Trigonometry and modelling 7C

1 $\sin 2A = \sin(A+A) = \sin A\cos A + \cos A\sin A$
$= 2\sin A\cos A$

2 a $\cos 2A = \cos(A+A)$
$= \cos A\cos A - \sin A\sin A$
$= \cos^2 A - \sin^2 A$

b i $\cos 2A = \cos^2 A - \sin^2 A$
Use $\cos^2 A + \sin^2 A = 1$ to simplify, so
$\cos 2A = \cos^2 A - (1-\cos^2 A)$
$= 2\cos^2 A - 1$

ii $\cos 2A = \cos^2 A - \sin^2 A$
$= (1-\sin^2 A) - \sin^2 A$
$= 1 - 2\sin^2 A$

3 $\tan 2A = \tan(A+A)$
$= \dfrac{\tan A + \tan A}{1 - \tan A\tan A} = \dfrac{2\tan A}{1-\tan^2 A}$

4 a $2\sin 10°\cos 10° = \sin 20°$
(using $2\sin A\cos A \,^{\circ}\, \sin 2A$)

b $1 - 2\sin^2 25° = \cos 50°$
using $\cos 2A \equiv 1 - 2\sin^2 A$

c $\cos^2 40° - \sin^2 40° = \cos 80°$
using $\cos 2A \equiv \cos^2 A - \sin^2 A$

d $\dfrac{2\tan 5°}{1-\tan^2 5°} = \tan 10°$
using $\tan 2A \equiv \dfrac{2\tan A}{1-\tan^2 A}$

e $\dfrac{1}{2\sin 24.5°\cos 24.5°} = \dfrac{1}{\sin 49°}$
$= \operatorname{cosec} 49°$

f $6\cos^2 30° - 3 = 3(2\cos^2 30° - 1)$
$= 3\cos 60°$

g $\dfrac{\sin 8°}{\sec 8°} = \sin 8°\cos 8°$
$= \dfrac{1}{2}(2\sin 8°\cos 8°) = \dfrac{1}{2}\sin 16°$

h $\cos^2\dfrac{\pi}{16} - \sin^2\dfrac{\pi}{16} = \cos\dfrac{2\pi}{16} = \cos\dfrac{\pi}{8}$

5 a $2\sin 22.5°\cos 22.5° = \sin 2\times 22.5°$
$= \sin 45° = \dfrac{\sqrt{2}}{2}$

b $2\cos^2 15° - 1 = \cos(2\times 15°)$
$= \cos 30° = \dfrac{\sqrt{3}}{2}$

c $(\sin 75° - \cos 75°)^2$
$= \sin^2 75° + \cos^2 75° - 2\sin 75°\cos 75°$
$= 1 - \sin(2\times 75°)$
as $\sin^2 75° + \cos^2 75° = 1$, and this gives
$(\sin 75° - \cos 75°)^2 = 1 - \sin 150°$
$= 1 - \dfrac{1}{2} = \dfrac{1}{2}$

d $\dfrac{2\tan\frac{\pi}{8}}{1-\tan^2\frac{\pi}{8}} = \tan\left(2\times\dfrac{\pi}{8}\right) = \tan\dfrac{\pi}{8} = 1$

6 a $(\sin A + \cos A)^2$
$= \sin^2 A + 2\sin A\cos A + \cos^2 A$
$= 1 + \sin 2A$

b $\left(\sin\dfrac{\pi}{8} + \cos\dfrac{\pi}{8}\right)^2$
$= 1 + \sin\dfrac{\pi}{4} = 1 + \dfrac{\sqrt{2}}{2} = \dfrac{2+\sqrt{2}}{4}$

7 a $\cos^2 3\theta - \sin^2 3\theta = \cos(2\times 3\theta) = \cos 6\theta$

b $6\sin 2q\cos 2q = 3(2\sin 2q\cos 2q)$
$= 3\sin(2\times 2\theta)$
$= 3\sin 4\theta$

c $\dfrac{2\tan\frac{\theta}{2}}{1-\tan^2\frac{\theta}{2}} = \tan\left(2\times\dfrac{\theta}{2}\right) = \tan\theta$

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7 d $2-4\sin^2\frac{\theta}{2}=2\left(1-2\sin^2\left(\frac{\theta}{2}\right)\right)$

$=2\cos\left(2\times\frac{\theta}{2}\right)=2\cos\theta$

e $\sqrt{1+\cos 2q}=\sqrt{1+(2\cos^2 q-1)}$

$=\sqrt{2\cos^2 q}$

$=\sqrt{2}\cos q$

f $\sin^2\theta\cos^2\theta=\frac{1}{4}(4\sin^2\theta\cos^2\theta)$

$=\frac{1}{4}(2\sin\theta\cos\theta)^2$

$=\frac{1}{4}\sin^2 2\theta$

g $4\sin q\cos q\cos 2q=2(2\sin q\cos q)\cos 2q$

$=2\sin 2\theta\cos 2\theta$

$=\sin 4\theta$

As $\sin 2A=2\sin A\cos A$ with $A=2\theta$

h $\frac{\tan q}{\sec^2 q-2}=\frac{\tan q}{(1+\tan^2 q)-2}$

$=\frac{\tan q}{\tan^2 q-1}$

$=-\frac{\tan q}{1-\tan^2 q}$

$=-\frac{1}{2}\left(\frac{2\tan\theta}{1-\tan^2\theta}\right)$

$=-\frac{1}{2}\tan 2\theta$

i $\cos^4 q-2\sin^2 q\cos^2 q+\sin^4 q$

$=(\cos^2 q-\sin^2 q)^2$

$=\cos^2 2q$

8 $p=2\cos\theta \Rightarrow \cos\theta=\frac{p}{2}$

$\cos 2\theta=q$

Using $\cos 2\theta=2\cos^2\theta-1$

$\Rightarrow q=2\left(\frac{p}{2}\right)^2-1$

$\Rightarrow q=\frac{p^2}{2}-1$

9 a $\cos^2 q=x,\ \cos 2q=1-y$

Using $\cos 2\theta=2\cos^2\theta-1$

$\Rightarrow 1-y=2x-1$

$\Rightarrow y=2-2x=2(1-x)$

Any form of this equation is correct

b $y=\cot 2\theta \Rightarrow \tan 2\theta=\frac{1}{y}$

$x=\tan\theta$

Using $\tan 2\theta=\frac{2\tan\theta}{1-\tan^2\theta}$

$\Rightarrow \frac{1}{y}=\frac{2x}{1-x^2}$

$\Rightarrow 2xy=1-x^2$

Any form of this equation is correct

c $x=\sin q,\ y=2\sin q\cos q$

$\Rightarrow y=2x\cos\theta$

$\Rightarrow \cos\theta=\frac{y}{2x}$

Using $\sin^2\theta+\cos^2\theta\equiv 1$

$\Rightarrow x^2+\frac{y^2}{4x^2}=1$

$\Rightarrow 4x^4+y^2=4x^2$

or $y^2=4x^2(1-x^2)$

Any form of this equation is correct

d $x=3\cos 2\theta+1 \Rightarrow \cos 2\theta=\frac{x-1}{3}$

$y=2\sin\theta \Rightarrow \sin\theta=\frac{y}{2}$

Using $\cos 2\theta=1-2\sin^2\theta$

$\Rightarrow \frac{x-1}{3}=1-\frac{2y^2}{4}=1-\frac{y^2}{2}$

Multiplying both sides by 6 gives

$2(x-1)=6-3y^2$

$\Rightarrow 3y^2=6-2(x-1)=8-2x$

$\Rightarrow y^2=\frac{2(4-x)}{3}$

Any form of this equation is correct

10 $\cos 2x=2\cos^2 x-1$

$\cos 2x=2\left(\frac{1}{4}\right)^2-1=\frac{1}{8}-1=-\frac{7}{8}$

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11 $\cos 2q = 1 - 2\sin^2 q$

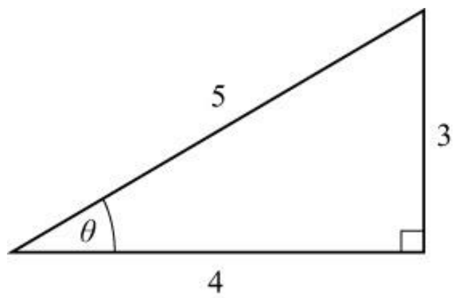
So $\frac{23}{25} = 1 - 2\sin^2\theta$

$\Rightarrow 2\sin^2\theta = 1 - \frac{23}{25} = \frac{2}{25}$

$\Rightarrow \sin^2\theta = \frac{1}{25}$

$\Rightarrow \sin\theta = \pm\frac{1}{5}$

12 Draw a right-angled triangle with θ as one of the angles. The hypotenuse is 5



So $\sin\theta = \frac{3}{5}$, $\cos\theta = \frac{4}{5}$, $\tan\theta = \frac{3}{4}$

a i $\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta} = \frac{\frac{3}{2}}{1-\frac{9}{16}}$

$= \frac{\frac{3}{2}}{\frac{7}{16}} = \frac{3}{2}\times\frac{16}{7} = \frac{24}{7}$

ii $\sin 2\theta = 2\sin\theta\cos\theta = 2\times\frac{3}{5}\times\frac{4}{5} = \frac{24}{25}$

iii $\cos 2\theta = \cos^2\theta - \sin^2\theta = \frac{16}{25} - \frac{9}{25} = \frac{7}{25}$

b

$\sin 4\theta = 2\sin 2\theta\cos 2\theta$

$= 2\times\frac{24}{25}\times\frac{7}{25} = \frac{336}{625}$

13 a i $\cos 2A = 2\cos^2 A - 1$

$= 2\left(-\frac{1}{3}\right)^2 - 1 = \frac{2}{9} - 1 = -\frac{7}{9}$

ii $\cos 2A = 1 - 2\sin^2 A$

$\Rightarrow -\frac{7}{9} = 1 - 2\sin^2 A$

$\Rightarrow 2\sin^2 A = 1 + \frac{7}{9} = \frac{16}{9}$

$\Rightarrow \sin^2 A = \frac{8}{9}$

$\Rightarrow \sin A = \pm\frac{\sqrt{8}}{3} = \pm\frac{2\sqrt{2}}{3}$

But A is in the second quarter so $\sin A$ is positive, and the solution is

$\sin A = \frac{2\sqrt{2}}{3}$

iii $\operatorname{cosec} 2A = \frac{1}{\sin 2A} = \frac{1}{2\sin A\cos A}$

$= \frac{1}{2\times\frac{2\sqrt{2}}{3}\times\left(-\frac{1}{3}\right)}$

$= -\frac{9}{4\sqrt{2}} = -\frac{9\sqrt{2}}{8}$

b $\tan 2A = \frac{\sin 2A}{\cos 2A} = \frac{-\frac{4\sqrt{2}}{9}}{-\frac{7}{9}}$

$= -\frac{4\sqrt{2}}{9}\times -\frac{9}{7} = \frac{4\sqrt{2}}{7}$

14 Using $\tan\theta = \frac{2\tan\frac{\theta}{2}}{1-\tan^2\frac{\theta}{2}}$

$\Rightarrow \frac{3}{4} = \frac{2\tan\frac{\theta}{2}}{1-\tan^2\frac{\theta}{2}}$

$\Rightarrow 3 - 3\tan^2\frac{\theta}{2} = 8\tan\frac{\theta}{2}$

$\Rightarrow 3\tan^2\frac{\theta}{2} + 8\tan\frac{\theta}{2} - 3 = 0$

$\Rightarrow \left(3\tan\frac{\theta}{2} - 1\right)\left(\tan\frac{\theta}{2} + 3\right) = 0$

so $\tan\frac{\theta}{2} = \frac{1}{3}$ or $\tan\frac{\theta}{2} = -3$

But $\pi < \theta < \frac{3\pi}{2}$ so $\frac{\pi}{2} < \frac{\theta}{2} < \frac{3\pi}{4}$

As $\frac{\theta}{2}$ is in the second quadrant, so $\tan\frac{\theta}{2}$ is negative, and the solution is $\tan\frac{\theta}{2} = -3$

15 $\cos x + \sin x = m$

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$\cos x - \sin x = n$

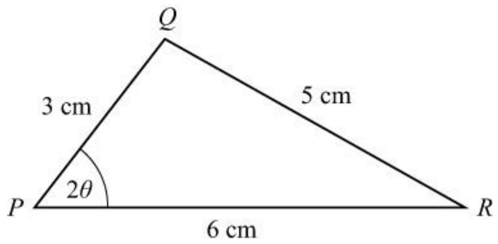
Multiply the equations

$(\cos x + \sin x)(\cos x - \sin x) = mn$

$\Rightarrow \cos^2 x - \sin^2 x = mn$

$\Rightarrow \cos 2x = mn$

16



a Using cosine rule with

$$\cos P = \frac{q^2 + r^2 - p^2}{2qr}$$

$$\cos 2\theta = \frac{36 + 9 - 25}{2 \times 6 \times 3} = \frac{20}{36} = \frac{5}{9}$$

b Using $\cos 2q = 1 - 2\sin^2 q$

$$\frac{5}{9} = 1 - 2\sin^2\theta$$

$$\Rightarrow 2\sin^2\theta = 1 - \frac{5}{9} = \frac{4}{9}$$

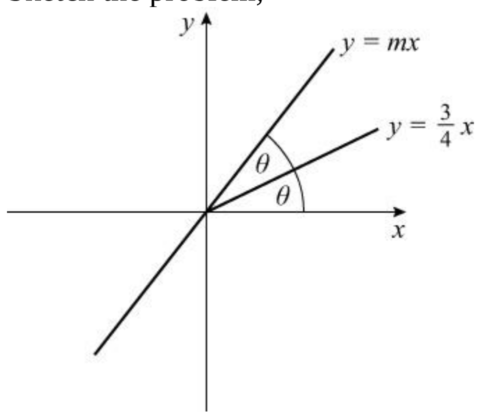
$$\Rightarrow \sin^2\theta = \frac{2}{9}$$

$$\Rightarrow \sin\theta = \pm\frac{\sqrt{2}}{3}$$

As 2θ is acute, θ must be in the first quadrant so $\sin\theta$ is positive, so

$$\sin\theta = \frac{\sqrt{2}}{3}$$

17 Sketch the problem,



a The gradient of line l is $\frac{3}{4}$, which is $\tan\theta$.

So $\tan\theta = \frac{3}{4}$

b The gradient of $y = mx$ is m and as $y = \frac{3}{4}x$ bisects the angle between $y = mx$ and the x-axis

$$m = \tan 2\theta = \frac{2\tan\theta}{1 - \tan^2\theta}$$

$$= \frac{2 \times \frac{3}{4}}{1 - \left(\frac{3}{4}\right)^2} = \frac{\frac{3}{2}}{\frac{7}{16}} = \frac{3}{2} \times \frac{16}{7} = \frac{24}{7}$$

18 a $\cos 2A = \cos(A + A)$

$= \cos A \cos A - \sin A \sin A$

$= \cos^2 A - \sin^2 A$

$= \cos^2 A - (1 - \cos^2 A)$

$= 2\cos^2 A - 1$

b The lines intersect when

$4\cos 2x = 6\cos^2 x - 3\sin 2x$

This equation can be written as

$\cos 2x + 3\cos 2x = 6\cos^2 x + 3\sin 2x$

Use the fact that $3\cos 2x = 6\cos^2 x - 3$, so the equation becomes

$\cos 2x + 6\cos^2 x - 3$

$= 6\cos x^2 - 3\sin 2x$

$\Rightarrow \cos 2x - 3 = 3\sin 2x$

$\Rightarrow \cos 2x + 3\sin 2x - 3 = 0$

19 $\tan 2A \equiv \frac{\sin 2A}{\cos 2A} \equiv \frac{2\sin A\cos A}{\cos^2 A - \sin^2 A}$

$$\equiv \frac{\frac{2\sin A\cos A}{\cos^2 A}}{\frac{\cos^2 A - \sin^2 A}{\cos^2 A}}$$

$$\equiv \frac{\frac{2\sin A}{\cos A}}{1 - \frac{\sin^2 A}{\cos^2 A}}$$

$$\equiv \frac{2\tan A}{1 - \tan^2 A}$$

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