Trigonometric Functions 6E

1 **a** $\arccos(0)$ is the angle α in $0 \le \alpha \le \pi$ for which $\cos\alpha = 0$

Refer to graph of $y = \cos\theta \Rightarrow \alpha = \frac{\pi}{2}$

So $\arccos(0) = \frac{\pi}{2}$

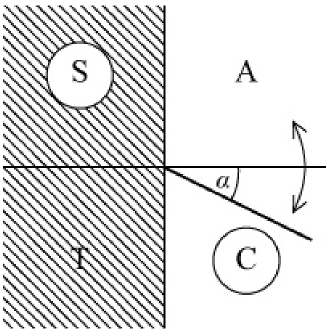
b $\arcsin(1)$ is the angle α in $-\frac{\pi}{2} \le \alpha \le \frac{\pi}{2}$ for which $\sin\alpha = 1$

Refer to graph of $y = \sin\theta \Rightarrow \alpha = \frac{\pi}{2}$

So $\arcsin(1) = \frac{\pi}{2}$

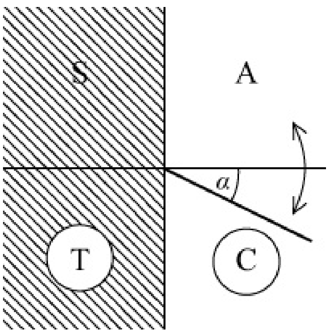
c $\arctan(-1)$ is the angle α in $-\frac{\pi}{2} < \alpha < \frac{\pi}{2}$ for which $\tan\alpha = -1$

So $\arctan(-1) = -\frac{\pi}{4}$



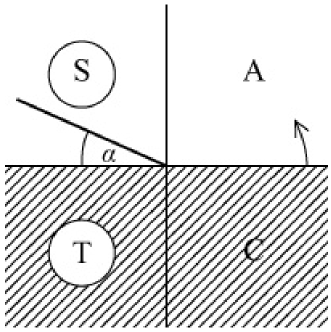
d $\arcsin\left(-\frac{1}{2}\right)$ is the angle α in $-\frac{\pi}{2} \le \alpha \le \frac{\pi}{2}$ for which $\sin\alpha = -\frac{1}{2}$

So $\arcsin\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$



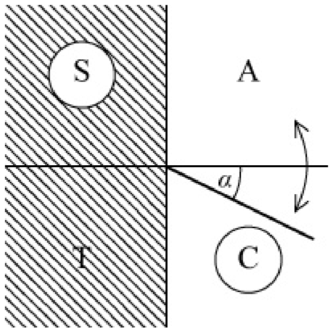
e $\arccos\left(-\frac{1}{\sqrt{2}}\right)$ is the angle α in $0 \le \alpha \le \pi$ for which $\cos\alpha = -\frac{1}{\sqrt{2}}$

So $\arccos\left(-\frac{1}{\sqrt{2}}\right) = \frac{3\pi}{4}$



f $\arctan\left(-\frac{1}{\sqrt{3}}\right)$ is the angle α in $-\frac{\pi}{2} < \alpha < \frac{\pi}{2}$ for which $\tan\alpha = -\frac{1}{\sqrt{3}}$

So $\arctan\left(-\frac{1}{\sqrt{3}}\right) = -\frac{\pi}{6}$



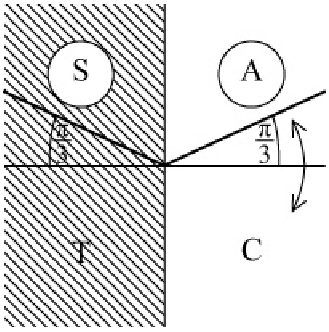
g $\arcsin\left(\sin\frac{\pi}{3}\right)$ is the angle α in $-\frac{\pi}{2} \le \alpha \le \frac{\pi}{2}$ for which $\sin\alpha = \sin\frac{\pi}{3}$

So $\arcsin\left(\sin\frac{\pi}{3}\right) = \frac{\pi}{3}$

1 h $\arcsin\left(\sin\frac{2\pi}{3}\right)$ is the angle α in

$-\frac{\pi}{2}\leq\alpha\leq\frac{\pi}{2}$ for which $\sin\alpha=\sin\frac{2\pi}{3}$

So $\arcsin\left(\sin\frac{2\pi}{3}\right)=\frac{\pi}{3}$



2 a $\arcsin\left(\frac{1}{2}\right)+\arcsin\left(-\frac{1}{2}\right)=\frac{\pi}{6}+\left(-\frac{\pi}{6}\right)=0$

b $\arccos\left(\frac{1}{2}\right)-\arccos\left(-\frac{1}{2}\right)=\frac{\pi}{3}-\frac{2\pi}{3}=-\frac{\pi}{3}$

c $\arctan(1)-\arctan(-1)=\frac{\pi}{4}-\left(-\frac{\pi}{4}\right)=\frac{\pi}{2}$

3 a $\sin\left(\arcsin\frac{1}{2}\right)$

$\arcsin\frac{1}{2}=\alpha$ where $\sin\alpha=\frac{1}{2}$,

and $-\frac{\pi}{2}\leq\alpha\leq\frac{\pi}{2}$

So $\arcsin\frac{1}{2}=\frac{\pi}{6}$

$\Rightarrow\ \sin\left(\arcsin\frac{1}{2}\right)=\sin\frac{\pi}{6}=\frac{1}{2}$

b $\sin\left(\arcsin\left(-\frac{1}{2}\right)\right)$

$\arcsin\left(-\frac{1}{2}\right)=\alpha$

where $\sin\alpha=-\frac{1}{2}$, $-\frac{\pi}{2}\leq\alpha\leq\frac{\pi}{2}$

So $\arcsin\left(-\frac{1}{2}\right)=-\frac{\pi}{6}$

$\Rightarrow\ \sin\left(\arcsin\left(-\frac{1}{2}\right)\right)=\sin\left(-\frac{\pi}{6}\right)=-\frac{1}{2}$

c $\tan\left(\arctan(-1)\right)$

$\arctan(-1)=\alpha$

where $\tan\alpha=-1,\ -\frac{\pi}{2}<\alpha<\frac{\pi}{2}$

So $\arctan(-1)=-\frac{\pi}{4}$

$\Rightarrow\ \tan\left(\arctan(-1)\right)=\tan\left(-\frac{\pi}{4}\right)=-1$

d $\cos(\arccos\ 0)$

$\arccos 0=\alpha$ where $\cos\alpha=0,\ 0\leq\alpha\leq\pi$

So $\arccos 0=\frac{\pi}{2}$

$\Rightarrow\ \cos(\arccos 0)=\cos\frac{\pi}{2}=0$

4 a $\sin\left(\arccos\frac{1}{2}\right)$

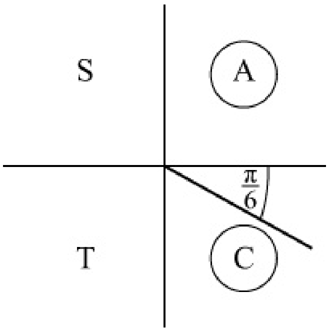
$\arccos\frac{1}{2}=\frac{\pi}{3}$

$\sin\frac{\pi}{3}=\frac{\sqrt{3}}{2}$

4 b $\cos\left(\arcsin\left(-\frac{1}{2}\right)\right)$

$\arcsin\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$

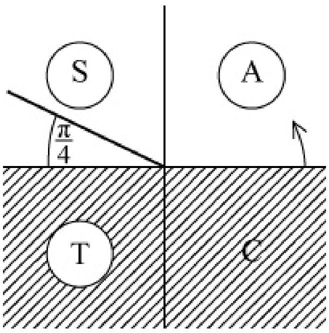
$\cos\left(-\frac{\pi}{6}\right) = +\frac{\sqrt{3}}{2}$



c $\tan\left(\arccos\left(-\frac{\sqrt{2}}{2}\right)\right)$

$\arccos\left(-\frac{\sqrt{2}}{2}\right) = \alpha$

where $\cos\alpha = -\frac{\sqrt{2}}{2}$, $0 \le \alpha \le \pi$



So $\arccos\left(-\frac{\sqrt{2}}{2}\right) = \frac{3\pi}{4}$

$\tan\frac{3\pi}{4} = -1$

d $\sec(\arctan\sqrt{3})$

$\arctan\sqrt{3} = \frac{\pi}{3}$

(the angle whose $\tan$ is $\sqrt{3}$)

$\sec\frac{\pi}{3} = \frac{1}{\cos\frac{\pi}{3}} = \frac{1}{\frac{1}{2}} = 2$

e $\operatorname{cosec}(\arcsin(-1))$

$\arcsin(-1) = \alpha$

where $\sin\alpha = -1$, $-\frac{\pi}{2} \le \alpha \le \frac{\pi}{2}$

So $\arcsin(-1) = -\frac{\pi}{2}$

$\Rightarrow \operatorname{cosec}(\arcsin(-1)) = \frac{1}{\sin\left(-\frac{\pi}{2}\right)}$

$= \frac{1}{-1} = -1$

f $\sin\left(2\arcsin\left(\frac{\sqrt{2}}{2}\right)\right)$

$\arcsin\frac{\sqrt{2}}{2} = \frac{\pi}{4}$

So $\sin\left(2\arcsin\left(\frac{\sqrt{2}}{2}\right)\right) = \sin\frac{\pi}{2} = 1$

5 As k is positive, the first two positive solutions of $\sin x = k$ are $\arcsin k$ and $\pi - \arcsin k$ i.e. α and $\pi - \alpha$
(Try a few examples, taking specific values for k).

6 a $\arcsin x$ is the angle α in $-\frac{\pi}{2} \le \alpha \le \frac{\pi}{2}$ such that $\sin\alpha = x$

In this case $x = \sin k$ where $0 < k < \frac{\pi}{2}$

As $\sin$ is an increasing function

$\sin 0 < x < \sin\frac{\pi}{2}$

$\Rightarrow 0 < x < 1$

b i $\cos k = \pm\sqrt{1-\sin^2 k} = \pm\sqrt{1-x^2}$

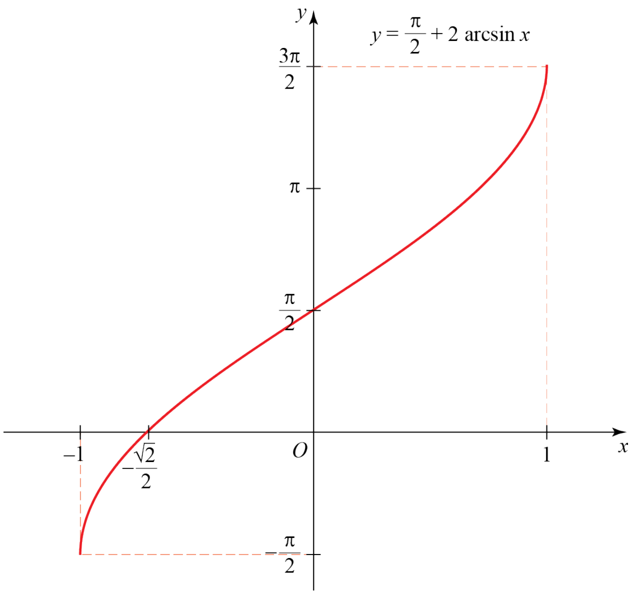
k is in the 1st quadrant $\Rightarrow \cos k > 0$

So $\cos k = \sqrt{1-x^2}$

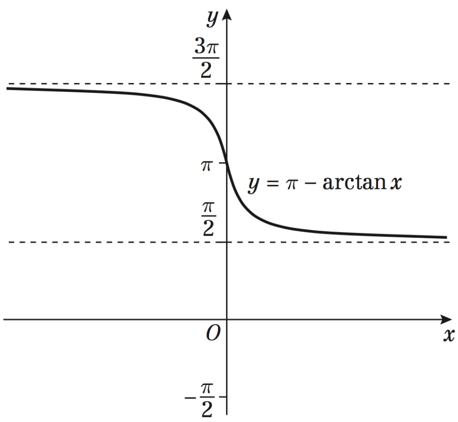
ii $\tan k = \frac{\sin k}{\cos k} = \frac{x}{\sqrt{1-x^2}}$

6 c k is now in the 4th quadrant, where $\cos k$ is positive. So the value of $\cos k$ remains the same and there is no change to $\tan k$.

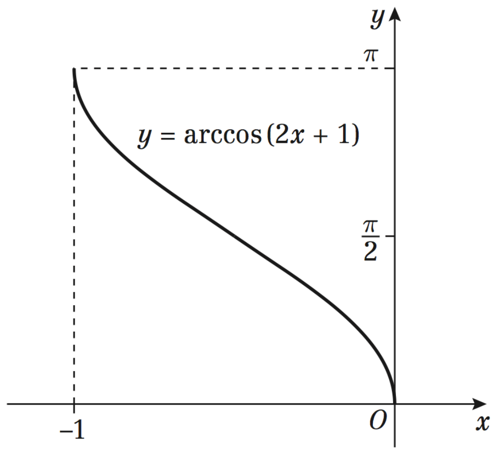
7 a The graph of $y = \frac{\pi}{2} + 2\arcsin x$ is $y = \arcsin x$ stretched by a scale factor 2 in the y direction and then translated by the vector $\begin{pmatrix} 0 \\ \frac{\pi}{2} \end{pmatrix}$



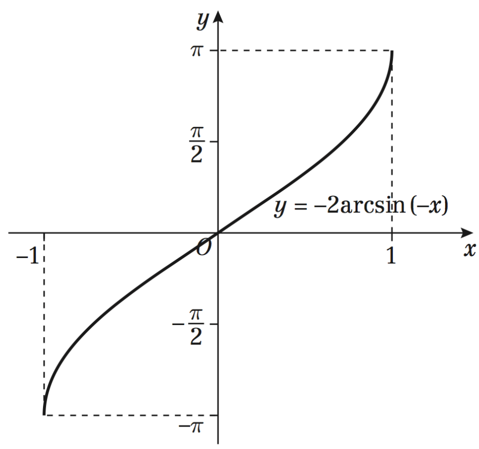
b The graph of $y = \pi - \arctan x$ is $y = \arctan x$ reflected in the x-axis and then translated by the vector $\begin{pmatrix} 0 \\ \pi \end{pmatrix}$



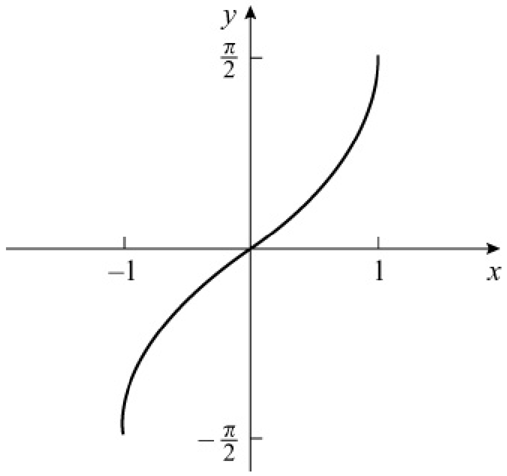
7 c The graph of $y = \arccos(2x+1)$ is $y = \arccos x$ translated by the vector $\begin{pmatrix} -1 \\ 0 \end{pmatrix}$ and then stretched by scale factor $\frac{1}{2}$ in the x direction



d The graph of $y = -2\arcsin(-x)$ is $y = \arcsin x$ reflected in the y-axis, then reflected in the x-axis and then stretched by a scale factor 2 in the y direction



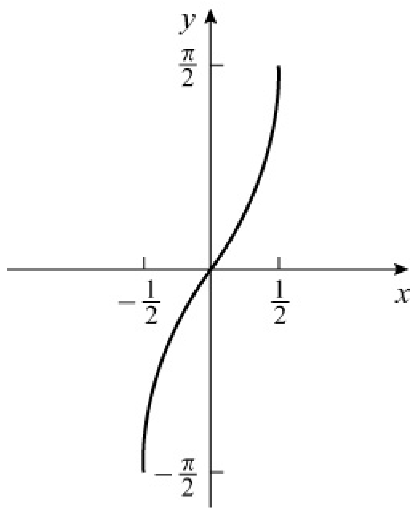
8 a $y = \arcsin x$



Range is $-\frac{\pi}{2} \le f(x) \le \frac{\pi}{2}$

b The graph of $y = f(2x)$ is the graph of $y = f(x)$ stretched in the x direction by scale factor $\frac{1}{2}$

$y = g(x)$



c $g: x \mapsto \arcsin 2x$

The domain is $-\frac{1}{2} \le x \le \frac{1}{2}$

d Let $y = \arcsin 2x$

$\Rightarrow 2x = \sin y$

$\Rightarrow x = \frac{1}{2}\sin y$

So $g^{-1}: x \mapsto \frac{1}{2}\sin x, \; -\frac{\pi}{2} \le x \le \frac{\pi}{2}$

9 a Let $y = \arccos x$

As $0 \le x \le 1 \Rightarrow 0 \le y \le \frac{\pi}{2}$

$\cos y = x$, and using $\cos^2 y + \sin^2 y \equiv 1$

$\Rightarrow \sin^2 y = 1 - \cos^2 y$

$\Rightarrow \sin y = \sqrt{1 - \cos^2 y} = \sqrt{1 - x^2}$

Note, $\sin y \ge 0$ since $0 \le y \le \frac{\pi}{2}$

so $\sin y \ne -\sqrt{1 - x^2}$

$\sin y = \sqrt{1 - x^2}$

$\Rightarrow y = \arcsin \sqrt{1 - x^2}$

Therefore, $\arccos x = \arcsin \sqrt{1 - x^2}$

for $0 \le x \le 1$

b For $-1 \le x \le 0$, $\frac{\pi}{2} \le \arccos x \le \pi$

But arcsin has a range of $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

So $\arccos x \ne \arcsin \sqrt{1 - x^2}$,

for $-1 \le x \le 0$

An alternative approach is to provide a counterexample.

Let $x = -\frac{1}{\sqrt{2}}$

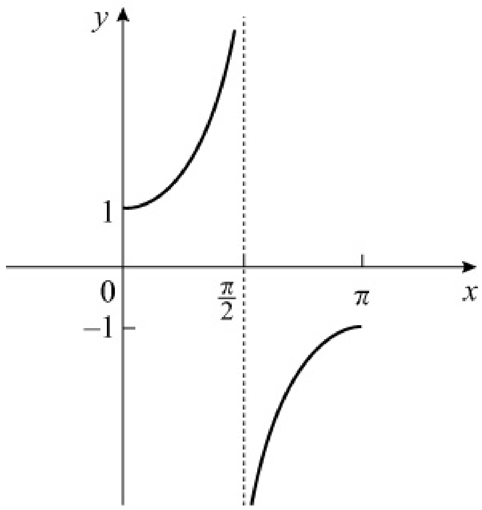
$\arccos x = \frac{3\pi}{4}$

$\arcsin \sqrt{1 - x^2} = \arcsin \frac{1}{\sqrt{2}} = \frac{\pi}{4}$

So $\arccos x \ne \arcsin \sqrt{1 - x^2}$

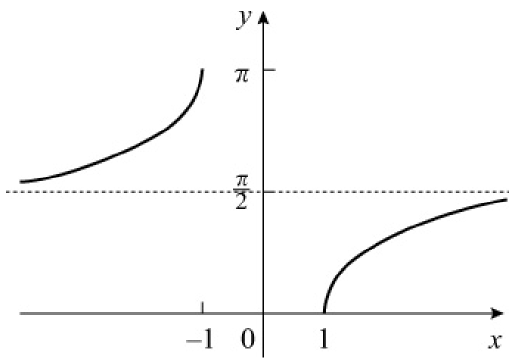
Challenge

a $y = \sec x$



b Reflect the graph drawn for part (a) in the line $y = x$

$y = \text{arc}\sec x,\ x \leq -1,\ x \geq 1$



Range is $0 \leq \text{arc}\sec x \leq \pi$, for $\text{arc}\sec x \neq \dfrac{\pi}{2}$