1)(-2)(-3)}{3!}x^3+\ldots\right)$$
$$=3\left(1-x+x^2-x^3+\ldots\right)$$
$$=3-3x+3x^2-3x^3+\ldots$$

$$2\left(1+\frac{1}{2}x\right)^{-1}=2\left(1+(-1)\left(\frac{1}{2}x\right)+\frac{(-1)(-2)}{2!}\left(\frac{1}{2}x\right)^2+\frac{(-1)(-2)(-3)}{3!}\left(\frac{1}{2}x\right)^3+\ldots\right)$$
$$=2\left(1-\frac{1}{2}x+\frac{1}{4}x^2-\frac{1}{8}x^3+\ldots\right)$$
$$=2-x+\frac{1}{2}x^2-\frac{1}{4}x^3+\ldots$$
$$9+\frac{3}{1+x}-\frac{4}{2+x}=9+3-3x+3x^2-3x^3+\ldots-\left(2-x+\frac{1}{2}x^2-\frac{1}{4}x^3+\ldots\right)$$
$$=10-2x+\frac{5}{2}x^2-\frac{11}{4}x^3+\ldots$$

Equating coefficients gives:
$$B=\frac{5}{2}, C=-\frac{11}{4}$$

15 b $q(0.1) = \frac{9(0.1)^2 + 26(0.1) + 20}{(1+0.1)(2+0.1)} = 9.822510823$

Using the expansion:

$q(0.1) \approx 10 - 2(0.1) + \frac{5}{2}(0.1)^2 - \frac{11}{4}(0.1)^3 = 9.82225$

Percentage error $= \frac{9.822510823 - 9.82225}{9.822510823} \times 100 = 0.0027\%$

Challenge

$f(x) = \frac{1}{\sqrt{1+3x^2}} = (\sqrt{1+3x^2})^{-1}$

$= (1+3x^2)^{-\frac{1}{2}}$ Use binomial expansion with $n = -\frac{1}{2}$ and $x = 3x^2$

$= 1 + \left(-\frac{1}{2}\right)(3x^2) + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)(3x^2)^2}{2!} + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)(3x^2)^3}{3!} + \ldots$

$= 1 - \frac{3x^2}{2} + \frac{27x^4}{8} - \frac{135x^6}{16} + \ldots$

Valid for $|3x^2| < 1$