Trigonometric Functions 6D

1 a Use $1+\tan^2\theta=\sec^2\theta$

with θ replaced with $\frac{1}{2}\theta$

$1+\tan^2\left(\frac{1}{2}\theta\right)=\sec^2\left(\frac{1}{2}\theta\right)$

b $(\sec\theta-1)(\sec\theta+1)$ (multiply out)

$$=\sec^2\theta-1$$
$$=(1+\tan^2\theta)-1$$
$$=\tan^2\theta$$

c $\tan^2\theta(\operatorname{cosec}^2\theta-1)$

$$=\tan^2\theta\left((1+\cot^2\theta)-1\right)$$
$$=\tan^2\theta\cot^2\theta$$
$$=\tan^2\theta\times\frac{1}{\tan^2\theta}$$
$$=1$$

d $(\sec^2\theta-1)\cot\theta$

$$=\tan^2\theta\cot\theta$$
$$=\tan^2\theta\times\frac{1}{\tan\theta}$$
$$=\tan\theta$$

e $(\operatorname{cosec}^2\theta-\cot^2\theta)^2$

$$=\left((1+\cot^2\theta)-\cot^2\theta\right)^2$$
$$=1^2=1$$

f $2-\tan^2\theta+\sec^2\theta$

$$=2-\tan^2\theta+(1+\tan^2\theta)$$
$$=2-\tan^2\theta+1+\tan^2\theta$$
$$=3$$

g $\dfrac{\tan\theta\sec\theta}{1+\tan^2\theta}$

$$=\frac{\tan\theta\sec\theta}{\sec^2\theta}$$
$$=\frac{\tan\theta}{\sec\theta}$$
$$=\tan\theta\cos\theta$$
$$=\frac{\sin\theta}{\cos\theta}\times\cos\theta$$
$$=\sin\theta$$

h $(1-\sin^2\theta)(1+\tan^2\theta)$

$$=\cos^2\theta\times\sec^2\theta$$
$$=\cos^2\theta\times\frac{1}{\cos^2\theta}$$
$$=1$$

i $\dfrac{\operatorname{cosec}\theta\cot\theta}{1+\cot^2\theta}$

$$=\frac{\operatorname{cosec}\theta\cot\theta}{\operatorname{cosec}^2\theta}$$
$$=\frac{1}{\operatorname{cosec}\theta}\times\cot\theta$$
$$=\frac{\sin\theta}{1}\times\frac{\cos\theta}{\sin\theta}$$
$$=\cos\theta$$

j $\sec^4\theta-2\sec^2\theta\tan^2\theta+\tan^4\theta$

$$=(\sec^2\theta-\tan^2\theta)^2 \quad \text{(factorise)}$$
$$=\left((1+\tan^2\theta)-\tan^2\theta\right)^2$$
$$=1^2=1$$

k $4\operatorname{cosec}^2 2\theta+4\operatorname{cosec}^2 2\theta\cot^2 2\theta$

$$=4\operatorname{cosec}^2 2\theta(1+\cot^2 2\theta)$$
$$=4\operatorname{cosec}^2 2\theta\operatorname{cosec}^2 2\theta$$
$$=4\operatorname{cosec}^4 2\theta$$

2 $\operatorname{cosec}x=\dfrac{k}{\operatorname{cosec}x}$

$$\Rightarrow\ \operatorname{cosec}^2 x=k$$
$$\Rightarrow\ 1+\cot^2 x=k$$
$$\Rightarrow\ \cot^2 x=k-1$$
$$\Rightarrow\ \cot x=\pm\sqrt{k-1}$$

3 a $\cot\theta = \sqrt{3} \qquad 90° < \theta < 180°$

$\Rightarrow \cot^2\theta = 3$

$\Rightarrow 1 + \cot^2\theta = 1 + 3 = 4$

$\Rightarrow \operatorname{cosec}^2\theta = 4$

$\Rightarrow \sin^2\theta = \frac{1}{4}$

$\Rightarrow \sin\theta = \frac{1}{2}$

(as θ is in 2nd quadrant, $\sin\theta$ is positive)

b Using $\sin^2\theta + \cos^2\theta = 1$

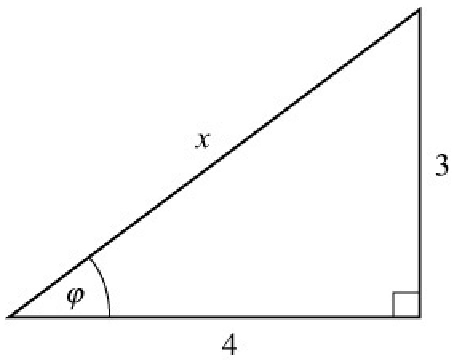
$\Rightarrow \cos^2\theta = 1 - \sin^2\theta = 1 - \frac{1}{4} = \frac{3}{4}$

$\Rightarrow \cos\theta = -\frac{\sqrt{3}}{4}$

(as θ is in 2nd quadrant, $\cos\theta$ is negative)

4 $\tan\theta = \frac{3}{4} \qquad 180° < \theta < 270°$

Draw a right-angled triangle where $\tan\varphi = \frac{3}{4}$



Using Pythagoras' theorem, $x = 5$

So $\cos\varphi = \frac{4}{5}$ and $\sin\varphi = \frac{3}{5}$

As θ is in the 3rd quadrant, both $\sin\theta$ and $\cos\theta$ are negative.

a $\sec\theta = \frac{1}{\cos\theta} = -\frac{1}{\cos\varphi} = -\frac{5}{4}$

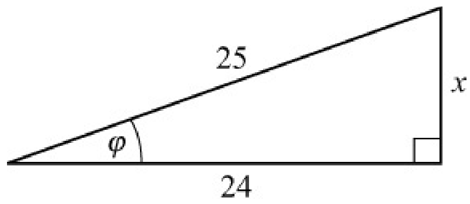
b $\cos\theta = -\cos\varphi = -\frac{4}{5}$

c $\sin\theta = -\sin\varphi = -\frac{3}{5}$

5 $\cos\theta = \frac{24}{25}$, θ reflex

As $\cos\theta$ is positive and θ reflex, θ is in the 4th quadrant.

Use right-angled triangle where $\cos\varphi = \frac{24}{25}$



Using Pythagoras' theorem,

$25^2 = x^2 + 24^2$

$\Rightarrow x^2 = 25^2 - 24^2 = 49$

$\Rightarrow x = 7$

So $\tan\varphi = \frac{7}{24}$ and $\sin\varphi = \frac{7}{25}$

As θ is in the 4th quadrant, both $\tan\theta$ and $\sin\theta$ are negative

a $\tan\theta = -\frac{7}{24}$

b $\operatorname{cosec}\theta = \frac{1}{\sin\theta} = -\frac{1}{\frac{7}{25}} = -\frac{25}{7}$

6 a LHS $\equiv \sec^4\theta - \tan^4\theta$

$\equiv (\sec^2\theta - \tan^2\theta)(\sec^2\theta + \tan^2\theta)$

(difference of two squares)

$\equiv (1)(\sec^2\theta + \tan^2\theta)$

(as $1 + \tan^2\theta \equiv \sec^2\theta$

$\Rightarrow \sec^2\theta - \tan^2\theta \equiv 1$)

$\equiv \sec^2\theta + \tan^2\theta \equiv$ RHS

b LHS $\equiv \operatorname{cosec}^2 x - \sin^2 x$

$\equiv (1 + \cot^2 x) - (1 - \cos^2 x)$

$\equiv 1 + \cot^2 x - 1 + \cos^2 x$

$\equiv \cot^2 x + \cos^2 x \equiv$ RHS

6 c LHS $\equiv \sec^2 A(\cot^2 A - \cos^2 A)$

$$\equiv \frac{1}{\cos^2 A}\left(\frac{\cos^2 A}{\sin^2 A} - \cos^2 A\right)$$

$$\equiv \frac{1}{\sin^2 A} - 1 \equiv \text{cosec}^2 A - 1$$

(use $1 + \cot^2\theta = \text{cosec}^2\theta$)

$$\equiv 1 + \cot^2 A - 1$$

$$\equiv \cot^2 A \equiv \text{ RHS}$$

d RHS $\equiv (\sec^2\theta - 1)(1 - \sin^2\theta)$

$$\equiv \tan^2\theta \times \cos^2\theta$$

(use $1 + \tan^2\theta \equiv \sec^2\theta$ and $\cos^2\theta + \sin^2\theta \equiv 1$)

$$\equiv \frac{\sin^2\theta}{\cos^2\theta} \times \cos^2\theta \equiv \sin^2\theta$$

$$\equiv 1 - \cos^2\theta \equiv \text{ LHS}$$

e LHS $\equiv \dfrac{1 - \tan^2 A}{1 + \tan^2 A} \equiv \dfrac{1 - \tan^2 A}{\sec^2 A}$

$$\equiv \frac{1}{\sec^2 A}(1 - \tan^2 A)$$

$$\equiv \cos^2 A\left(1 - \frac{\sin^2 A}{\cos^2 A}\right)$$

$$\equiv \cos^2 A - \sin^2 A$$

$$\equiv (1 - \sin^2 A) - \sin^2 A$$

$$\equiv 1 - 2\sin^2 A \equiv \text{ RHS}$$

f RHS $\equiv \sec^2\theta\,\text{cosec}^2\theta$

$$\equiv \sec^2\theta(1 + \cot^2\theta)$$

$$\equiv \sec^2\theta + \frac{1}{\cos^2\theta} \times \frac{\cos^2\theta}{\sin^2\theta}$$

$$\equiv \sec^2\theta + \frac{1}{\sin^2\theta}$$

$$\equiv \sec^2\theta + \text{cosec}^2\theta \equiv \text{ LHS}$$

Alternatively

LHS $\equiv \sec^2\theta + \text{cosec}^2\theta \equiv \dfrac{1}{\cos^2\theta} + \dfrac{1}{\sin^2\theta}$

$$\equiv \frac{\sin^2\theta + \cos^2\theta}{\cos^2\theta\sin^2\theta} \equiv \frac{1}{\cos^2\theta\sin^2\theta}$$

$$\equiv \sec^2\theta\,\text{cosec}^2\theta \equiv \text{ RHS}$$

g LHS $\equiv \text{cosec}\,A\sec^2 A$

$$\equiv \text{cosec}\,A(1 + \tan^2 A)$$

$$\equiv \text{cosec}\,A + \frac{1}{\sin A} \times \frac{\sin^2 A}{\cos^2 A}$$

$$\equiv \text{cosec}\,A + \frac{\sin A}{\cos^2 A}$$

$$\equiv \text{cosec}\,A + \frac{\sin A}{\cos A} \times \frac{1}{\cos A}$$

$$\equiv \text{cosec}\,A + \tan A\sec A \equiv \text{ RHS}$$

h LHS $\equiv (\sec\theta - \sin\theta)(\sec\theta + \sin\theta)$

$$\equiv \sec^2\theta - \sin^2\theta$$

$$\equiv (1 + \tan^2\theta) - (1 - \cos^2\theta)$$

$$\equiv 1 + \tan^2\theta - 1 + \cos^2\theta$$

$$\equiv \tan^2\theta + \cos^2\theta \equiv \text{ RHS}$$

7 $3\tan^2\theta + 4\sec^2\theta = 5$

$\Rightarrow\ 3\tan^2\theta + 4(1 + \tan^2\theta) = 5$

$\Rightarrow\ 3\tan^2\theta + 4 + 4\tan^2\theta = 5$

$\Rightarrow\ 7\tan^2\theta = 1$

$\Rightarrow\ \tan^2\theta = \dfrac{1}{7}$

$\Rightarrow\ \cot^2\theta = 7$

$\Rightarrow\ \text{cosec}^2\theta - 1 = 7$

$\Rightarrow\ \text{cosec}^2\theta = 8$

$\Rightarrow\ \sin^2\theta = \dfrac{1}{8}$

As θ is obtuse (in the 2nd quadrant), $\sin\theta$ is positive.

So $\sin\theta = \sqrt{\dfrac{1}{8}} = \dfrac{1}{2\sqrt{2}} = \dfrac{\sqrt{2}}{4}$

8 a $\sec^2\theta = 3\tan\theta \qquad 0 \le \theta \le 360°$

$\Rightarrow 1+\tan^2\theta = 3\tan\theta$

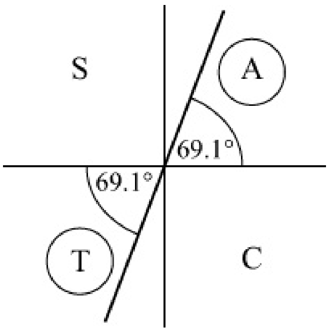
$\Rightarrow \tan^2\theta - 3\tan\theta + 1 = 0$

$\tan\theta = \dfrac{3\pm\sqrt{5}}{2}$

(equation does not factorise)

For $\tan\theta = \dfrac{3+\sqrt{5}}{2}$,

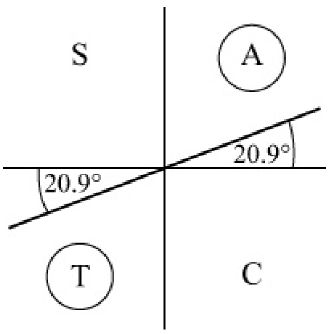
calculator value is 69.1° (3 s.f.)



Solutions are 69.1°, 249°

For $\tan\theta = \dfrac{3-\sqrt{5}}{2}$,

calculator value is 20.9° (3 s.f.)



Solutions are 20.9°, 201°

Set of solutions: 20.9°, 69.1°, 201°, 249° (3 s.f.)

b $\tan^2\theta - 2\sec\theta + 1 = 0 \qquad -\pi \le \theta \le \pi$

$\Rightarrow (\sec^2\theta - 1) - 2\sec\theta + 1 = 0$

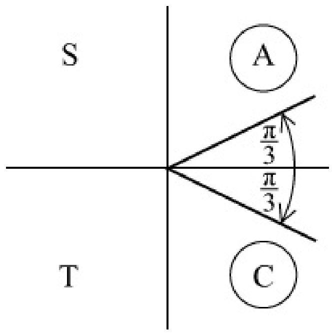
$\Rightarrow \sec^2\theta - 2\sec\theta = 0$

$\Rightarrow \sec\theta(\sec\theta - 2) = 0$

$\Rightarrow \sec\theta = 2 \quad$ (as $\sec\theta$ cannot be 0)

$\Rightarrow \cos\theta = \dfrac{1}{2}$

$\Rightarrow \theta = -\dfrac{\pi}{3},\ \dfrac{\pi}{3}$



c $\operatorname{cosec}^2\theta + 1 = 3\cot\theta \qquad -180° \le \theta \le 180°$

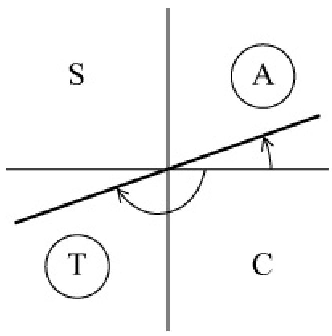
$\Rightarrow (1+\cot^2\theta) + 1 = 3\cot\theta$

$\Rightarrow \cot^2\theta - 3\cot\theta + 2 = 0$

$\Rightarrow (\cot\theta - 1)(\cot\theta - 2) = 0$

$\Rightarrow \cot\theta = 1$ or $\cot\theta = 2$

$\Rightarrow \tan\theta = 1$ or $\tan\theta = \dfrac{1}{2}$



$\tan\theta = 1 \Rightarrow \theta = -135°,\ 45°$

$\tan\theta = \dfrac{1}{2} \Rightarrow \theta = -153°,\ 26.6°$ (3 s.f.)

8 d $\cot\theta = 1 - \operatorname{cosec}^2\theta \qquad 0 \le \theta \le 2\pi$

$\Rightarrow \cot\theta = 1-(1+\cot^2\theta)$

$\Rightarrow \cot\theta = -\cot^2\theta$

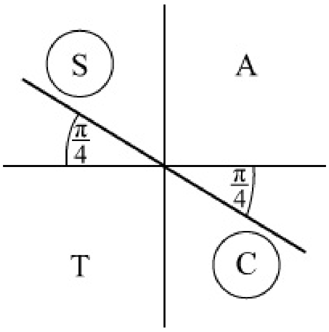
$\Rightarrow \cot^2\theta + \cot\theta = 0$

$\Rightarrow \cot\theta(\cot\theta+1) = 0$

$\Rightarrow \cot\theta = 0$ or $\cot\theta = -1$

For $\cot\theta = 0$ refer to graph: $\theta = \frac{\pi}{2}, \frac{3\pi}{2}$

For $\cot\theta = -1, \ \tan\theta = -1$



So $\theta = \frac{3\pi}{4}, \frac{7\pi}{4}$

Set of solutions: $\frac{\pi}{2}, \frac{3\pi}{4}, \frac{3\pi}{2}, \frac{7\pi}{4}$

e $3\sec\frac{1}{2}\theta = 2\tan^2\frac{1}{2}\theta \qquad 0 \le \theta \le 360°$

$\Rightarrow 3\sec\frac{1}{2}\theta = 2\left(\sec^2\frac{1}{2}\theta - 1\right)$

(use $1+\tan^2 A \equiv \sec^2 A$ with $A = \frac{1}{2}\theta$)

$\Rightarrow 2\sec^2\frac{1}{2}\theta - 3\sec\frac{1}{2}\theta - 2 = 0$

$\Rightarrow \left(2\sec\frac{1}{2}\theta+1\right)\left(\sec\frac{1}{2}\theta-2\right)=0$

$\Rightarrow \sec\frac{1}{2}\theta = -\frac{1}{2}$ or $\sec\frac{1}{2}\theta = 2$

Only $\sec\frac{1}{2}\theta = 2$ applies as

$\sec A \le -1$ or $\sec A \ge 1$

$\Rightarrow \cos\frac{1}{2}\theta = \frac{1}{2}$

As $0 \le \theta \le 360°$ so $0 \le \frac{1}{2}\theta \le 180°$

Calculator value is 60°

This is the only value in the interval.

So $\frac{1}{2}\theta = 60°$

$\Rightarrow \theta = 120°$

f $(\sec\theta - \cos\theta)^2 = \tan\theta - \sin^2\theta \qquad 0 \le \theta \le \pi$

$\Rightarrow \sec^2\theta - 2\sec\theta\cos\theta + \cos^2\theta$

$= \tan\theta - \sin^2\theta$

$\Rightarrow \sec^2\theta - 2 + \cos^2\theta = \tan\theta - \sin^2\theta$

$\left(\sec\theta\cos\theta = \frac{1}{\cos\theta}\times\cos\theta = 1\right)$

$\Rightarrow (1+\tan^2\theta) - 2 + (\cos^2\theta + \sin^2\theta)$

$= \tan\theta$

$\Rightarrow 1 + \tan^2\theta - 2 + 1 = \tan\theta$

$\Rightarrow \tan^2\theta - \tan\theta = 0$

$\Rightarrow \tan\theta(\tan\theta - 1) = 0$

$\Rightarrow \tan\theta = 0$ or $\tan\theta = 1$

$\tan\theta = 0 \Rightarrow \theta = 0, \pi$

$\tan\theta = 1 \Rightarrow \theta = \frac{\pi}{4}$

Set of solutions: $0, \frac{\pi}{4}, \pi$

g $\tan^2 2\theta = \sec 2\theta - 1 \qquad 0 \le \theta \le 180°$

$\Rightarrow \sec^2 2\theta - 1 = \sec 2\theta - 1$

$\Rightarrow \sec^2 2\theta - \sec 2\theta = 0$

$\Rightarrow \sec 2\theta(\sec 2\theta - 1) = 0$

$\Rightarrow \sec 2\theta = 0$ (not possible)

or $\sec 2\theta = 1$

$\Rightarrow \cos 2\theta = 1 \qquad 0 \le 2\theta \le 360°$

Refer to graph of $y = \cos\theta$

$\Rightarrow 2\theta = 0°, 360°$

$\Rightarrow \theta = 0°, 180°$

8 h $\sec^2\theta - (1+\sqrt{3})\tan\theta + \sqrt{3} = 1$

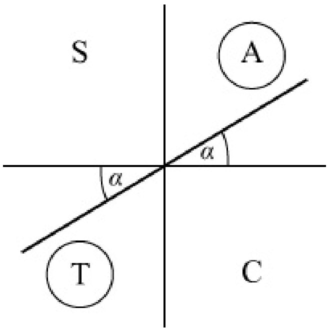
for $0 \le \theta \le 2\pi$

$\Rightarrow (1+\tan^2\theta) - (1+\sqrt{3})\tan\theta + \sqrt{3} = 1$

$\Rightarrow \tan^2\theta - (1+\sqrt{3})\tan\theta + \sqrt{3} = 0$

$\Rightarrow (\tan\theta - \sqrt{3})(\tan\theta - 1) = 0$

$\Rightarrow \tan\theta = \sqrt{3}$ or $\tan\theta = 1$



First answer for $\tan\theta = \sqrt{3}$ is $\frac{\pi}{3}$

Second solution is $\pi + \frac{\pi}{3} = \frac{4\pi}{3}$

First answer for $\tan\theta = 1$ is $\frac{\pi}{4}$

Second solution is $\pi + \frac{\pi}{4} = \frac{5\pi}{4}$

Set of solutions: $\frac{\pi}{4}, \frac{\pi}{3}, \frac{5\pi}{4}, \frac{4\pi}{3}$

9 a $\tan^2 k = 2\sec k$

$\Rightarrow (\sec^2 k - 1) = 2\sec k$

$\Rightarrow \sec^2 k - 2\sec k - 1 = 0$

$\Rightarrow \sec k = \frac{2 \pm \sqrt{8}}{2} = \frac{2 \pm 2\sqrt{2}}{2} = 1 \pm \sqrt{2}$

As $\sec k$ has no values between -1 and 1

$\sec k = 1 + \sqrt{2}$

b $\cos k = \frac{1}{1+\sqrt{2}} = \frac{\sqrt{2}-1}{\left(1+\sqrt{2}\right)\left(\sqrt{2}-1\right)}$

$= \frac{\sqrt{2}-1}{2-1} = \sqrt{2}-1$

c Solutions of

$\tan^2 k = 2\sec k,\ \ 0 \le k \le 360°$

are solutions of $\cos k = \sqrt{2} - 1$

Calculator solution is 65.5° (1 d.p.)

$\Rightarrow k = 65.5°,\ 360° - 65.5°$

$= 65.5°,\ 294.5°$ (1 d.p.)

10 a As $a = 4\sec x$

$\Rightarrow \sec x = \frac{a}{4}$

$\Rightarrow \cos x = \frac{4}{a}$

As $\cos x = b$

$\Rightarrow b = \frac{4}{a}$

b $c = \cot x$

$\Rightarrow c^2 = \cot^2 x$

$\Rightarrow \frac{1}{c^2} = \tan^2 x$

$\Rightarrow \frac{1}{c^2} = \sec^2 x - 1$

(use $1 + \tan^2 x \equiv \sec^2 x$)

$\Rightarrow \frac{1}{c^2} = \frac{a^2}{16} - 1 \qquad \left(\sec x = \frac{a}{4}\right)$

$\Rightarrow 16 = a^2c^2 - 16c^2$ (multiply by $16c^2$)

$\Rightarrow c^2(a^2 - 16) = 16$

$\Rightarrow c^2 = \frac{16}{a^2 - 16}$

11 a $x = \sec\theta + \tan\theta$

$\frac{1}{x} = \frac{1}{\sec\theta + \tan\theta}$

$= \frac{\sec\theta - \tan\theta}{(\sec\theta + \tan\theta)(\sec\theta - \tan\theta)}$

$= \frac{\sec\theta - \tan\theta}{\sec^2\theta - \tan^2\theta}$

$= \sec\theta - \tan\theta$

(as $1 + \tan^2\theta \equiv \sec^2\theta$

$\Rightarrow \sec^2\theta - \tan^2\theta \equiv 1$)

11 b $x+\frac{1}{x}=\sec\theta+\tan\theta+\sec\theta-\tan\theta$

$=2\sec\theta$

$\Rightarrow\left(x+\frac{1}{x}\right)^2=4\sec^2\theta$

$\Rightarrow\ x^2+2x\times\frac{1}{x}+\frac{1}{x^2}=4\sec^2\theta$

$\Rightarrow\ x^2+\frac{1}{x^2}+2=4\sec^2\theta$

12 $2\sec^2\theta-\tan^2\theta=p$

$\Rightarrow\ 2(1+\tan^2\theta)-\tan^2\theta=p$

$\Rightarrow\ 2+2\tan^2\theta-\tan^2\theta=p$

$\Rightarrow\ \tan^2\theta=p-2$

$\Rightarrow\ \cot^2\theta=\frac{1}{p-2}\quad(p\neq 2)$

$\text{cosec}^2\,\theta=1+\cot^2\theta=1+\frac{1}{p-2}$

$=\frac{(p-2)+1}{p-2}=\frac{p-1}{p-2},\ p\neq 2$