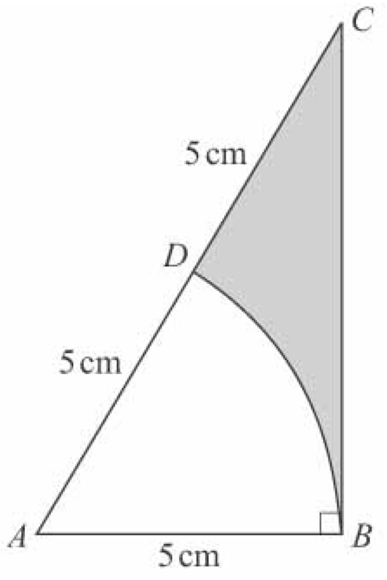
Radians Mixed exercise

1



a In the right-angled triangle ABC :

$\cos \angle BAC = \frac{BA}{AC} = \frac{5}{10} = \frac{1}{2}$

so $\angle BAC = \frac{\pi}{3}$

b Area of triangle ABC

$= \frac{1}{2} \times AB \times AC \times \sin \angle BAC$

$= \frac{1}{2} \times 5 \times 10 \times \sin \frac{\pi}{3} = 21.650...\text{cm}^2$

Area of sector DAB

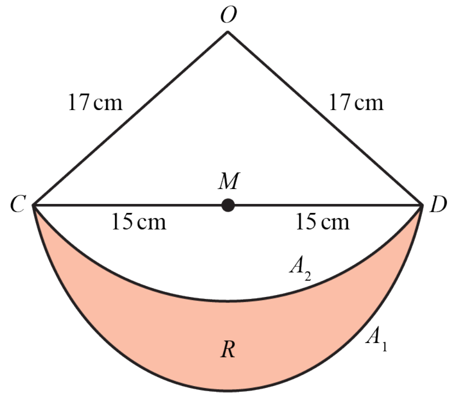
$= \frac{1}{2} \times 5^2 \times \frac{\pi}{3} = 13.089...\text{cm}^2$

Area of shaded region

$=$ area of $\triangle ABC$ $-$ area of sector DAB

$= 21.650... - 13.089... = 8.56\,\text{cm}^2$ (3 s.f.)

2



a Using Pythagoras' theorem to find OM :

$OM^2 = 17^2 - 15^2 = 64 \Rightarrow OM = 8\,\text{cm}$

Area of $\triangle OCD = \frac{1}{2} \times CD \times OM$

$= \frac{1}{2} \times 30 \times 8 = 120\,\text{cm}^2$

b Area of shaded region R

$=$ area of semicircle CDA_1

$-$ area of segment CDA_2

Area of semicircle CDA_1

$= \frac{1}{2} \times \pi \times 15^2 = 353.429...\text{cm}^2$

Area of segment CDA_2

$=$ area of sector OCD

$-$ area of triangle OCD

$= \frac{1}{2} \times 17^2 \times \angle COD - 120$

In right-angled triangle COM:

$\sin \angle COM = \frac{CM}{OC} = \frac{15}{17}$

so $\angle COM = 1.0808...$

hence $\angle COD = 2.1616...$

So area of segment CDA_2

$= \frac{1}{2} \times 17^2 \times 2.1616... - 120$

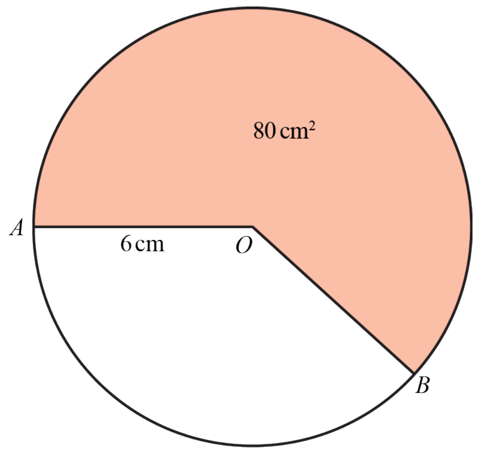
$= 192.362...\text{cm}^2$

So area of shaded region R

$= 353.429... - 192.362...$

$= 161.07\,\text{cm}^2$ (2 d.p.)

3



a Reflex angle $AOB = (2\pi - \theta)$ rad
Area of shaded sector

$= \frac{1}{2} \times 6^2 \times (2\pi - \theta)$

$= (36\pi - 18\theta)\text{ cm}^2$

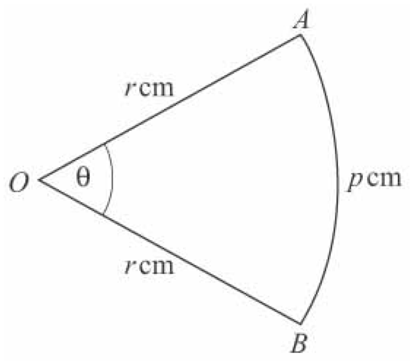
So $80 = 36\pi - 18\theta$

$\Rightarrow 18\theta = 36\pi - 80$

$\Rightarrow \theta = \frac{36\pi - 80}{18} = 1.839$ (3 d.p.)

b Length of minor arc AB
$= 6\theta = 6 \times 1.8387... = 11.03\text{ cm}$ (2 d.p.)

4



a Using $l = r\theta$:

$p = r\theta \Rightarrow \quad \theta = \frac{p}{r}$

b Area of sector
$= \frac{1}{2}r^2\theta = \frac{1}{2}r^2 \times \frac{p}{r} = \frac{1}{2}pr\text{ cm}^2$

c $4.65 \leqslant r < 4.75,\ 5.25 \leqslant p < 5.35$
Least possible value for area of sector

$= \frac{1}{2} \times 5.25 \times 4.65 = 12.207\text{ cm}^2$ (3 d.p.)

(Note: Least possible value is 12.20625, so 12.207 should be given, not 12.206)

d Maximum possible value of θ

$= \frac{\max p}{\min r} = \frac{5.35}{4.65} = 1.1505...$

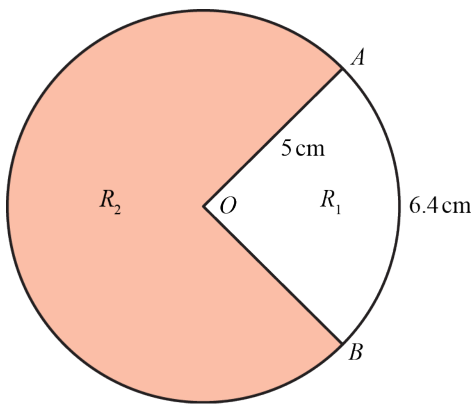
So give 1.150 (3 d.p.)

Minimum possible value of θ

$= \frac{\min p}{\max r} = \frac{5.25}{4.75} = 1.1052...$

So give 1.106 (3 d.p.)

5



a Using $l = r\theta$:

$6.4 = 5\theta \Rightarrow \theta = \frac{6.4}{5} = 1.28\text{ rad}$

b Using area of sector $= \frac{1}{2}r^2\theta$:

$R_1 = \frac{1}{2} \times 5^2 \times 1.28 = 16$

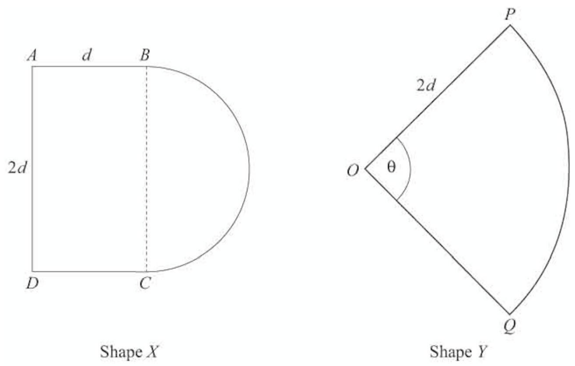
c R_2 = area of circle $- R_1$

$= \pi \times 5^2 - 16 = 62.5398...$

So $\frac{R_1}{R_2} = \frac{16}{62.5398...} = \frac{1}{3.908...} = \frac{1}{p}$

$\Rightarrow p = 3.91$ (3 s.f.)

6



a Area of shape X
$= \text{area of rectangle} + \text{area of semicircle}$
$= (2d^2 + \frac{1}{2}\pi d^2)\ \text{cm}^2$

Area of shape $Y = \frac{1}{2}(2d)^2\theta = 2d^2\theta\ \text{cm}^2$

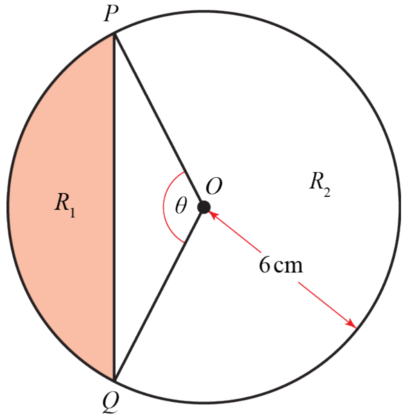
Since $X = Y$:
$2d^2 + \frac{1}{2}\pi d^2 = 2d^2\theta$
Divide by $2d^2$:
$1 + \frac{\pi}{4} = \theta$

b Perimeter of shape X
$= (d + 2d + d + \pi d)\ \text{cm}$ with $d = 3$
$= (3\pi + 12)\ \text{cm}$

c Perimeter of shape Y
$= (2d + 2d + 2d\theta)\ \text{cm}$
with $d = 3$ and $\theta = 1 + \frac{\pi}{4}$
$= 12 + 6\left(1 + \frac{\pi}{4}\right)$
$= \left(18 + \frac{3\pi}{2}\right)\ \text{cm}$

d Difference
$= \left(18 + \frac{3\pi}{2}\right) - (3\pi + 12)$
$= 6 - \frac{3\pi}{2}$
$= 1.287...\ \text{cm}$
$= 12.9\ \text{mm}$ (3 s.f.)

7

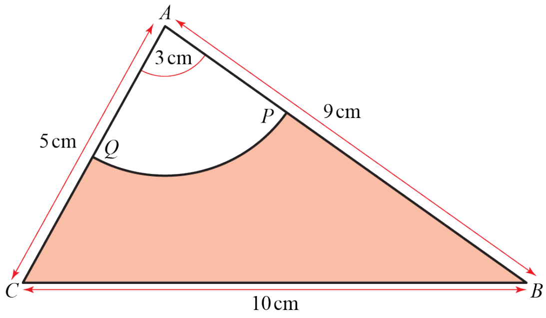


a Area of segment R_1
$= \text{area of sector } OPQ$
$\quad - \text{area of triangle } OPQ$
$\Rightarrow A_1 = \frac{1}{2} \times 6^2 \times \theta - \frac{1}{2} \times 6^2 \times \sin\theta$
$\Rightarrow A_1 = 18(\theta - \sin\theta)$

b $A_2 = \text{area of circle} - A_1$
$= \pi \times 6^2 - 18(\theta - \sin\theta)$
$= 36\pi - 18(\theta - \sin\theta)$

Since $A_2 = 3A_1$:
$36\pi - 18(\theta - \sin\theta) = 3 \times 18(\theta - \sin\theta)$
$36\pi - 18(\theta - \sin\theta) = 54(\theta - \sin\theta)$
$36\pi = 72(\theta - \sin\theta)$
$\pi = \theta - \sin\theta$
$\sin\theta = \theta - \frac{\pi}{2}$

8



a Using the cosine rule in $\triangle ABC$:

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

$$\Rightarrow \cos \angle BAC = \frac{5^2 + 9^2 - 10^2}{2 \times 5 \times 9} = 0.0\dot{6}$$

$$\Rightarrow \angle BAC = 1.50408...$$

$$= 1.504 \text{ rad (3 d.p.)}$$

b i Using the sector area formula:

$$\text{area of sector} = \frac{1}{2}r^2\theta$$

$$\Rightarrow \text{area of sector } APQ$$

$$= \frac{1}{2} \times 3^2 \times 1.504 = 6.77 \text{ cm}^2 \text{ (3 s.f.)}$$

ii Area of shaded region $BPQC$

$= \text{area of } \triangle ABC - \text{area of sector } APQ$

$$= \frac{1}{2} \times 5 \times 9 \times \sin 1.504 - \frac{1}{2} \times 3^2 \times 1.504$$

$$= 15.681...$$

$$= 15.7 \text{ cm}^2 \text{ (3 s.f.)}$$

iii Perimeter of shaded region $BPQC$

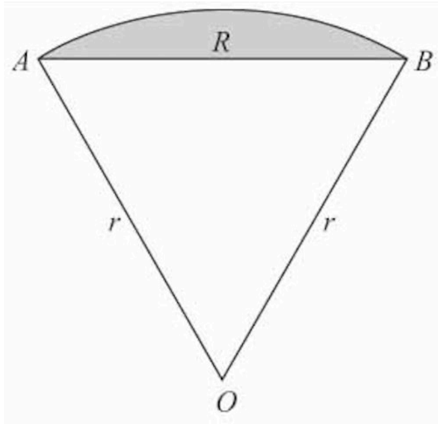
$= QC + CB + BP + \text{arc length } PQ$

$= 2 + 10 + 6 + (3 \times 1.504)$

$= 22.51...$

$= 22.5 \text{ cm (3 s.f.)}$

9



a Area of sector $= \frac{1}{2}r^2\theta = \frac{1}{2}r^2 \times 1.5 \text{ cm}^2$

So $\frac{3}{4}r^2 = 15$

$$\Rightarrow r^2 = \frac{60}{3} = 20$$

$$\Rightarrow r = \sqrt{20} = \sqrt{4 \times 5} = \sqrt{4} \times \sqrt{5} = 2\sqrt{5}$$

b Arc length $AB = r(1.5) = 3\sqrt{5}$ cm

Perimeter of sector

$= AO + OB + \text{arc length } AB$

$= 2\sqrt{5} + 2\sqrt{5} + 3\sqrt{5}$

$= 7\sqrt{5}$

$= 15.7 \text{ cm (3 s.f.)}$

c Area of segment R

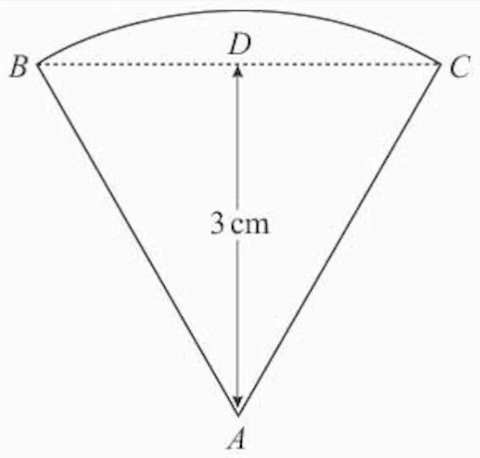
$= \text{area of sector} - \text{area of } \triangle AOB$

$$= 15 - \frac{1}{2}r^2 \sin 1.5$$

$$= 15 - 10 \sin 1.5$$

$$= 5.025 \text{ cm}^2 \text{ (3 d.p.)}$$

10



a Using the right-angled $\triangle ABD$, with

$\angle ABD = \frac{\pi}{3}$:

$$\sin\frac{\pi}{3} = \frac{3}{AB}$$

$$\Rightarrow AB = \frac{3}{\sin\frac{\pi}{3}} = \frac{3}{\frac{\sqrt{3}}{2}}$$

$$= 3 \times \frac{2}{\sqrt{3}} = 2\sqrt{3} \text{ cm}$$

b Area of badge = area of sector

$$= \frac{1}{2} \times (2\sqrt{3})^2 \theta \text{ where } \theta = \frac{\pi}{3}$$

$$= \frac{1}{2} \times 4 \times 3 \times \frac{\pi}{3}$$

$$= 2\pi \text{ cm}^2$$

c Perimeter of badge

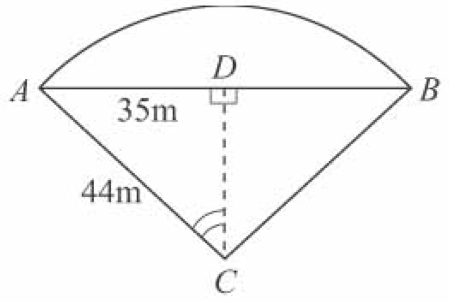
$= AB + AC +$ arc length BC

$$= 2\sqrt{3} + 2\sqrt{3} + 2\sqrt{3} \times \frac{\pi}{3}$$

$$= 2\sqrt{3}\left(2 + \frac{\pi}{3}\right)$$

$$= \frac{2\sqrt{3}}{3}(6 + \pi) \text{ cm}$$

11



a Using the right-angled $\triangle ADC$:

$$\sin\angle ACD = \frac{35}{44}$$

So $\angle ACD = \sin^{-1}\frac{35}{44}$

and $\angle ACB = 2\sin^{-1}\frac{35}{44}$

$$\Rightarrow \angle ACB = 1.8395...$$

$$= 1.84 \text{ rad (2 d.p.)}$$

b i Length of railway track

= length of arc AB

$= 44 \times 1.8395...$

$= 80.9$m (3 s.f.)

ii Shortest distance from C to AB is DC.
Using Pythagoras' theorem:

$$DC^2 = 44^2 - 35^2$$

$$DC = \sqrt{44^2 - 35^2} = 26.7\text{m (3 s.f.)}$$

iii Area of region

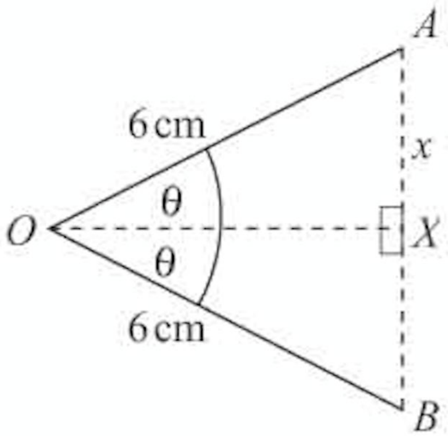
= area of segment

= area of sector ABC – area of $\triangle ABC$

$$= \frac{1}{2} \times 44^2 \times 1.8395... - \frac{1}{2} \times 70 \times DC$$

$$= 847\text{m}^2 \text{ (3 s.f.)}$$

12



a In right-angled $\triangle OAX$ (see diagram):

$\frac{x}{6} = \sin\theta$

$\Rightarrow x = 6\sin\theta$

So $AB = 2x = 12\sin\theta$ $(AB = DC)$

Perimeter of the cross-section

$= \text{arc length } AB + AD + DC + BC$

$= 6 \times 2\theta + 4 + 12\sin\theta + 4 \text{ cm}$

$= (8 + 12\theta + 12\sin\theta) \text{ cm}$

So $2(7 + \pi) = 8 + 12\theta + 12\sin\theta$

$\Rightarrow 14 + 2\pi = 8 + 12\theta + 12\sin\theta$

$\Rightarrow 12\theta + 12\sin\theta - 6 = 2\pi$

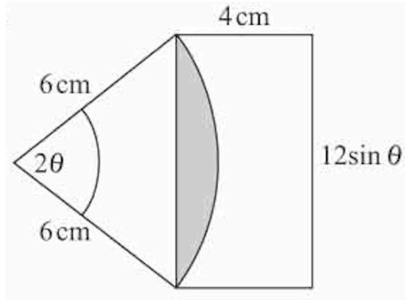
Divide by 6:

$2\theta + 2\sin\theta - 1 = \frac{\pi}{3}$

b When $\theta = \frac{\pi}{6}$,

$$2\theta + 2\sin\theta - 1 = \frac{\pi}{3} + \left(2 \times \frac{1}{2}\right) - 1$$
$$= \frac{\pi}{3}$$

c



Area of the cross-section
= area of rectangle $ABCD$
− area of shaded segment

Area of rectangle $= 4 \times 12 \times \sin\frac{\pi}{6}$
$= 24 \text{ cm}^2$

Area of shaded segment
= area of sector − area of triangle

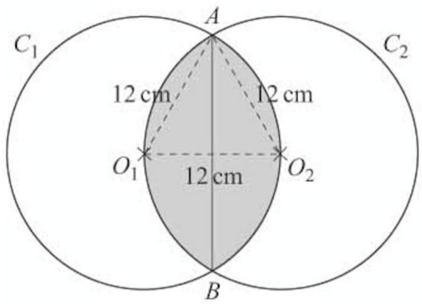
$= \frac{1}{2} \times 6^2 \times \frac{\pi}{3} - \frac{1}{2} \times 6^2 \times \sin\frac{\pi}{3}$

$= 3.261\ldots \text{cm}^2$

So area of cross-section

$= 20.7 \text{ cm}^2$ (3 s.f.)

13



a $O_1A = O_2A = 12$, as they are radii of their respective circles.
$O_1O_2 = 12$, as O_2 is on the circumference of C_1 and hence is a radius (and vice versa).
Therefore $\triangle AO_1O_2$ is equilateral.

So $\angle AO_1O_2 = \frac{\pi}{3}$

and $\angle AO_1B = 2 \times \angle AO_1O_2 = \frac{2\pi}{3}$

13 b Consider arc AO_2B of circle C_1.

Using arc length $= r\theta$:

arc length $AO_2B = 12 \times \frac{2\pi}{3} = 8\pi$ cm

Perimeter of R

$= $ arc length $AO_2B +$ arc length AO_1B

$= 2 \times 8\pi = 16\pi$ cm

c Consider the segment AO_2B in circle C_1.

Area of segment AO_2B

$=$ area of sector $O_1AB -$ area of $\triangle O_1AB$

$$= \frac{1}{2} \times 12^2 \times \frac{2\pi}{3} - \frac{1}{2} \times 12^2 \times \sin\frac{2\pi}{3}$$

$= 88.442\ldots \text{ cm}^2$

Area of region R

$=$ area of segment AO_2B

$+$ area of segment AO_1B

$= 2 \times 88.442\ldots$

$= 177 \text{ cm}^2$ (3 s.f.)

14 a The student has used an angle measured in degrees – it needs to be measured in radians to use that formula.

b

$$50° = \frac{50}{180} \times \pi \text{ rad}$$

$$\frac{1}{2}r^2\theta = \frac{1}{2} \times 3^2 \times \frac{5}{18}\pi$$

$$= \frac{5}{4}\pi \text{ cm}^2$$

15 a

$$\frac{\cos\theta - 1}{\theta\tan 2\theta} \approx \frac{\left(1 - \frac{\theta^2}{2}\right) - 1}{\theta \times 2\theta}$$

$$= \frac{-\frac{\theta^2}{2}}{2\theta^2}$$

$$= \frac{-\theta^2}{4\theta^2}$$

$$= -\frac{1}{4}$$

b

$$\frac{2(1-\cos\theta) - 1}{\tan\theta - 1} \approx \frac{2\left(1 - \left(1 - \frac{\theta^2}{2}\right)\right) - 1}{\theta - 1}$$

$$= \frac{2 \times \frac{\theta^2}{2} - 1}{\theta - 1}$$

$$= \frac{\theta^2 - 1}{\theta - 1}$$

$$= \frac{(\theta-1)(\theta+1)}{\theta - 1}$$

$$= \theta + 1$$

16 a

$$\frac{7 + 2\cos 2\theta}{\tan 2\theta + 3} \approx \frac{7 + 2\left(1 - \frac{(2\theta)^2}{2}\right)}{2\theta + 3}$$

$$= \frac{7 + 2\left(1 - \frac{4\theta^2}{2}\right)}{2\theta + 3}$$

$$= \frac{9 - 4\theta^2}{2\theta + 3}$$

$$= \frac{(3 + 2\theta)(3 - 2\theta)}{2\theta + 3}$$

$$= 3 - 2\theta$$

b 3

17 a When θ is small:

LHS $= 32\cos 5\theta + 203\tan 10\theta$

$$\approx 32\left(1 - \frac{(5\theta)^2}{2}\right) + 203(10\theta)$$

$$= 32 - 16(25\theta^2) + 2010\theta$$

So $32 - 400\theta^2 - 2030\theta = 182$

$$400\theta^2 + 2030\theta + 150 = 0$$

$$40\theta^2 + 203\theta + 15 = 0$$

b $40\theta^2 + 203\theta + 15 = 0$

$(40\theta - 3)(\theta - 5) = 0$

$$\theta = \frac{3}{40}, 5$$

17 c $\theta = 5$ is not a valid solution, as 5 is not 'small'. $\frac{3}{40}$ is 'small', so this solution is valid.

18 $\cos^4\theta - \sin^4\theta$
$= (\cos^2\theta - \sin^2\theta)(\cos^2\theta + \sin^2\theta)$
$= \cos^2\theta - \sin^2\theta$
$= 1 - \sin^2\theta - \sin^2\theta$
$= 1 - 2\sin^2\theta$
$\approx 1 - 2\theta^2$

19 a $3\sin\theta = 2,\ 0 \leqslant \theta \leqslant \pi$

$\sin\theta = \dfrac{2}{3}$

$\theta = 0.730, 2.41$

b $\sin\theta = -\cos\theta,\ -\pi \leqslant \theta \leqslant \pi$

$\tan\theta = -1$

$\theta = -\dfrac{\pi}{4}, \dfrac{3\pi}{4}$

c $\tan\theta + \dfrac{1}{\tan\theta} = 2, 0 \leqslant \theta \leqslant 2\pi$

$\tan^2\theta + 1 = 2\tan\theta$

$\tan^2\theta - 2\tan\theta + 1 = 0$

$(\tan\theta - 1)^2 = 0$

$\tan\theta = 1$

$\theta = \dfrac{\pi}{4}, \dfrac{5\pi}{4}$

d $2\sin^2\theta - \sin\theta - 1 = \sin^2\theta,\ -\pi \leqslant \theta \leqslant \pi$

$\sin^2\theta - \sin\theta - 1 = 0$

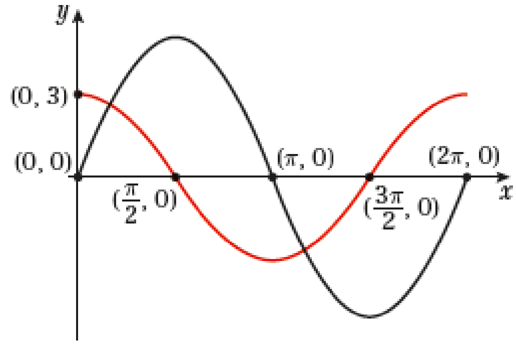
$\sin\theta = \dfrac{1 \pm \sqrt{5}}{2}$

$\sin\theta = 1.618$ (no solutions)

or $\sin\theta = -0.618$

$\Rightarrow \theta = -0.666, -2.48$

20 a



b The curves intersect twice in the given range, so the equation has two solutions.

c $5\sin x = 3\cos x$

$\tan x = \dfrac{3}{5}$

$x = 0.540, 3.68$

21 a $4\sin\theta - \cos\left(\dfrac{\pi}{2} - \theta\right) = 4\sin\theta - \sin\theta$

$= 3\sin\theta$

b $4\sin\theta - \cos\left(\dfrac{\pi}{2} - \theta\right) = 1,\ 0 \leqslant \theta \leqslant 2\pi$

$3\sin\theta = 1$

$\sin\theta = \dfrac{1}{3}$

$\theta = 0.340, 2.80$

22 $\dfrac{\sin 2x + 0.5}{1 - \sin 2x} = 2,\ \ 0 < x < \dfrac{3\pi}{2}$

$\sin 2x + 0.5 = 2 - 2\sin 2x$

$3\sin 2x = 1.5$

$\sin 2x = 0.5$

Let $X = 2x$

$\sin X = 0.5,\ \ 0 < X < 3\pi$

$X = \dfrac{\pi}{6}, \dfrac{5\pi}{6}, \dfrac{13\pi}{6}, \dfrac{17\pi}{6}$

$x = \dfrac{\pi}{12}, \dfrac{5\pi}{12}, \dfrac{13\pi}{12}, \dfrac{17\pi}{12}$

23 a Cosine can be negative, so do not reject $-\dfrac{1}{\sqrt{2}}$

b Rearranged incorrectly, so incorrectly square rooted.

c
$$2\cos^2 x = 1$$
$$\cos^2 x = \frac{1}{2}$$
$$\cos x = \pm\frac{1}{\sqrt{2}}$$
$$x = -\frac{3\pi}{4}, -\frac{\pi}{4}, \frac{\pi}{4}, \frac{3\pi}{4}$$

24 a Not all solutions have been calculated. There will be four solutions in the given interval.

b $2\tan 2x = 5,\ 0 \leqslant x \leqslant 2\pi$
Let $X = 2x$
$2\tan X = 5,\ 0 \leqslant X \leqslant 4\pi$
$\tan X = 2.5$
$X = 1.19, 4.33, 7.47, 10.6$
$x = 0.595, 2.17, 3.74, 5.31$

25 a
$$5\sin x = 1 + \cos^2 x$$
$$5\sin x = 1 + 2(1 - \sin^2 x)$$
$$2\sin^2 x + 5\sin x - 3 = 0$$

b $2\sin^2 x + 5\sin x - 3 = 0,\ 0 \leqslant x \leqslant 2\pi$
$(2\sin x - 1)(\sin x + 3) = 0$
$\sin x = 0.5 \Rightarrow x = \dfrac{\pi}{6}, \dfrac{5\pi}{6}$
or $\sin x = -3$ (no solution)

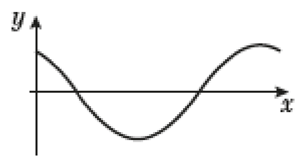
26 a
$$4\sin^2 x + 9\cos x - 6 = 0$$
$$4(1 - \cos^2 x) + 9\cos x - 6 = 0$$
$$4 - 4\cos^2 x + 9\cos x - 6 = 0$$
$$4\cos^2 x - 9\cos x + 2 = 0$$

b $4\cos^2 x - 9\cos x + 2 = 0,\ 0 \leqslant \theta \leqslant 4\pi$
$(4\cos x - 1)(\cos x - 2) = 0$
$\cos x = 2$ (no solution)
or $\cos x = 0.25 \Rightarrow x = 1.3, 5.0, 7.6, 11.2$

27 a
$$\tan 2x = 5\sin 2x$$
$$\frac{\sin 2x}{\cos 2x} = 5\sin 2x$$
$$\sin 2x = 5\sin 2x\cos 2x$$
$$(1 - 5\cos 2x)\sin 2x = 0$$

b $\sin 2x(1 - 5\cos 2x) = 0,\ 0 \leqslant x \leqslant \pi$
Let $X = 2x$
$\sin X(1 - 5\cos X) = 0,\ 0 \leqslant X \leqslant 2\pi$
$\sin X = 0 \Rightarrow X = 0, \pi, 2\pi$
or $\cos X = 0.2 \Rightarrow X = 1.37, 4.91$
$x = 0, 0.7, \dfrac{\pi}{2}, 2.5, \pi$

28 a



b $\left(0, \dfrac{\sqrt{3}}{2}\right), \left(\dfrac{\pi}{3}, 0\right), \left(\dfrac{4\pi}{3}, 0\right)$

c $\cos\left(x + \dfrac{\pi}{6}\right) = 0.65,\ 0 \leqslant x \leqslant 2\pi$
Let $X = x + \dfrac{\pi}{6}$
$\cos X = 0.65,\ \dfrac{\pi}{6} \leqslant x \leqslant \dfrac{13\pi}{6}$
$X = 0.863, 5.42$
$x = 0.34, 4.90$

29 $\sin\left(3x+\frac{\pi}{3}\right)=0.45, 0 \le x \le \pi$

Let $X = 3x+\frac{\pi}{3}, \frac{\pi}{3} \le X \le \frac{10\pi}{3}$

$\sin X = 0.45$

$X = 2.67, 6.75, 8.96$

$x = 0.54, 1.90$ or 2.64 (2 d.p.)

Challenge

a $9\sin\theta\tan\theta + 25\tan\theta = 6$

When θ is small:

$\text{LHS} \approx 9\theta^2 + 25\theta$

so $9\theta^2 + 25\theta = 6$

$9\theta^2 + 25\theta - 6 = 0$

$(9\theta - 2)(\theta + 3) = 0$

$\theta = \frac{2}{9}$ or $\theta = -3$

$\theta = \frac{2}{9}$ is 'small', so this value is valid.

$\theta = -3$ is not 'small', so this value is not valid. 'Small' in this context is 'close to 0'.

b $2\tan\theta + 3 = 5\cos 4\theta$

When θ is small:

$\text{LHS} \approx 2\theta + 3$ and $\text{RHS} \approx 5\left(1-\frac{(4\theta)^2}{2}\right)$

so $2\theta + 3 = 5 - 40\theta^2$

$40\theta^2 + 2\theta - 2 = 0$

$20\theta^2 + \theta - 1 = 0$

$(4\theta + 1)(5\theta - 1) = 0$

$\theta = -\frac{1}{4}, \theta = \frac{1}{5}$

Both values of θ could be considered 'small' in this case so both solutions are valid.

c $\sin 4\theta = 37 - 2\cos 2\theta$

When is small:

$\text{LHS} \approx 4\theta$

and $\text{RHS} \approx 37 - 2\left(1-\frac{(2\theta)^2}{2}\right)$

so $4\theta = 37 - 2 + 4\theta^2$

$4\theta^2 - 4\theta + 35 = 0$

$b^2 - 4ac < 0$

So there are no solutions.